SOME PROPERTIES OF UNILATERAL DIFFERENTIABLE FUNCTIONS OF TWO VARIABLES

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(Received: 25.07.2011; accepted: 29.12.2011)

Abstract

The notion of unilateral differentiability of functions of two variables is introduced by O. Dzagnidze. Some properties of such functions are considered.

Key words and phrases: Unilateral differentiable functions of two variables, partial differential, symmetrical differentiability.

AMS subject classification: 26B05, 26B35.

1 The Unilateral Differentiability of Functions of Two Variables

Let $U(0)$ and $U^0(0) = U(0) \setminus \{0\}$ denote the neighborhood and the punctured neighborhood of the point $O = (0, 0)$. We use the following sets ([1, p. 43]):

$$\begin{gathered} A_1^+ = \{(h,k) \in U(0) : h > 0\}, \quad A_2^+ = \{(0,k) \in U(0) : k > 0\}, \\ A_1^- = \{(h,k) \in U(0) : h < 0\}, \quad A_2^- = \{(0,k) \in U(0) : k < 0\}, \\ A_{12}^+ = A_1^+ \cup A_2^+, \quad A_{12}^- = A_1^- \cup A_2^- . \end{gathered}$$

It is obvious that

$$A_{12}^+ \cap A_{12}^- = \varnothing \quad \text{and} \quad A_{12}^+ \cup A_{12}^- = U^0(0). \tag{1.1}$$

Let us introduce the following two definitions.

Definition 1.1. ([2]) A function $f(x, y)$ is called right-differentiable at the point $p_0 = (x_0, y_0)$ if the equality

$$\lim_{\substack{(h,k)\to(0,0)\\(h,k)\in A_{12}^+}} \frac{f(x_0+h, y_0+k) - f(x_0, y_0) - A^+h - B^+k}{|h| + |k|} = 0 \tag{1.2}$$

is fulfilled for some finite numbers A^+ and B^+, and the linear function $A^+h + B^+k$ for $(h,k) \in A_{12}^+$ is called a right-differential of f at the point p_0, denoted by $d^+f(p_0)$ and we write

$$d^+f(p_0) = A^+h + B^+k. \tag{1.3}$$

Definition 1.2. ([2]) A function $f(x,y)$ is called left-differentiable at the point $p_0 = (x_0, y_0)$ if there exist finite numbers A^- and B^- such that the equality

$$\lim_{\substack{(h,k)\to(0,0)\\(h,k)\in A_{12}^-}} \frac{f(x_0+h, y_0+k) - f(x_0,y_0) - A^-h - B^-k}{|h|+|k|} = 0 \tag{1.4}$$

is fulfilled, and the linear function $A^-h + B^-k$ is called a left-differential of f at the point p_0, denoted by $d^-f(p_0)$, $(h,k) \in A_{12}^-$, and we write

$$d^-f(p_0) = A^-h + B^-k. \tag{1.5}$$

The next two propositions are obvious.

Proposition 1.1. ([2]) A differentiable at a point p_0 function $f(x,y)$ is bilaterally differentiable at p_0 and the equalities

$$\begin{aligned} d^+f(p_0) &= df(p_0), \quad d^-f(p_0) = df(p_0), \\ A^+ = A^- &= f'_{\hat{x}}(p_0), \quad B^+ = B^- = f'_{\hat{y}}(p_0) \end{aligned} \tag{1.6}$$

are fulfilled.

Proposition 1.2. ([2]) If a function $f(x,y)$ is bilaterally differentiable at a point p_0 and the equalities $A^+ = A^-$ and $B^+ = B^-$ are fulfilled, then f is differentiable at p_0 and

$$A^+ = f'_{\hat{x}}(p_0) = A^-, \quad B^+ = f'_{\hat{y}}(p_0) = B^-. \tag{1.7}$$

2 A Symmetrical Differential and Unilateral Differentials of Functions of Two Variables

Definition 2.1. ([3]) A function $\varphi(x,y)$ is called symmetrically differentiable at a point (x_0,y_0) if there exist finite constants A and B with the property

$$\lim_{(h,k)\to(0,0)} \frac{\varphi(x_0+h, y_0+k) - \varphi(x_0-h, y_0-k) - 2Ah - 2Bk}{|h|+|k|} = 0. \tag{2.1}$$

When a function is symmetrically differentiable at a point (x_0, y_0), then the coefficients of its symmetrical differential at (x_0, y_0)

$$d^{sym}\varphi(x_0, y_0) = A\,dx + B\,dy \tag{2.2}$$

are symmetrical partial derivatives of φ at the point (x_0, y_0) with respect to the variables x and y:

$$\begin{gathered} A = \varphi_x^{(1)}(x_0, y_0), \quad B = \varphi_y^{(1)}(x_0, y_0), \\ d^{sym}\varphi(x_0, y_0) = \varphi_x^{(1)}(x_0, y_0)\,dx + \varphi_y^{(1)}(x_0, y_0)\,dy. \end{gathered} \tag{2.3}$$

A symmetrical differential and unilateral differentials are related in the manner as follows

Theorem 2.1. *Let a function f have, at a point $p_0 = (x_0, y_0)$, two unilateral differentials $d^+f(p_0)$ and $d^-f(p_0)$. Then there exists at p_0 its symmetrical differential $d^{sym}f(p_0)$ and the equality*

$$d^{sym}f(p_0) = \frac{1}{2}\left[d^-f(p_0) + d^+f(p_0)\right] \tag{2.4}$$

is fulfilled.

Proof. Assume that equalities (1.2) and (1.3) are fulfilled. Since the point
$(-h, -k)$ belongs to the set A_{12}^- and the function f is left-differentiable at the point p_0, the equality

$$\lim_{\substack{(h,k)\to(0,0)\\(-h,-k)\in A_{12}^-}} \frac{f(x_0-h, y_0-k) - f(x_0, y_0) - A^-(-h) - B^-(-k)}{|-h| + |-k|} = 0 \tag{2.5}$$

is fulfilled too.

Moreover, we have the equality

$$\begin{aligned} &\frac{f(x_0+h, y_0+k) - f(x_0, y_0) - A^+h - B^+k}{|h| + |k|} \\ &- \frac{f(x_0-h, y_0-k) - f(x_0, y_0) - A^-(-h) - B^-(-k)}{|-h| + |-k|} \\ &= \frac{f(x_0+h, y_0+k) - f(x_0-h, y_0-k) - 2\,\frac{A^++A^-}{2}\,h - 2\frac{B^++B^-}{2}\,k}{|h| + |k|}. \end{aligned} \tag{2.6}$$

The left-hand side of this equality tends to zero by virtue of equalities (1.2) and (2.5) when $(h, k) \in A_{12}^+$ and $(h, k) \to (0, 0)$. Thus the right-hand side of (2.6) also tends to zero and

$$\begin{aligned} d^{sym}f(p_0) &= \frac{1}{2}\,(A^+ - A^-)\,dx + \frac{1}{2}\,(B^+ - B^-)\,dy = \frac{1}{2}\,(A^+dx + B^+dx) \\ &+ \frac{1}{2}\,(A^-dx + B^-dx) = \frac{1}{2}\,d^+f(p_0) + \frac{1}{2}\,d^-f(p_0). \end{aligned}$$

The theorem is proved. □

3 The Properties of a Unilaterally Differentiable Function of Two Variables

We will investigate a unilaterally differentiable function with respect to each independent variable. For this we need to introduce some definitions.

Definition 3.1. The following limits

$$\partial_x^+ f(p_0) = \lim_{h\to 0+} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}, \tag{3.1}$$

$$\partial_y^+ f(p_0) = \lim_{k\to 0+} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k} \tag{3.2}$$

are called the right partial derivatives of a function $f(x, y)$ with respect to the variables x and y at the point $p_0 = (x_0, y_0)$. The left partial derivatives of a function $f(x, y)$ with respect to x and y at the point $p_0 = (x_0, y_0)$ are defined analogously:

$$\partial_x^- f(p_0) = \lim_{h\to 0-} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}, \tag{3.3}$$

$$\partial_y^- f(p_0) = \lim_{k\to 0-} \frac{f(x_0, y_0+k) - f(x_0, y_0)}{k}. \tag{3.4}$$

Putting $k = 0$ in equality (1.2) we obtain

Proposition 3.1. If a function $f(x, y)$ is right-differentiable at a point $p_0 = (x_0, y_0)$, then for the constants A^+ and B^+ from equality (1.3) we have

$$A^+ = \partial_x^+ f(p_0), \;\; B^+ = \partial_y^+ f(p_0) \;\text{ and}$$
$$d^+ f(p_0) = \partial_x^+ f(p_0)\, dx + \partial_y^+ f(p_0)\, dy. \tag{3.5}$$

Analogously we have

Proposition 3.2. If a function $f(x, y)$ is left-differentiable at a point $p_0 = (x_0, y_0)$, then for the constants A^- and B^- from equality (1.5) we have

$$A^- = \partial_x^- f(p_0), \;\; B^- = \partial_y^- f(p_0) \;\text{ and}$$
$$d^- f(p_0) = \partial_x^- f(p_0)\, dx + \partial_y^- f(p_0)\, dy. \tag{3.6}$$

Further we need the following

Definition 3.2. ([1, pp. 83-84]) The limits

$$\partial_{\widehat{x}}^+ f(p_0) = \lim_{\substack{h\to 0+ \\ |k|\le ch}} \frac{f(x_0+h, y_0+k) - f(x_0, y_0+k)}{h}, \tag{3.7}$$

$$\partial_{\widehat{y}}^+ f(p_0) = \lim_{\substack{k\to 0+ \\ k\ge l|h|}} \frac{f(x_0+h, y_0+k) - f(x_0+h, y_0)}{k}, \tag{3.8}$$

if they do not depend on arbitrary constants $c > 0$ and $l > 0$, are called the right angular partial derivatives of a function $f(x, y)$ at the point $p_0 = (x_0, y_0)$ with respect to the variables x and y.

We define analogously the left angular partial derivatives of a function f at the point p_0 with respect to x and y:

$$\partial_{\widehat{x}}^- f(p_0) = \lim_{\substack{h \to 0- \\ |k| \le -ch}} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h}, \quad c > 0, \tag{3.9}$$

$$\partial_{\widehat{y}}^- f(p_0) = \lim_{\substack{k \to 0- \\ k \ge -l|h|}} \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k}, \quad l > 0. \tag{3.10}$$

Theorem 3.1. *Let a function $f(x, y)$ be right-differentiable at a point $p_0 = (x_0, y_0)$ and let the equality*

$$B^+ = \partial_y^- f(p_0) \tag{3.11}$$

be fulfilled. Then there exist finite $\partial_y f(p_0)$, $\partial_{\widehat{x}}^+ f(p_0)$ and the equality

$$d^+ f(p_0) = \partial_{\widehat{x}}^+ f(p_0)\, dx + \partial_y f(p_0)\, dy \tag{3.12}$$

is fulfilled.

Proof. From equalities (3.5) and (3.11) we obtain

$$B^+ = \partial_y f(p_0) \tag{3.13}$$

and we are to consider only those points (h, k) from the set A_{12}^+ where $h > 0$. Taking into account the equality $A^+ = \partial_x^+ f(p_0)$ from (3.5) and equality (3.13), equality (1.2) implies that to each number $\varepsilon_1 > 0$ there corresponds a number $\delta_1 = \delta_1(\varepsilon_1, p_0) > 0$ with the property

$$\begin{aligned} &\big|f(x_0 + h, y_0 + k) - f(x_0, y_0) - A^+ h - k\partial_y f(p_0)\big| \\ &\qquad < \varepsilon_1(h + |k|), \quad 0 < h < \delta_1, \quad |k| < \delta_1. \end{aligned} \tag{3.14}$$

Along with this, equality (3.13) implies the fulfillment of the inequality

$$|f(x_0, y_0 + k) - f(x_0, y_0) - k\partial_y f(p_0)| < \varepsilon_1 |k|, \quad |k| < \delta_2. \tag{3.15}$$

Using estimates (3.14) and (3.15), from the equality

$$\begin{aligned} &f(x_0 + h, y_0 + k) - f(x_0, y_0 + k) - A^+ h \\ &= \big[f(x_0 + h, y_0 + k) - f(x_0, y_0) - A^+ h - k\partial_y f(p_0)\big] \\ &- [f(x_0, y_0 + k) - f(x_0, y_0) - k\partial_y f(p_0)] \end{aligned} \tag{3.16}$$

we obtain

$$\begin{aligned}&\left|f(x_0+h,y_0+k)-f(x_0,y_0+k)-A^+h\right|\\&<\varepsilon_1(h+|k|)+\varepsilon_1|k|,\quad 0<h<\delta_3,\quad |k|<\delta_3,\end{aligned}\tag{3.17}$$

where $\delta_3=\min\{\delta_1,\delta_2\}$. Hence we have the estimate

$$\begin{aligned}&\left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h}-A^+\right|\\&<2\varepsilon_1\left(1+\frac{|k|}{h}\right),\quad 0<h<\delta_3,\quad |k|<\delta_3,\end{aligned}\tag{3.18}$$

Let us take an arbitrary constant $c>0$ and let the tendency $(h,k)\to(0,0)$ obey the condition $|k|\le ch$. If now for $\varepsilon>0$ given in any manner we put $\varepsilon_1=\frac{\varepsilon}{2(1+c)}$ in (3.18), then we obtain

$$\left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0+k)}{h}-A^+\right|<\varepsilon,\tag{3.19}$$

when $0<h<\delta$, $|k|<\delta$ and $|k|\le ch$, where $\delta=\delta(c,\varepsilon,p_0)>0$. This means that there exists the finite $\partial^+_{\widehat{x}}f(p_0)$ and the equality

$$A^+=\partial^+_{\widehat{x}}f(p_0)\tag{3.20}$$

is fulfilled.

Now equality (3.12) follows from equalities (1.3), (3.20) and (3.13). □

Remark 3.1. From (3.7) it obviously follows that the definition of the number $\partial^+_{\widehat{x}}f(p_0)$ contains both positive and negative values of the increment k. Therefore the equality $B^+=\partial^+_y f(p_0)$ from relations (3.5) is not sufficient for the fulfilment of inequality (3.15) without which it is impossible to establish the existence of the number $\partial^+_{\widehat{x}}f(p_0)$. This is the reason for which Theorem 3.1 contains the assumption as to the fulfillment of equality (3.11).

Corollary 3.1. If a function $f(x,y)$ is right-differentiable at a point p_0 and there exists the finite $\partial_y f(p_0)$, then there exists the finite $\partial^+_{\widehat{x}}f(p_0)$ and the equality

$$d^+f(p_0)=\partial^+_{\widehat{x}}f(p_0)\,dx+\partial_y f(p_0)\,dy\tag{3.21}$$

is fulfilled.

The following statement is proved analogously.

Theorem 3.2. *Let a function $f(x,y)$ be left-differentiable at a point $p_0=(x_0,y_0)$ and let the equality*

$$B^-=\partial^+_y f(p_0)\tag{3.22}$$

be fulfilled. Then there exist finite $\partial_y f(p_0)$, $\partial^-_{\widehat{x}} f(p_0)$ *and*

$$d^- f(p_0) = \partial^-_{\widehat{x}} f(p_0)\, dx + \partial_y f(p_0)\, dy. \tag{3.23}$$

Corollary 3.2. Assume that a function $f(x, y)$ is left-differentiable at a point $p_0 = (x_0, y_0)$ and let there exist the finite $\partial_y f(p_0)$. Then there exists the finite $\partial^-_{\widehat{x}} f(p_0)$ and

$$d^- f(p_0) = \partial^-_{\widehat{x}} f(p_0)\, dx + \partial_y f(p_0)\, dy. \tag{3.24}$$

4 Unilateral differentials and unilateral strong partial derivatives

Now we need the following

Definition 4.1. ([1, p. 82]) The right strong partial derivatives of a function $f(x, y)$ at a point $p_0 = (x_0, y_0)$ with respect to the variables x and y are defined by the equalities

$$\partial^+_{[x]} f(p_0) = \lim_{\substack{(h,k)\to(0,0)\\ h>0}} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h}, \tag{4.1}$$

$$\partial^+_{[y]} f(p_0) = \lim_{\substack{(h,k)\to(0,0)\\ k>0}} \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k}. \tag{4.2}$$

We define analogously the left strong partial derivatives with respect to the variables x and y for a function $f(x, y)$ at a point $p_0 = (x_0, y_0)$:

$$\partial^-_{[x]} f(p_0) = \lim_{\substack{(h,k)\to(0,0)\\ h<0}} \frac{f(x_0 + h, y_0 + k) - f(x_0, y_0 + k)}{h}, \tag{4.3}$$

$$\partial^-_{[y]} f(p_0) = \lim_{\substack{(h,k)\to(0,0)\\ k<0}} \frac{f(x_0 + h, y_0 + k) - f(x_0 + h, y_0)}{k}. \tag{4.4}$$

Theorem 4.1. *A sufficient condition for a function* $f(x, y)$ *to be right-differentiable at a point* $p_0 = (x_0, y_0)$ *is the existence of finite* $\partial^+_{[x]} f(p_0)$ *and* $\partial_y f(p_0)$. *If this condition is fulfilled, then we have*

$$d^+ f(p_0) = \partial^+_{[x]} f(p_0)\, dx + \partial_y f(p_0)\, dy. \tag{4.5}$$

proof. The existence of finite $\partial^+_{[x]}f(p_0)$ and $\partial_y f(p_0)$ is equivalent to the fulfillment of the inequalities

$$\left|f(x_0+h,y_0+k)-f(x_0,y_0+k)-h\partial^+_{[x]}f(p_0)\right| < \varepsilon h,\ \ 0<h<\delta_1,\ |k|<\delta_1, \tag{4.6}$$

$$|f(x_0,y_0+k)-f(x_0,y_0)-k\partial_y f(p_0)| < \varepsilon|k|, \quad |k|<\delta_2. \tag{4.7}$$

By virtue of these inequalities, from the equality

$$\begin{aligned} &f(x_0+h,y_0+k)-f(x_0,y_0)-h\partial^+_{[x]}f(p_0)-k\partial_y f(p_0)\\ &=\left[f(x_0+h,y_0+k)-f(x_0,y_0+k)-h\partial^+_{[x]}f(p_0)\right]\\ &+[f(x_0,y_0+k)-f(x_0,y_0)-k\partial_y f(p_0)]\end{aligned}$$

we obtain the estimate

$$\left|f(x_0+h,y_0+k)-f(x_0,y_0)-h\partial^+_{[x]}f(p_0)-k\partial_y f(p_0)\right| < \varepsilon(h+|k|), \tag{4.8}$$

when $h+|k|<\min\{\delta_1,\delta_2\}$. Inequality (4.8) implies the fulfillment of equality (1.2) for the numbers $A^+=\partial^+_{[x]}f(p_0)$ and $B^+=\partial_y f(p_0)$. Hence, according to (1.3), equality (4.5) is valid. □

We prove analogously

Theorem4.2. *A sufficient condition for a function $f(x,y)$ to be left-differentiable at a point $p_0=(x_0,y_0)$ is the existence of finite $\partial^-_{[x]}f(p_0)$ and $\partial_y f(p_0)$. Under this assumption we have*

$$d^-f(p_0)=\partial^-_{[x]}f(p_0)\,dx+\partial_y f(p_0)\,dy. \tag{4.9}$$

Note that from Theorems 4.1 and 4.2 we obtain the following known condition for the existence of a differential.

Theorem4.3. ([1, p. 80]) *If a function $f(x,y)$ has finite $\partial_{[x]}f(p_0)$ and $\partial_y f(p_0)$ at a point $p_0=(x_0,y_0)$, then the function f is differentiable at p_0 and equality*

$$df(p_0)=\partial_{[x]}f(p_0)\,dx+\partial_y f(p_0)\,dy \tag{4.10}$$

is fulfilled.

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