GALERKIN METHOD FOR ONE NONLINEAR INTEGRO-DIFFERENTIAL EQUATION WITH MIXED BOUNDARY CONDITIONS

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Abstract

Galerkin finite element method for the approximation of a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. Initial-boundary value problem with mixed boundary condition is investigated. The convergence of the finite element scheme is proved. The rate of convergence is given too.

Key words and phrases: Nonlinear integro-differential equations, finite elements, Galerkin method.

AMS subject classification: 45K05, 35K55, 65N30, 65M60.

1 Introduction

The goal of this paper is a study of Galerkin finite element method for approximation of a nonlinear integro-differential equation arising in mathematical modeling of the process of a magnetic field penetrating into a substance. If the coefficient of thermal heat capacity and electroconductivity of the substance highly dependent on temperature, then the Maxwell's system [1], that describe above-mentioned process, can be rewritten in the following form [2]:

$$\frac{\partial W}{\partial t} = -rot\left[a\left(\int_0^t |rotW|^2\, d\tau\right) rotW\right], \tag{1.1}$$

where $W = (W_1, W_2, W_3)$ is a vector of the magnetic field and the function $a = a(\sigma)$ is defined for $\sigma \in [0, \infty)$. Let us consider magnetic field W, with the form $W = (0, 0, u)$, where $u = u(x, t)$ is a scalar function of time and of one spatial variables. Then $rotW = \left(0, -\frac{\partial u}{\partial x}, 0\right)$ and system (1.1) will take the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial u}{\partial x}\right)^2 d\tau\right)\frac{\partial u}{\partial x}\right]. \tag{1.2}$$

Note that (1.2) is complex, but special cases of such type models were investigated, see [2]-[12]. The existence of global solutions for initial-boundary value problems of such models have been proven in [2]-[5],[11] by using the Galerkin and compactness methods [13],[14]. The asymptotic behavior of the solutions of (1.2) have been the subject of intensive research in recent years, (see e.g. [11],[15]-[21]).

In [8] some generalization of equations of type (1.1) is proposed. Assume the temperature of the considered body is depending on time, but independent of the space coordinates. If the magnetic field again has the form $W = (0, 0, u)$ and $u = u(x, t)$, then the same process of penetration of the magnetic field into the material is modeled by the following integro-differential equation [8]

$$\frac{\partial u}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial u}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 u}{\partial x^2}. \tag{1.3}$$

The asymptotic behavior of solutions of the initial-boundary value problem for the equation (1.3) and the convergence of the finite difference approximation for the case $a(\sigma) = 1 + \sigma$ with first kind boundary conditions were studied in [18]. The asymptotic behavior of solutions of the initial-boundary value problem for the equation (1.3) with mixed boundary conditions are considered in [22].

Note that in [18], [23]-[26] difference schemes for (1.2),(1.3) type models were also investigated. Difference schemes for one nonlinear parabolic integro-differential model similar to (1.2) were studied in [27] and [28].

The purpose of this study is the Galerkin finite element approximation of the initial-boundary value problem with mixed boundary conditions for (1.3) in the case $a(\sigma) = 1 + \sigma$.

2 Statement of Problem and Main Results

Consider the following initial-boundary value problem:

$$\frac{\partial u}{\partial t} = (1 + \sigma(t))\frac{\partial^2 u}{\partial x^2} + f(x, t), \quad (x, t) \in (0, 1) \times (0, T), \tag{2.1}$$

$$u(0,t)=0, \quad \left.\frac{\partial u}{\partial x}\right|_{x=1}=0, \quad t\geq 0, \tag{2.2}$$

$$u(x,0)=u_0(x), \quad x\in[0,1], \tag{2.3}$$

where

$$\sigma(t)=\int_0^t\int_0^1\left(\frac{\partial u}{\partial x}\right)^2 dxd\tau$$

and $u_0(x)$ is a given function.

We use the usual spaces C^k, L_p, H^k and norm

$$||u(\cdot,t)||_r=\left\{\int_0^1\sum_{i=0}^r\left|\frac{\partial^i u(x,t)}{\partial x^i}\right|^2 dx\right\}^{1/2}.$$

Let us assume that $u=u(x,t)$, is a solution of problem (2.1)-(2.3) such that $u(\cdot,t)$, $\frac{\partial u(\cdot,t)}{\partial x}$, $\frac{\partial u(\cdot,t)}{\partial t}$, $\frac{\partial^2 u(\cdot,t)}{\partial t\,\partial x}$ are all in $C^0([0,\infty);\ L_2(0,1))$, while $\frac{\partial^2 u(\cdot,t)}{\partial t^2}$, is in $L_2((0,\infty);\ L_2(0,1))$.

It is easy to obtain the continuous dependence of solutions on initial data. Indeed, by multiplying equation (2.1) by u, after simple transformations, we get the following estimate

$$\|u\|_0\leq\|u_0\|_0.$$

The following theorems are strengthening the result above [22].

Theorem 2.1. *If* $u_0\in H^2(0,1)$, $u_0(0)=0$, $\left.\frac{\partial u_0(x)}{\partial x}\right|_{x=1}=0$, *then for the solution of problem (2.1)-(2.3) the following estimate is true*

$$\|u\|_0+\left\|\frac{\partial u}{\partial x}\right\|_0\leq C\exp\left(-\frac{t}{2}\right).$$

Here and below C denotes positive constants.

Theorem 2.2. *If* $u_0\in H^3(0,1)$, $u_0(0)=0$, $\left.\frac{\partial u_0(x)}{\partial x}\right|_{x=1}=0$, *then for the solution of problem (2.1)-(2.3) the following relations hold:*

$$\left|\frac{\partial u(x,t)}{\partial x}\right|\leq C\exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial u(x,t)}{\partial t}\right|\leq C\exp\left(-\frac{t}{2}\right).$$

These theorems guarantee the continuous dependence of solution on initial data.

Remark: The existence of globally defined solutions of problems (2.1)-(2.3) can be obtained by a routine procedure. One first establishes the existence of local solutions on a maximal time interval and then uses the derived a priori estimates to show that the solutions cannot escape in finite time. This approach is used very often (see, for example, [13], [14]).

The uniqueness of solutions of problem (2.1)-(2.3) can be proven as well. Indeed, let u, $\bar{u}$, be two solutions of problem (2.1)-(2.3) and $z(x,t) = u(x,t) - \bar{u}(x,t)$. We have

$$\frac{\partial z}{\partial t} = [1+\sigma(t)]\,\frac{\partial^2 u}{\partial x^2} - [1+\bar{\sigma}(t)]\,\frac{\partial^2 \bar{u}}{\partial x^2}, \tag{2.4}$$

where

$$\bar{\sigma}(t) = \int_0^t \int_0^1 \left(\frac{\partial \bar{u}}{\partial x}\right)^2 dx d\tau.$$

Multiplying (2.4) by z and integrating, after some transformations we get

$$\int_0^1 z^2 dx + \frac{1}{2}\,[\sigma(t) - \bar{\sigma}(t)]^2 \le 0.$$

From this we immediately get $z(x,t) \equiv 0$, which proves the uniqueness of the solution.

Now we are going to study Galerkin finite element method for the investigated problem. One of the ingredients of finite-element method is a variational formulation of problem. Let us denote by H the linear space of functions u satisfying (2.2) and

$$||u(\cdot,t)||_1 < \infty.$$

The variational formulation of problem (2.1)-(2.3) can be stated as follows: find a function $u(x,t) \in H$ for which

$$< v, \frac{\partial u}{\partial t} > + < (1+\sigma(t))\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x} > = < f, v >, \tag{2.5}$$

and

$$< v, u(x,0) > = < v, u_0(x) >, \qquad \forall v \in H, \tag{2.6}$$

where $< p(x), q(x) > = \int\limits_0^1 p(x)q(x)dx$.

To approximate the solution of (2.5),(2.6) we require that u lies in a finite-dimensional subspace S_h of H for each t (see e.g. [29]).

The approximation $u^h \in S_h$ to u is defined by the following variational analog of (2.5),(2.6):

Find a $u^h \in S_h$ such that

$$< v^h, \frac{\partial u^h}{\partial t} > + < (1+\sigma_h(t))\frac{\partial u^h}{\partial x}, \frac{\partial v^h}{\partial x} >=< f, v^h >, \qquad (2.7)$$

and

$$< v^h, u^h(x,0) >=< v^h, u_0(x) >, \qquad \forall v^h \in S_h, \qquad (2.8)$$

where

$$\sigma_h(t) = \int\limits_0^t \int\limits_0^1 \left(\frac{\partial u^h}{\partial x}\right)^2 dx d\tau.$$

Once a basis has been selected for S_h, (2.7),(2.8) are equivalent to a set of N integro-differential equations, where N is a dimension of S_h.

Using the technic of the work [30], where same question for initial-boundary value problem with first kind boundary conditions for equation (2.1) is studied, the following statement is proved.

Theorem 2.3. *The error in the finite element approximation u^h generated by (2.7),(2.8) satisfies the relation*

$$|||u-u^h|||_1 \le h^{j-1}C\left\{\frac{1}{2}h^2||u_0||_0^2 + Ch^2|||u_t|||_0^2\right.$$

$$\left. +C^2\left[1+h^{2(j-1)}|||u|||_0^2\right]|||u|||_0^2 + C^3\left(h^{j-1}|||u|||_0^2 + []u[]\right)^2\right\}^{1/2}, \quad j>1,$$

where

$$|||E|||_r = \int\limits_0^T \int\limits_0^1 \sum_{i=0}^r \left|\frac{\partial^i E(x,t)}{\partial x^i}\right|^2 dx dt$$

and

$$[]u[] = \int\limits_0^T \int\limits_0^1 |u|\, dx d\tau.$$

For the numerical solution of (2.7),(2.8) we let $\phi_1(x), \ldots, \phi_N(x)$ be a basis for S_h. Therefore $u^h \in S_h$ can be represented by

$$u^h(x,t) = \sum_{j=1}^N u_j(t)\phi_j(x).$$

Since (2.7),(2.8) are valid for all $v^h \in S_h$, one can let $v^h = \phi_k$. This yields the following system for the weight $\mathbf{u}(t)$:

$$M\dot{\mathbf{u}} + K(u)\mathbf{u} = \mathbf{F}, \qquad (2.9)$$

$$M\mathbf{u}(0) = \mathbf{U}, \tag{2.10}$$

where

$$M_{jk} =< \phi_k, \phi_j >,$$

$$K(u)_{jk} =< (1+\sigma_h(t))\phi'_k, \phi'_j >,$$

$$\mathbf{F}_j =< \phi_j, f >, \qquad \mathbf{U}_j =< \phi_j, u_0 > .$$

Solving the system (2.9),(2.10), we have carried out various experiments and in all cases we noticed the agreement with the theoretical results.

References

1. L. Landau, E. Lifschitz, *Electrodynamics of Continuous Media. Course of Theoretical Physics,* Vol. 8. (Translated from the Russian) Pergamon Press, Oxford-London-New York-Paris; Addision-Wesley Publishing Co., Inc., Reading, Mass., 1960; Russian original: Gosudarstv. Izdat. Tehn-Teor. Lit., Moscow, 1957.

2. D.G. Gordeziani, T.A. Dzhangveladze, T.K. Korshia, Existence and uniqueness of the solution of a class of nonlinear parabolic problems, (Russian) *Differ. Uravn.*, **19**, (1983), 1197–1207. English translation: *Diff. Equ.*, **19**, (1983), 887–895.

3. T. A. Dzhangveladze, First boundary-value problem for a nonlinear equation of parabolic type, (Russian) *Dokl. Akad. Nauk SSSR*, **269**, (1983), 839–842. English translation: *Soviet Phys. Dokl.*, **28**, (1983), 323–324.

4. T. A. Dzhangveladze, *An Investigation of the First Boundary-Value Problem for Some Nonlinear Parabolic Integrodifferential Equations*, (Russian) Tbilisi State University, Tbilisi, 1983.

5. T. A. Dzhangveladze, A nonlinear integro-differential equation of parabolic type, (Russian) *Differ. Uravn.,* **21**, (1985), 41–46. English translation: *Diff. Eq.,* **21**, (1985), 32–36.

6. G. Laptev, Quasilinear parabolic equations which contains in coefficients Volterra's operator, (Russian) *Math. Sbornik*, **136**, (1988), 530–545. English translation: *Sbornik Math.*, **64**, (1989), 527–542.

7. G. Laptev, Mathematical singularities of a problem on the penetration of a magnetic field into a substance, (Russian) *Zh. Vychisl. Mat. Mat. Fiz.*, **28**, (1988), 1332–1345. English translation: *U.S.S.R. Comput. Math. Math. Phys.*, **28**, (1990), 35–45.

8. G. Laptev, *Quasilinear Evolution Partial Differential Equations with Operator Coefficients*, (Russian) Doctoral Dissertation, Moscow, 1990.

9. Y. Lin, H. M. Yin, Nonlinear parabolic equations with nonlinear functionals, *J. Math. Anal. Appl.*, **168**, (1992), 28–41.

10. N. Long, A. Dinh, Nonlinear parabolic problem associated with the penetration of a magnetic field into a substance, *Math. Mech. Appl. Sci.*, **16**, (1993), 281–295.

11. T. A. Jangveladze, On one class of nonlinear integro-differential equations, *Rep. Semin. I. Vekua Inst. Appl. Math.*, **23**, (1997), 51–87.

12. Y. Bai, P. Zhang, On a class of Volterra nonlinear equations of parabolic type, *Appl. Math. Comp.* **216**, (2010), 236–240.

13. M. Vishik, Solvability of boundary-value problems for quasi-linear parabolic equations of higher orders, (Russian) *Math. Sb. (N.S.)*, **59(101)**, (1962), suppl. 289–325.

14. J. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non-linéaires,* Dunod, Gauthier-Villars. Paris, 1969.

15. T. A. Jangveladze, Z. V. Kiguradze, Asymptotics of a solution of a nonlinear system of diffusion of a magnetic field into a substance, (Russian) *Sibirsk. Mat. Zh.*, **47**, (2006), 1058–1070. English translation: *Siberian Math. J.*, **47**, (2006), 867–878.

16. T. A. Dzhangveladze, Z. V. Kiguradze, On the stabilization of solutions of an initial-boundary value problem for a nonlinear integro-differential equation, (Russian) *Differ. Uravn.*, **43**, (2007), 833–840. English translation: *Diff. Equ.*, **43**, (2007), 854–861.

17. T. A. Dzhangveladze, Z. V. Kiguradze, Asymptotic behavior of the solution to nonlinear integro-differential diffusion equation, (Russian) *Differ. Uravn.*, **44**, (2008), 517–529. English translation: *Diff. Equ.*, **44**, (2008), 538–550.

18. T. Jangveladze, Z. Kiguradze, B. Neta, Large time behavior of solutions and finite difference scheme to a nonlinear integro-differential equation, *Comput. Math. Applic.*, **57**, (2009), 799-811.

19. T. Jangveladze, Z. Kiguradze, B. Neta, Large time behavior of solutions to a nonlinear integro-differential system, *J. Math. Anal. Appl.*, **351**, (2009), 382–391.

20. M. Aptsiauri, T. Jangveladze, Z. Kiguradze, On asymptotic behavior of solution of one nonlinear one-dimensional integro-differential analogue of Maxwell's system, *Rep. Enlarged Sess. Semin I.Vekua Appl. Math.*, **23**, (2009), 3–6.

21. T. Jangveladze, Z. Kiguradze, B. Neta, Large time asymptotic and numerical solution of a nonlinear diffusion model with memory, *Comput. Math. Appl.*, **59**, (2010), 254–273.

22. T. Jangveladze, Z. Kiguradze, Large time behavior of the solution to an initial-boundary value problem with mixed boundary conditions for a nonlinear integro-differential equation, *Cent. Eur. J. Math.*, **9**, (2011), 866–873.

23. T. A. Jangveladze, Convergence of a difference scheme for a nonlinear integro-differential equation, *Proc. I. Vekua Inst. Appl. Math.*, **48**, (1998), 38–43.

24. Z. Kiguradze, The difference scheme for one system of nonlinear partial differential equations, *Rep. Enlarged Sess. Semin I. Vekua Appl. Math.*, **14**, (1999), 67–70.

25. Z. V. Kiguradze, Finite difference scheme for a nonlinear integro-differential system, *Proc. I. Vekua Inst. Appl. Math.*, **50-51**, (2000-2001), 65–72.

26. T. Jangveladze, Z. Kiguradze, B. Neta, Finite difference approximation of a nonlinear integro-differential system, *Appl. Math. Comput.*, **215**, (2009), 615–628.

27. B. Neta, Numerical solution of a nonlinear integro-differential equation, *J. Math. Anal. Appl.*, **89**, (1982), 598-611.

28. B. Neta, J. O. Igwe, Finite differences versus finite elements for solving nonlinear integro-differential equations, *J. Math. Anal. Appl.*, **112**, (1985), 607–618.

29. B. Neta, Finite element approxsimation of a nonlinear parabolic problem, *Comput. Math. Appl.*, **4**, (1978), 247–255.

30. T. Jangveladze, Z. Kiguradze, B. Neta, Galerkin finite element method for one nonlinear integro-differential model, *Appl. Math. Comput.*, **217**, (2011), 6883–6892.