AN ENHANCED NUMERICAL SOLUTION OF THE LORENZ SYSTEM BY MEANS OF THE DIFFERENTIAL QUADRATURE METHOD

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Abstract

The differential quadrature method (DQM) which is indeed a higher order implicit Runge-Kutta method, is applied to the Lorenz system. Numerical comparisons are made between the DQM and the conventional fourth-order Runge-Kutta method (RK4). It is concluded that the DQM can be used as an efficient tool for handling the Lorenz system.

Key words and phrases: Differential quadrature method (DQM), Lorenz system, Fourth-order Runge-Kutta method

AMS subject classification: 65P20

For almost 50 years, the Lorenz attractor [1], with its intriguing double-lobed shape and chaotic behavior, has symbolized order within chaos in dynamical systems. The Lorenz attractor dates from 1963, when the meteorologist Edward Lorenz published an analysis of a simple system of three differential equations that he had extracted from a model of atmospheric convection. He pointed out that they possessed some surprising features. In particular, the equations are 'sensitive to initial conditions', meaning that tiny differences at the start become amplified exponentially as time passes. Since then, the Lorenz System has been the subject of many articles [2, 3, 4], monographs [5], textbooks [6, 7], and university theses [8, 9].

In the present paper, we will consider the well-known Lorenz equations which can be expressed as follows [1]:

$$\frac{dx}{dt} = \sigma(y - x), \tag{1.1}$$

$$\frac{dy}{dt} = Rx - y - xz, \tag{1.2}$$

$$\frac{dz}{dt} = xy + bz, \tag{1.3}$$

where x, y, and z are dynamical variables of the Lorenz system; and σ, R, and b are the related constants. The Lorenz system can exhibit both chaotic and non-chaotic behaviors for distinct parameter values. Bifurcation studies show that with the parameters $\sigma = 10$ and $b = -8/3$ chaos sets in around the critical parameter value $R = Rcr = 27.74$ [1]. In other words the system (1-3) exhibit non-chaotic behavior when $R < Rcr$ and does chaotic behavior when $R > Rcr$.

Since analytical solution can not be found for the chaotic system (1-3), there has been a considerable effort to solve this system numerically. But numerical methods provide the solutions only at the discrete time points. Besides, they often need very small time step sizes to ensure the convergence and to arrive at an accurate solution. Thus, much attention has been paid to analytical asymptotic (i.e. semi-analytic) techniques such as the Adomian decomposition method (ADM) [10], Homotopy analysis method (HAM) [11], and more recently the multistage homotopy-perturbation method (MHPM) [12]. These semi-analytic methods give some promising results, but each of these methods has its own drawbacks and weaknesses. For example, when we use the ADM or the HAM we should then calculate some polynomials (say ADM/HAM polynomials). This procedure is often so cumbersome or the 'formula' obtained is often too complicated to understand and display clearly the principle features of the solution. In the literature, some researchers have also used the HPM to handle the nonlinear dynamical systems [10, 11]. However, as pointed out by Chowdhury et al. [12], this technique is not suitable for calculation of long-term solutions. In fact, the approximate solutions obtained using HPM, are generally not valid for long time durations [12]. To overcome the difficulties of the HPM, Chowdhury et al. [12] proposed a multistage HPM. In this technique, the time domain of interest is first divided into some time intervals (i.e., time elements/steps). Then, the HPM is applied to each time interval However, this technique produces high-accurate solutions, many calculations should be done to obtain the required polynomials coefficients. Thus, the major difficulty is not removed using this technique.

The above mentioned difficulties can be overcome by using the differential quadrature method (DQM). The DQ method, which was first introduced by Bellman et al [13] in the early 1970s, is an alternative method for directly solving the differential equations. The DQ method is basically based on the interpolation and derivation. It was also initiated from the idea of conventional integral quadrature. The DQ method approximates the derivative of a function at a certain point by means of a weighted linear sum of the function values in all discrete points at the domain of that variable. Since its introduction, the DQ method has been applied by many researchers to a variety of problems in engineering, mathematics, and physics and is gradually emerging as a distinct numerical technique. Compared to the low-order methods such as the finite element method and finite difference method, the DQ method can achieve very accurate solutions by using a considerably small number of sample points [13]. Another particular advantage of the DQ method lies in its ease of use and implementation. Due to the above-mentioned features, the DQ method has been applied extensively. Majority of these applications are related to statics or vibrations. The DQ method was initially proposed for the solution of initial-value problems, but less attention has been paid to the application of this technique to initial-value problems until the recent years [14-20].

In this paper, we apply the DQ method to the Lorenz system. To the authors' best knowledge, this is the first attempt in applying the DQ method to nonlinear systems of ODEs having chaotic behaviors such as the Lorenz system. Since for the Lorenz equation, a closed form of the analytic solution can not be found, the accuracy of the DQ method is then tested against conventional fourth-order Runge-Kutta method (RK4). The aim of this study is to compare the effectiveness of the DQ time integration method against the classical RK4 in producing solutions for the chaotic Lorenz system. It is shown that the DQ method produces much better accuracy than the RK4 using much larger time step sizes. Another particular advantage of the DQ method is its ability in providing us a continuous representation of the approximate solution, which allows better information of the solution over the time interval of interest. This characteristic distinguishes the DQ time integration scheme from the single step methods. Note that the RK4 only provides solutions in discretized form (i.e. only gives the solutions in discrete time points), thereby making it complicated in achieving a continuous representation of the approximate solution.

As pointed out earlier, the DQ method is basically based on the interpolation and derivation. Let $x(t)$ be a function which is approximated by the Lagrange interpolation functions $L_j(t)$, j = 1, 2,..., m, that is

$$x(t) = \sum_{j=1}^{m} x(t_j) \cdot L_j(t), \tag{1.4}$$

where m is the number of sample time points in the time domain (also the number of Lagrange interpolation functions), $x(t_j)$ are the function values at these points differentiating equation (1.1) with respect to time, we obtain

$$\dot{x}(t) = \frac{dx}{dt} = \sum_{j=1}^{m} x(t_j) \cdot \dot{L}_j(t) = \sum_{j=1}^{m} x(t_j) \frac{dL_j}{dt}. \tag{1.5}$$

From equation (1.5), the first-order derivative of the function $x(t)$ with respect to time at a time point t_i can be expressed as

$$\dot{x}(t_i) = \sum_{j=1}^{m} x(t_j) \cdot \dot{L}_j(t_i). \tag{1.6}$$

Equation (1.6) is, in fact, the quadrature rule

$$\begin{cases} \dot{x}(t_i) = \sum\limits_{j=1}^{m} A_{ij} x(t_j) \ \ or \\ \dot{x}_i = \sum\limits_{j=1}^{m} A_{ij} x_j, \end{cases} \tag{1.7}$$

which gives the first-order DQ weighting coefficients, Aij, as follows

$$A_{ik}^{(1.1)} = \begin{cases} \frac{M^{(1.1)}(t_i)}{(t_i - t_k) M^{(1.1)}(t_k)} & i \neq k, \quad i,k = 1,2,...,m \\ -\sum\limits_{j=1, j\neq i}^{m} A_{ij}^{(1.1)} & i = k, \quad i = 1,2,...,m \end{cases} \tag{1.8}$$

where $M(1.1)(t)$ is defined as

$$M^{(1.1)}(t_i) = \prod_{j=1, j\neq i}^{m} (t_i - t_j). \tag{1.9}$$

Obviously, the accuracy, stability, and rate of convergence of the DQ solutions are dictated by the choice of the sample time points. It is well known that non-uniformly spaced sample points (i.e, the Chebyshev-Gauss-Lobatto sample points) perform consistently much better than the equally spaced sample points. These points are given by

$$t_i = T/2 \left[1 - \cos\left(\frac{(i-1)\pi}{m-1} \right) \right], \quad i = 1,2,...,m \tag{1.10}$$

where T is the time span.

Now we derive the DQ analogs of the Lorenz system. From the DQ rule, the first-order derivative of the functions $x(t), y(t)$, and $z(t)$ at a time point t_i can be written as

$$\dot{x}_i = \sum_{j=1}^{m} A_{ij} x_j, \quad \dot{y}_i = \sum_{j=1}^{m} A_{ij} y_j, \quad \dot{z}_i = \sum_{j=1}^{m} A_{ij} z_j. \tag{1.11}$$

Substituting equation (1.9) in equations (1-3) yields

$$\begin{cases} \sum\limits_{j=1}^{m} A_{ij}x_j = \sigma(y_i - x_i), \\ \sum\limits_{j=1}^{m} A_{ij}y_j = Rx_i - y_i - x_iz_i, \qquad i = 1,2,...,m \\ \sum\limits_{j=1}^{m} A_{ij}z_j = x_iy_i + bz_i. \end{cases} \tag{1.12}$$

Let us take the initial conditions as follows

$$x(t=0) = x(t_1) = x_1 = x_0, \quad y_1 = y_0, \quad z_1 = z_0. \tag{1.13}$$

Applying the initial conditions in (1.12) yields

$$\begin{cases} \sum\limits_{j=2}^{m} A_{ij}x_j + A_{i1}x_0 = \sigma(y_i - x_i), \\ \sum\limits_{j=2}^{m} A_{ij}y_j + A_{i1}y_0 = Rx_i - y_i - x_iz_i, \qquad i = 2,3,...,m \\ \sum\limits_{j=2}^{m} A_{ij}z_j + A_{i1}z_0 = x_iy_i + bz_i. \end{cases} \tag{1.14}$$

Equation (1.14) is a system of nonlinear algebraic equations which can be easily and efficiently solved by iterative methods. In this work, we use the Newton method (i.e. Newton-Raphson method) to solve equation (1.14). Our numerical experiment for the present problem showed that the Newton method with only 3-5 iterations produced accurate solutions.

To test the accuracy and efficiency of the DQ time integration method and to provide a comparison of the results with those previously obtained by Guella et al. [10], and Chowdhury et al. [12] the parameters of the problem are chosen as: $\sigma = 10, b = -8/3$. The initial conditions of the problem are also considered as $x_0 = -15.8$, $y_0 = -17.48$ and $z_0 = 35.64$. We also demonstrate the accuracy and convergence of the DQ method for the solutions of both non-chaotic and chaotic systems. For the purpose of comparison, we will consider two cases: $R = 23.5$ where the system is non-chaotic and $R = 28$ where the system exhibits chaotic behavior. In addition to the above cases, we also consider two cases $R = 50$ and $R = 100$, corresponding to chaotic systems, in our attempt to show the applicability of DQ time integration method in prediction of behavior of chaotic systems. In solving the Lorenz equations, we apply the DQM as a step-by-step scheme (say DQEM: differential quadrature element method). By using DQEM, the long-term solutions can be efficiently and accurately obtained. In this case, the time domain of interest is divided into n equal

DQM time element with m sample time points. The total number of sample time points and the average time step can be obtained respectively as

$$\mathrm{M_{tot}} \;=\; \mathrm{n}\,(\mathrm{m}-1) \;+\; 1, \tag{1.15}$$

$$\Delta\mathrm{t} \;=\; \mathrm{T}/\,(\,\mathrm{M_{tot}}\;-1) \;=\; \mathrm{T}/\,(\,\mathrm{n}\,(\mathrm{m}-1))\,. \tag{1.16}$$

Note that the time step given in equation (1.16) is an average time step since the sample time points are taken non-uniformly spaced in the time domain (see equation (10)). The time domain is considered in this work (i.e. [0, T]) is [0, 5] or [0, 2.5].

First, we consider the non-chaotic case where $\sigma = 10$, $b = -8/3$ and $R = 23.5$. Figure 1 presents the convergence and accuracy of DQ solutions for this case. The use of Lagrange interpolation polynomials (in each time element) enables us to reach a continuous representation of the approximate solutions. A good convergence trend of solutions can be observed. However, when the total number of sample time points is too small (i.e. when the size of time steps are too large) a visible phase shift can be observed. On the other hand, the accuracy of solutions can be controlled by choosing the proper values of n and/or m. In other words, the accuracy of solutions can be improved by increasing n and/or m. Figure 2 presents the phase planes obtained using the DQ method and the RK4. The numerical simulations are done in the time interval $0 \le t \le 5$. By comparing the DQ solutions with those of RK4, one can conclude that the DQ method gives more accurate solutions than the RK4 using a considerably larger time step sizes.

As pointed out earlier, the system (1)-(3) with $R > 27.74$ exhibits chaotic behavior. For chaotic behaviors of Lorenz system, we consider three cases: $R = 28$, $R = 50$ and $R = 100$. When the system is chaotic, care should be taken in choosing a time step since the solutions are highly sensitive to time step. Figure 3 shows the convergence of DQ time integration method for the solutions of chaotic Lorenz system against the number of time elements, n, and the number of sample time points per time element, m, when $R = 28$. It can be observed that the DQ method encounters some large attenuation of amplitude and overshoot for long-term solutions when the time step is so large (i.e. when n or m is too small). However, by decreasing the time step (i.e. by increasing n or m) the solutions converge to the true solutions. In conclusion, the DQ time integration scheme may be possible to yield inaccurate solutions for chaotic systems with an inappropriate too large time step. Figure 4 shows the phase portraits of the Lorenz system, solutions by the $RK4$ and the DQ method. It can be seen that the results of the DQ method with $\Delta t = 0.0357$ are comparable in accuracy to those of RK4 with $\Delta t = 0.0125$. This demonstrates the superiority of DQ time integration method over the conventional RK4 for the solution of chaotic Lorenz system.

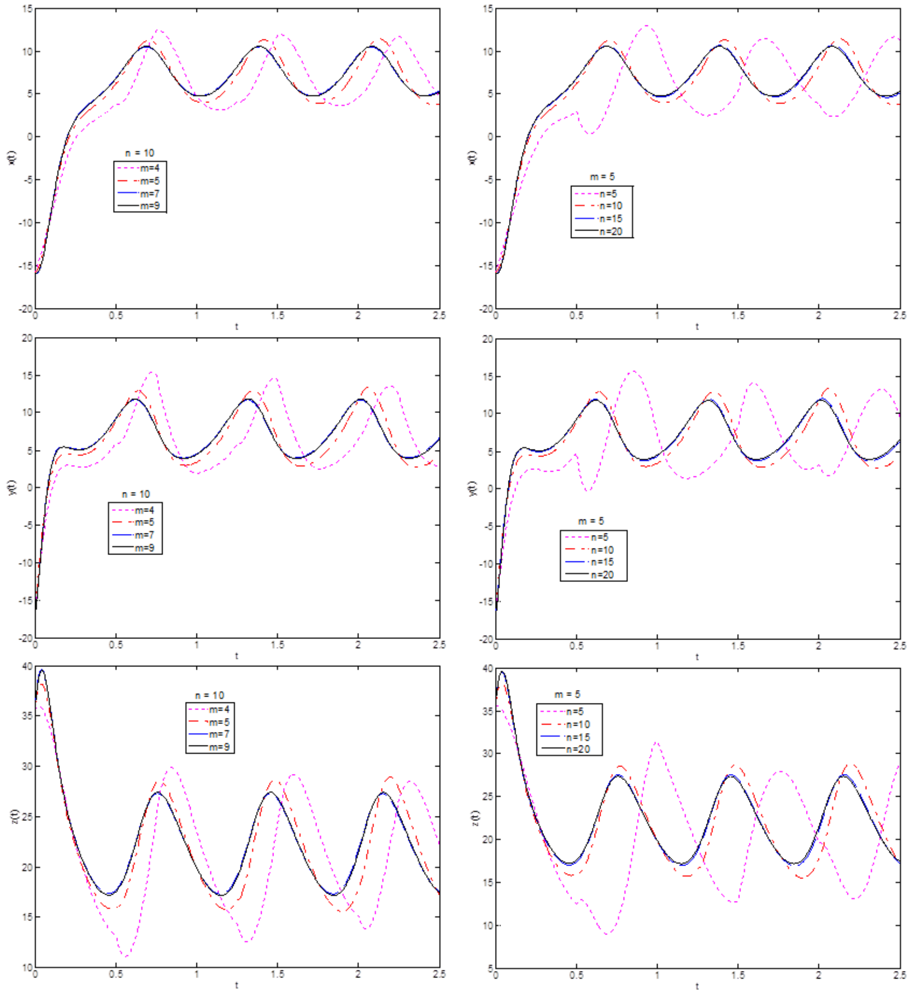

Figure 1. Convergence and accuracy of DQ time integration method for the solution of non-chaotic Lorenz system (R=23.5).

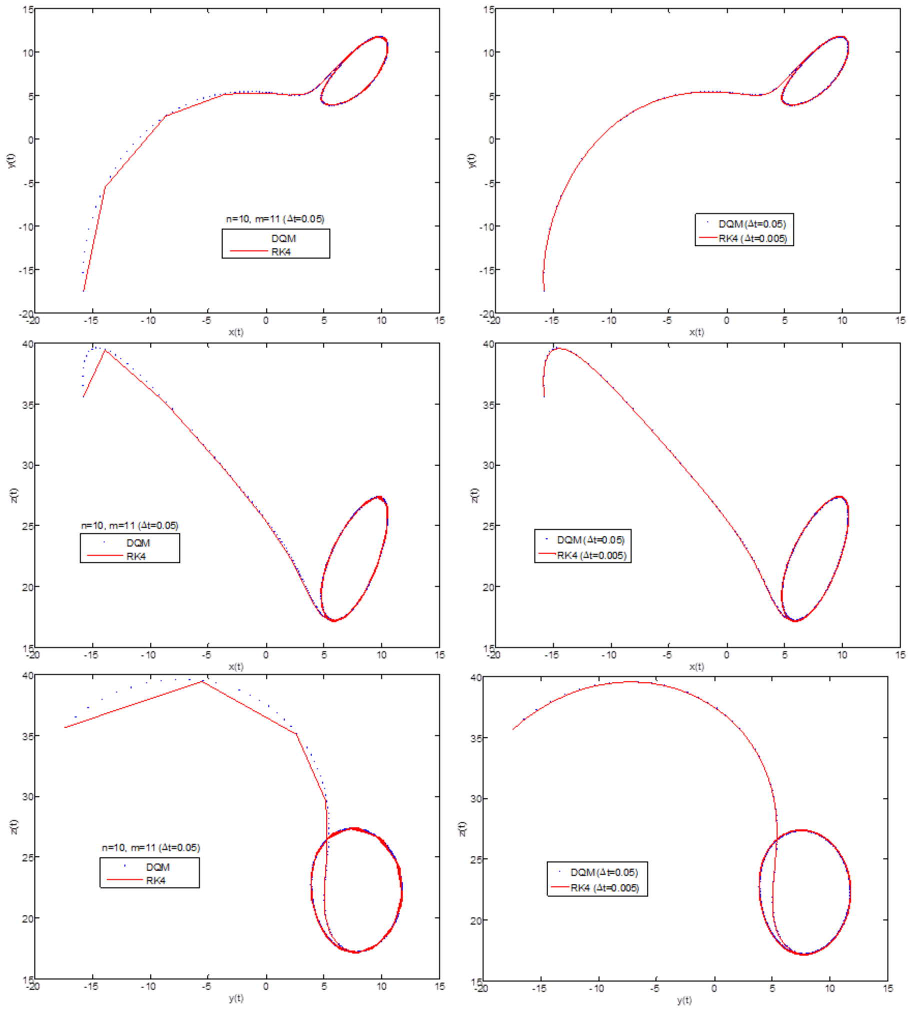

Figure 2. Phase portraits of the non-chaotic Lorenz system obtained using DQM and RK4 (R=23.5).

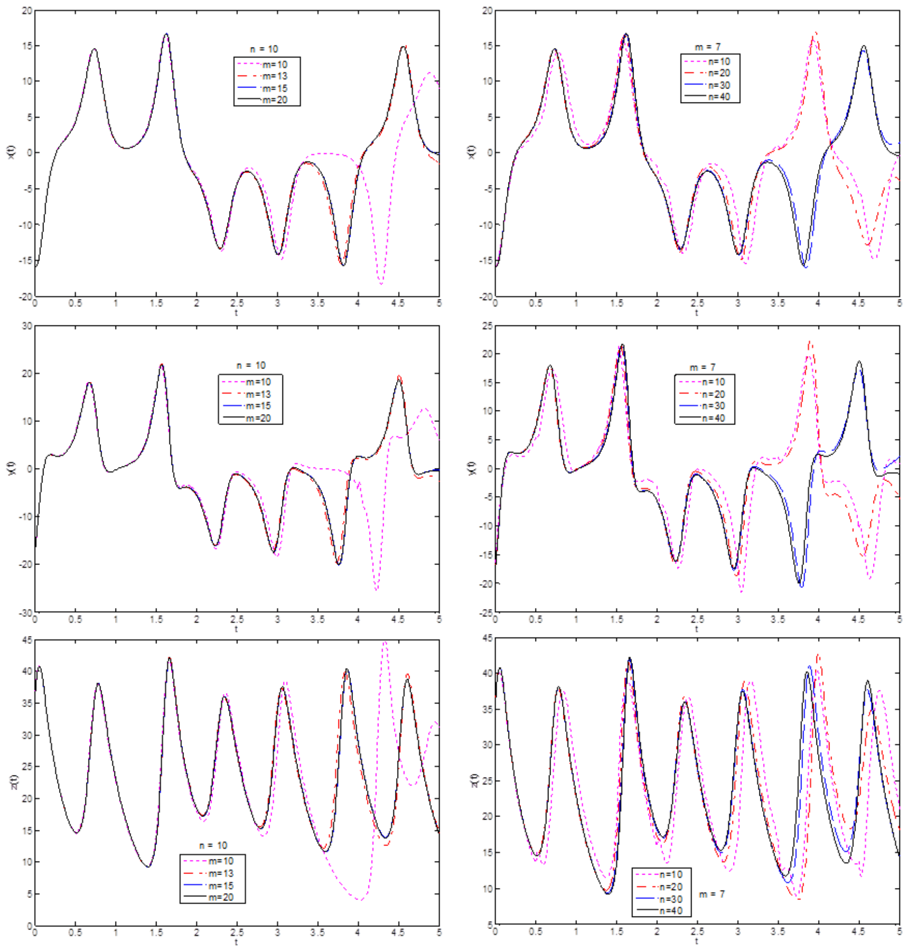

Figure 3. Convergence and accuracy of DQ time integration method for the solution of chaotic Lorenz system (R=28).

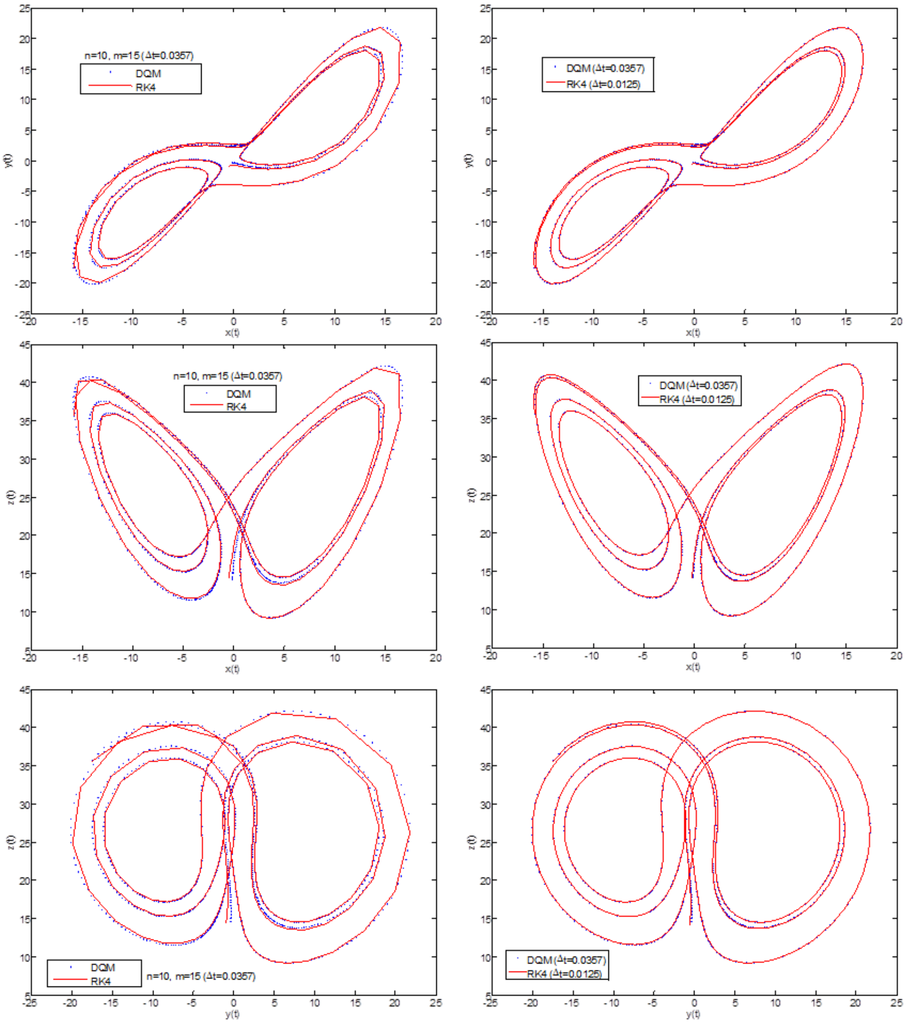

Figure 4. Phase portraits of chaotic Lorenz system obtained using DQM and RK4 (R=28).

Figure 5 presents the results for the chaotic Lorenz system with $R = 50$. Significant differences in numerical accuracy, amplitude attenuation, and phase shift are easily observed from figure 5 when using RK4 with $\Delta t = 0.0277$. It is also found that the DQ method also confronts some small attenuation of amplitude and overshoots for long-term solutions when $\Delta t = 0.0277$. However, the DQ solutions are better than those of RK4 in this case (i.e. when $\Delta t = 0.0277$). It is observed that both the DQ method and RK4 provide true solutions using sufficiently small time step $\Delta t = 0.0104$. By comparing the DQ solutions shown in figures 1, 3 and 5, one can conclude that as the parameter R increases, the size of time step required to achieve accurate solutions decreases (i.e. the total number of sample time points required to accurately obtain the solutions increase). This is actually due to the chaotic behavior of the Lorenz system. As the parameter R increases, the shape of dynamic responses becomes non-smoother and thus, a larger number of sample time points (i.e. a smaller size of time steps) are required to obtain the accurate solutions since the DQ method is basically based on the interpolation and derivation. The phase portraits of the Lorenz system obtained using DQ method and RK4 are given in figure 6. The DQ solutions are obtained using Δt = 0.01786 while those of RK4 are calculated using $\Delta t = 0.01786$ and $\Delta t = 0.0104$. It can be seen that the DQ solutions with Δt = 0.01786 are comparable in accuracy to RK4 solutions with $\Delta t = 0.0104$. It can also be observed the RK4 solutions have visible phase shift when $\Delta t = 0.01786$.

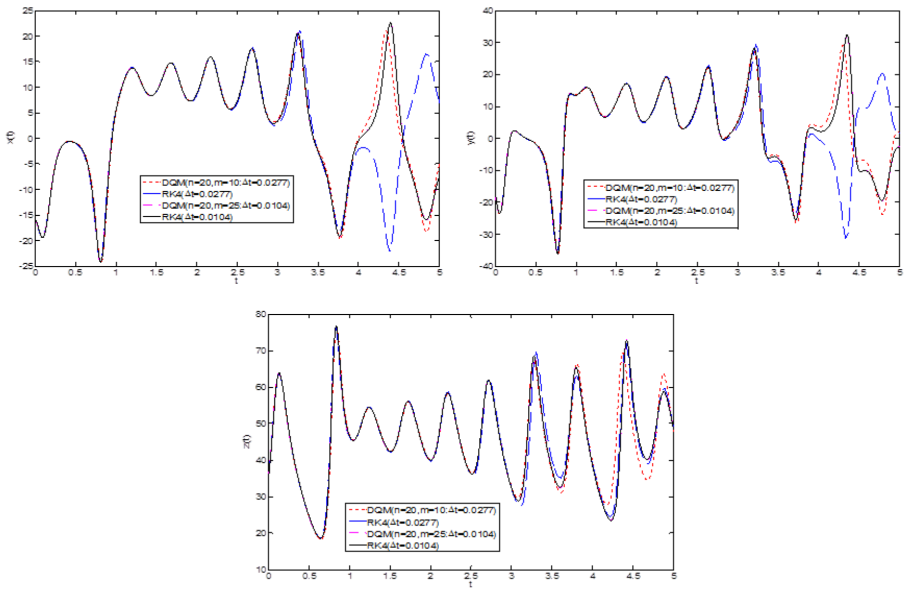

Figure 5. Accuracy of DQ time integration method for the solution of chaotic Lorenz system and comparisons with those of RK4 (R=50).

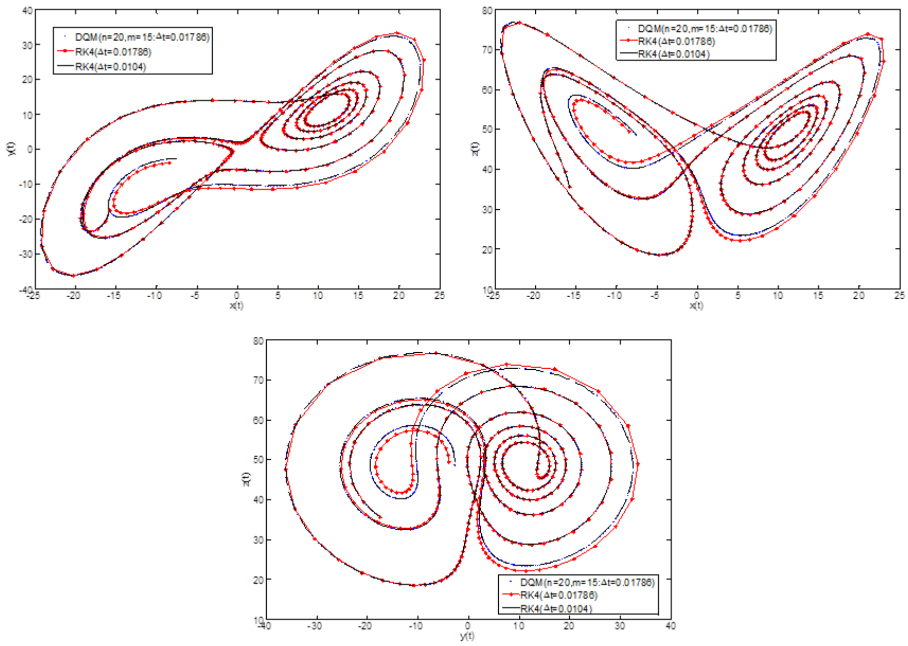

Figure 6. Phase portraits of chaotic Lorenz system obtained using DQM and RK4 (R=50).

Figures 7 and 8 illustrate the results for the chaotic Lorenz system with R=100. From figure 7, it can be observed that the DQ solutions with a rather large time step $\Delta t = 0.0185$ are comparable in accuracy to the RK4 solutions with a small time step $\Delta t = 0.00555$. Again, the solutions of the RK4 encounter a sharp drop of accuracy for the long-term response when the size of time step is large, (i.e. when $\Delta t = 0.0185$) as seen in figure 7. As pointed out earlier and as it was seen in figures 1, 3, 5, and 7, as the parameter R increases a smaller time steps should be used to ensure the convergence and arrive at accurate solutions. The phase portraits of the DQ solution with $\Delta \mathrm{t} = 0.0185$ and the RK4 with $\Delta t = 0.0185$ and $\Delta t = 0.00555$ are given in figure 8. It can be observed that as compared to the RK4, the DQ method produces better results using a much larger time step size. Note that the RK4 solutions with $\Delta t = 0.0185$ are not acceptable in accuracy in this case. In chaotic case even the round off error will affect the instantaneous response but the static response will be the same.

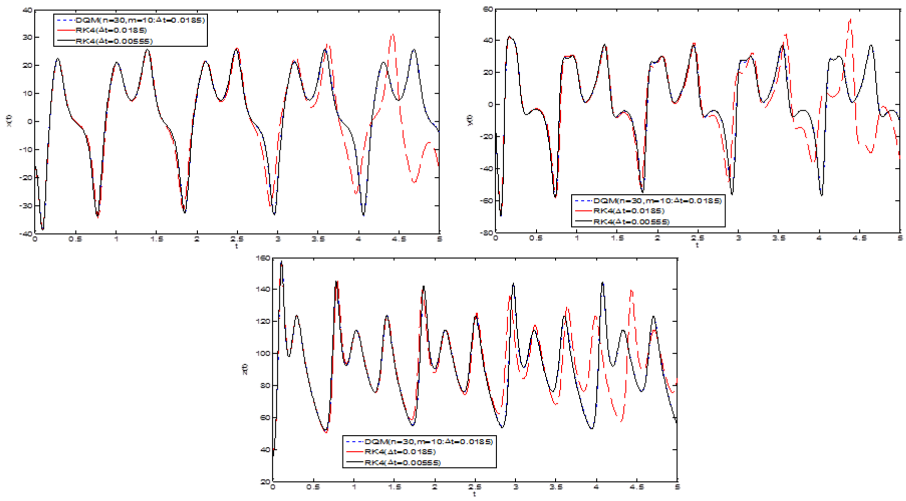

Figure 7. Accuracy of DQ time integration method for the solution of chaotic Lorenz system and comparisons with those of RK4 (R=100).

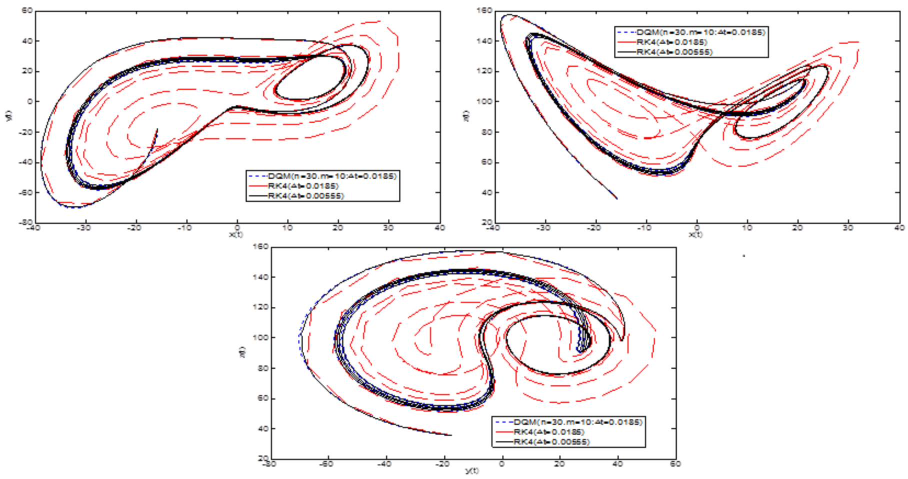

Figure 8. Phase portraits of chaotic Lorenz system obtained using DQM and RK4 (R=100).

In view of the foregoing discussions and comparison of the DQ time integration method, and those of RK4, it is concluded that the DQ method produces much better accuracy than the RK4 using much larger time step sizes (i.e. using considerable smaller number of sample time points). Thus, the DQ time integrations method seems to be an effective and promising tool for handling the Lorenz system.

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