INTEGRAL REPRESENTATIONS AND INCLUSION SETS IN $B^p(D)$ SPACE

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Abstract

Let $B^p(D)$ $(0 < p < 1)$ be the space of analytic in the unit disk D functions introduced by Duren, Romberg and Shields. By $A(D)$ denote the set of analytic functions in D which are continuous on $\overline{D}$ and by $H^p(D)$ denote the Hardy space of analytic in D functions.

The paper ascertains: 1) Which class the majorant function belongs to, when the outcome function belongs to$B^p(D)$ space; 2) Integral representations of $B^p(D)$ space functions are found; 3) Multipliers of inclusion are found from $B^p(D)$space to $H^2(D)$, $H^1(D)$ and $A(D)$spaces, i.e. conditions for fractional integrals to belong to $H^2(D)$, $H^1(D)$ and $A(D)$spaces are determined.

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1 Definitions and Preliminaries

Let us denote by $\mathbb{C}$ the space of complex numbers, and by D and T the open unit disk and the unit circumference on the planes $\mathbb{C}$, respectively, i.e. $D = \{z \in \mathbb{C} : \ |z| < 1\}$, $T = \{t \in \mathbb{C} : \ |t| = 1\}$.

Suppose that $t_0 \in T$ and $\alpha > 1$ is some fixed number. The set $\Delta(t_0) = \left\{z \in D : |z - t_0| < \frac{\alpha}{2}(1 - |z|^2)\right\}$ is called the Stolz angle with vertex at a point t_0.

We say that $z \in D$ tends nontangentialy (angularly) to a point $t_0 \in T$ if $\lim\limits_{z \to t_0} |z - t_0| = 0$ so that $z \in \Delta(t_0)$ and denote this situation by $z \widehat{\to} t_0$.

Let us consider the functions $f : D \to \mathbb{C}$ and $\varphi : T \to \mathbb{C}$. We say that φ is the angular (nontangential) boundary value of the function f at a point $t_0 \in T$ if $\lim\limits_{z \widehat{\to} t_0} f(z) = \varphi(t_0)$.

Let us denote by $A(D)$ the set of all analytic functions in the disk D which are continuous on the closed disk $\overline{D}$.

Assume that $I = [0, 1)$ and $dm_1(\omega) = (2\pi)^{-1}d\theta$ is the normed Lebesgue measure on the unit circumference T.

An analytic function $f : D \to \mathbb{C}$ is said to belong to the class $H^p(D)$ $(p > 0)$ if it satisfies the condition $\sup\limits_{r\in I} \int_T |f(r\omega)|^p dm_1(\omega) < \infty$.

It is known that (see e.g. [1], [2]): If $f \in H^p(D)$ $(p > 0)$, then there exists almost everywhere on T the angular limit $f^*(t) = f(e^{i\theta}) = \lim\limits_{z \widehat{\to} t} f(z)$, $t = e^{i\theta}$ and $f^* \in L^p(T)$;

If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$, then $f \in H^2(D) \iff \sum\limits_{n=0}^{\infty} |a_n|^2 < +\infty$.

We denote by $H(D)$ the space of all analytic functions in the disk D.

Assume that X and Y are some spaces of sequences of complex numbers. We say that a sequence $(\omega_n)_{n\geq1}$ is a multiplier from the space X in the space Y if $(\omega_n a_n)_{n\geq1} \in Y$ for every sequence $(a_n)_{n\geq1}$ from the space X.

If $f \in H(D)$ and $\alpha > 0$ is some number, then according to the definition introduced by Hardy-Littlewood a fractional integral and a derivative of order α of the function $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ are defined by the equalities (see e.g. [3], [4])

$$f_{[\alpha]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1)}{\Gamma(n+1+\alpha)}\, a_n z^n,$$
$$f^{[\alpha]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)}\, a_n z^n,$$

respectively, where Γ is the Euler function. Duren [3] showed that there exists a function $f \in A(D)$ such that for every number $\alpha > 0$ the function $f^{[\alpha]}$ has no boundary (radial) values on a set of positive measure on T. Therefore $f^{[\alpha]} \notin H^p$ holds for none of the values of p. Hayman [5] also showed that there exists a function $f \in N(D)$ such that $f_{[1]} \notin N(D)$, where $N(D)$ is the Nevanlinna class in D.

Assume that $0 < p < 1$. According to [3] a function $f \in H(D)$ is said to belong to the class $B^p(D)$ if

$$\|f\| = \int_0^1 \int_0^{2\pi} (1-r)^{\frac{1}{p}-2} |f(re^{i\theta})|\, dr\, d\theta < +\infty.$$

As is known (see [6], Ch. II, §9), for each power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with the convergence radius $r = 1$, there exists the majorant series

$$F(z) = \sum_{n=0}^{\infty} \psi(n) z^n, \tag{1}$$

$n|a_n| \leq \psi(n)$ $(n = \overline{0, \infty})$, which has almost everywhere on T angular boundary values, where $\psi : \mathbb{C} \to \mathbb{C}$ is an entire function of first order and minimal type.

2 Auxiliary Statements

Lemma 1. *If* $F_k(z) = \sum\limits_{n=1}^{\infty} n^k z^n$, *where* $k \in N$ *is some fixed natural number, then*

$$F_k(z) = \frac{p_k(z)}{(1-z)^{k+1}}, \tag{2}$$

where $p_k(z)$ *is a polynomial of degree* k.

Indeed, if we assume that $k = 1$, then $F_1(z) = \sum\limits_{n=1}^{\infty} n z^n$, this series is convergent in the disk D. It is obvious that

$$\begin{aligned} F_1(z) &= z \sum_{n=1}^{\infty} n z^{n-1} = z \sum_{n=1}^{\infty} (z^n)' = z \left(\sum_{n=1}^{\infty} z^n \right)' \\ &= z \cdot \left(\frac{z}{1-z} \right)' = \frac{z}{(1-z)^2}. \end{aligned} \tag{3}$$

Let us now assume that formula (2) is valid when $k = m$, i.e.

$$F_m(z) = \sum_{n=1}^{\infty} n^m z^n = \frac{p_m(z)}{(1-z)^{m+1}}\,. \tag{4}$$

Then by (4) we have

$$\begin{aligned} F_{m+1}(z) &= \sum_{n=1}^{\infty} n^{m+1} z^n = z \sum_{n=1}^{\infty} n^{m+1} z^{n-1} = z \sum_{n=1}^{\infty} (n^m z^n)' \\ &= z \left(\sum_{n=1}^{\infty} n^m z^n \right)' = z \left(\frac{p_m(z)}{(1-z)^{m+1}} \right)' = \frac{p_{m+1}(z)}{(1-z)^{m+2}}. \end{aligned}$$

Thus formula (2) is fulfilled $\forall k \in N$.

Lemma 2. *The function* $F(z)=\sum\limits_{n=1}^{\infty} n^k z^n$*, where* k *is some fixed natural number, belongs to the Hardy class* $H^p(D)$*,* $p\in\left(0,\frac{1}{k+1}\right)$*.*

Indeed, according to Lemma 1 it suffices to show that

$$(1-z)^{-(k+1)}\in H^p(D),\quad p\in\left(0,\frac{1}{k+1}\right).$$

Assume that $z=\rho e^{it}$. Then

$$|1-z|=\left|1-\rho e^{it}\right|=\sqrt{(1-\rho)^2+4\rho\sin^2\frac{t}{2}}\geq\sqrt{4\rho\sin^2\frac{t}{2}}=2\sqrt{\rho}\left|\sin\frac{t}{2}\right|$$
$$\geq 2\sqrt{\rho}\sin\frac{|t|}{2}\geq 2\sqrt{\frac{1}{2}}\cdot\frac{2}{\pi}\cdot\frac{|t|}{2}=\frac{\sqrt{2}}{\pi}\cdot|t|$$

when $\rho>\frac{1}{2}$ and $0\leq|t|<\pi$. Thus we obtain

$$\int_{-\pi}^{\pi}\frac{dt}{\left|1-\rho e^{it}\right|^{p(k+1)}}\leq\sup_{0\leq\rho<1}\int_{-\pi}^{\pi}\frac{dt}{\left|1-\rho e^{it}\right|^{p(k+1)}}$$
$$=\sup_{\left[0,\frac{1}{2}\right]\bigcup\left(\frac{1}{2},1\right)}\int_{-\pi}^{\pi}\frac{dt}{\left|1-\rho e^{it}\right|^{p(k+1)}}\leq\sup_{0\leq\rho\leq\frac{1}{2}}\int_{-\pi}^{\pi}\frac{dt}{\left|1-\rho e^{it}\right|^{p(k+1)}}$$
$$+\sup_{\frac{1}{2}<\rho<1}\int_{-\pi}^{\pi}\frac{dt}{\left|1-\rho e^{it}\right|^{p(k+1)}}<\sup_{0\leq\rho\leq\frac{1}{2}}\int_{-\pi}^{\pi}\frac{dt}{\left[(1-\rho)^2+4\rho\sin^2\frac{t}{2}\right]^{\frac{p(k+1)}{2}}}$$
$$+\int_{-\pi}^{\pi}\left(\frac{\pi}{2\sqrt{2}}\right)^{p(k+1)}\frac{dt}{|t|^{p(k+1)}}<\sup_{0\leq\rho\leq\frac{1}{2}}\int_{-\pi}^{\pi}\frac{dt}{(1-\rho)^{p(k+1)}}$$
$$+2\left(\frac{\pi}{\sqrt{2}}\right)^{p(k+1)}\int_{0}^{\pi}\frac{dt}{t^{p(k+1)}}=2\pi\cdot 2^{p(k+1)}+\left(\frac{\pi}{2\sqrt{2}}\right)^{p(k+1)}\int_{0}^{\pi}\frac{dt}{t^{p(k+1)}}.$$

The integral $\int\limits_0^{\pi}\frac{dt}{t^{p(k+1)}}$ is convergent if and only if $p(k+1)<1$. This implies the validity of Lemma 2.

Lemma 3. *If* $\varphi(z)=\sum\limits_{i=1}^{m}a_i z^i$*, then the function*

$$F(z)=\sum_{n=0}^{\infty}\varphi(n)z^n$$

belongs to the class $H^p(D)$*,* $p\in\left(0,\frac{1}{m+1}\right)$*.*

Lemma 3 immediately follows from Lemma 2.

Theorem A (see e.g. [3], [4]). *If* $f(z) = \sum\limits_{n=0}^{\infty} a_n(f) z^n \in B^p(D)$*, then*

$$a_n(f) = \overline{\overline{o}}\left(n^{\frac{1}{p}-1}\right),$$

where $a_n(f)$ *is the Taylor coefficient of the function* f*.*

3 Results

The aim of this paper is to study the following questions: 1) to which class does the majorant function (1) belong if the initial function belongs to $B^p(D)$? 2) what integral representation do functions of from the class $B^p(D)$ have? 3) What form do multipliers of inclusion from $B^p(D)$ to $H^2(D)$, $H^1(D)$ and $A(D)$ have or to what class do the fractional integrals H^2 of functions from the space B^p belong for H^1 and $A(D)$?

The answers to the above-posed questions are provided by the following theorems.

Theorem 1. *Assume that* $f \in B^p(D)$*. Then for the function* f *there exists a majorant function* $F(z) = \sum\limits_{n=0}^{\infty} \psi(n) z^n$ *such that* $F \in H^{\delta}(D)$*,* $\delta \in \left(0, \frac{1}{1+[p^{-1}]}\right)$*, where* $\psi : \mathbb{C} \to \mathbb{C}$ *is an entire function of first order and minimal type.*

Proof. According to Theorem A, for each $f \in B^p(D)$ we have

$$\lim_{n\to\infty} \left| a_n(f) n^{-[p^{-1}]} \right| = 0,$$

and therefore there exists a number $n_0 \in N$ such that for all $n \geq n_0$ the following inequality is fulfilled:

$$\forall n \geq n_0, \quad |a_n(f)| < n^{[p^{-1}]}.$$

Let us assume that $M = \max\limits_{0 \leq k \leq n_0} |a_k(f)|$. Then it is clear that

$$|a_n(f)| < n^{[p^{-1}]} + M + 1, \quad n = \overline{0, \infty}. \tag{5}$$

From inequality (5) it follows that the majorant function of f is

$$F(z) = \sum_{n=0}^{\infty} \left(n^{[p^{-1}]} + M + 1\right) z^n,$$

where $\psi(z) = z^{[p^{-1}]} + M + 1$.

By Lemma 2 we have $F \in H^{\delta}(D)$, $\quad \delta \in \left(0, \frac{1}{1+[p^{-1}]}\right)$. The theorem is proved. $\square$

Theorem 1 can be used for the integral representation of functions of the class $B^p(D)$.

As is known, for each analytic function $f : D \to \mathbb{C}$ there exists an entire function $g : \mathbb{C} \to \mathbb{C}$ and a square-summable function $\varphi : [0, 2\pi] \to \mathbb{C}$ such that

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} g\left(\frac{1}{1 - ze^{-i\theta}}\right) \varphi(e^{i\theta}) d\theta \tag{6}$$

(see [6], Ch. II, §9). It is clear that g and φ are in general the functions depending on f.

Theorem 2. *If $f \in B^p(D)$, then there exists a square summable function $\varphi : [0, 2\pi] \to \mathbb{C}$ such that*

$$f(z) = \frac{(1 + [p^{-1}])!}{2\pi} \int_0^{2\pi} \frac{\varphi\left(e^{i\theta}\right) d\theta}{(1 - ze^{-i\theta})^{[p^{-1}]+2}}, \quad z \in D. \tag{7}$$

Proof. Indeed, multiplying both sides of (5) by n, we obtain

$$n|a_n(f)| < n^{[p^{-1}]+1} + (1 + M)n. \tag{8}$$

From inequality (8) it follows that there exists a natural number m such that $\forall n \geq m$ the following inequality is fulfilled:

$$n|a_n(f)| < (n + 1)(n + 2) \cdots (n + [p^{-1}] + 1) = \psi(n). \tag{9}$$

Indeed, it is clear that

$$(n + 1)^{1+[p^{-1}]} < \psi(n). \tag{10}$$

It is likewise clear that inequality (9) will be fulfilled if

$$n^{1+[p^{-1}]} + (M + 1)n < (n + 1)^{1+[p^{-1}]},$$

but the latter inequality will be fulfilled for all those $n \in N$ which satisfy the inequality

$$(M + 1)n < (1 + [p^{-1}])n^{[p^{-1}]},$$

from which we obtain

$$n > \left(\frac{M + 1}{1 + [p^{-1}]}\right)^{\frac{1}{[p^{-1}]-1}} = \beta.$$

Assume that $m = 1 + [\beta]$, then it is clear that inequality (9) will be fulfilled $\forall n > m$. Let us show that

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{a_n(f)}{\psi(n)} z^n \in H^2(D).$$

Indeed, by inequality (9) we have

$$\sum_{n=0}^{\infty} \left|\frac{a_n(f)}{\psi(n)}\right|^2 = \sum_{n=0}^{m} \left|\frac{a_n(f)}{\psi(n)}\right|^2 + \sum_{n=m+1}^{\infty} \left|\frac{a_n(f)}{\psi(n)}\right|^2$$
$$< \sum_{n=0}^{m} \left|\frac{a_n(f)}{\psi(n)}\right|^2 + \sum_{n=m+1}^{\infty} \frac{1}{n^2} < +\infty.$$

Therefore $\varphi \in H^2(D)$ and the angular boundary values of φ are square-summable on $[0, 2\pi]$. If we assume that $\mu = 1 + [p^{-1}]$, then we also have that $z^\mu \varphi \in H^2(D)$. It also clearly follows that $\forall z \in D$

$$f(z) = \frac{d^\mu}{dz^\mu} \left(z^\mu \cdot \varphi(z)\right). \tag{11}$$

By the Cauchy formula (see [6], Ch. II, §5) we have

$$z^\mu \varphi(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\mu t} \varphi\left(e^{it}\right) dt}{1 - z e^{-it}}. \tag{12}$$

Using equalities (11) and (12) we obtain

$$f(z) = \frac{(1 + [p^{-1}])!}{2\pi} \int_0^{2\pi} \frac{\varphi(e^{it}) dt}{(1 - z e^{-it})^{2 + [p^{-1}]}}. \tag{13}$$

The theorem is proved. □

By this representation and taking into account the definition of a fractional integral, we easily make sure that the following theorem is valid.

Theorem 3. *If $f \in B^p(D)$, then a fractional integral of order $\alpha = 1 + [p^{-1}]$ of this function belongs to $H^2(D)$, i.e. $f_{[\alpha]} \in H^2(D)$, $\alpha = 1 + [p^{-1}]$.*

So that $\left(\omega_n = \frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])}\right)_{n \geq 0}$ is a multiplier from the space $B^p(D)$ to the space $H^2(D)$.

Proof. Indeed, if $f \in B^p(D)$, then by Theorem 2 there exists a function $\varphi \in H^2(D)$ such that $\forall z \in D$

$$f(z) = \frac{(1 + [p^{-1}])!}{2\pi} \int_0^{2\pi} \frac{\varphi(e^{it}) dt}{(1 - z e^{-it})^{2 + [p^{-1}]}}.$$

Applying the well-known binomial expansion, we obtain

$$f(z) = \frac{\left(1+\left[p^{-1}\right]\right)!}{2\pi} \sum_{n=0}^{\infty} \left[\frac{\Gamma\left(n+\left[p^{-1}\right]\right)}{\Gamma\left(1+n\right)} \int_0^{2\pi} \varphi\left(e^{it}\right) e^{-int} dt \right] z^n,$$

wherefrom

$$\begin{aligned} f_{[\alpha]}(z) &= \frac{(1+[p^{-1}])!}{2\pi} \\ &\times \left\{ \sum_{n=0}^{\infty} \left[\frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])} \cdot \frac{\Gamma(n+2+[p^{-1}])}{\Gamma(1+n)} \int_0^{2\pi} \varphi(e^{it}) e^{-int} dt \right] z^n \right\} \\ &= \frac{(1+[p^{-1}])!}{2\pi} \int_0^{2\pi} \left[\sum_{n=0}^{\infty} z^n e^{-int} \right] \varphi(e^{it}) dt \\ &= \frac{(1+[p^{-1}])!}{2\pi} \int_0^{2\pi} \frac{\varphi(e^{it}) dt}{1 - z e^{-it}} = (1+[p^{-1}])! \varphi(z) \in H^2. \end{aligned} \quad (14)$$

Here we have used Fichtenholz' theorem (see [6], Ch. II, §5). □

If $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n \in B^p(D)$, then Theorem 3 implies that

$$\left(\omega_n = \frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])} \right)_{n \geq 0}$$

is a multiplier of inclusion from the space $B^p(D)$ to the space $H^2(D)$ or to the space l^2, thus

$$\sum_{n=0}^{\infty} \left| \frac{\Gamma\left(1+n\right)}{\Gamma\left(n+2+[p^{-1}]\right)} \, a_n(f) \right|^2 < +\infty.$$

Thus we show that the following statement is true.

Corollary 1. If $f(z) = \sum\limits_{n=0}^{\infty} a_n(f) z^n$ and $f \in B^p(D)$, then

$$\left(\beta_n = \frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])} \, a_n(f) \right)_{n \geq 0} \in l^2.$$

Corollary 2. If $f(z) = \sum\limits_{n=0}^{\infty} a_n(f) z^n$ and $f \in B^p(D)$, then

$$\left(\gamma_n = \frac{\Gamma(1+n)}{n \cdot \Gamma(n+2+[p^{-1}])} \right)_{n \geq 0}$$

is a multiplier of inclusion from the space $B^p(D)$ to the space $l^1 = l$.

Indeed, this follows from the following inequality

$$|a_n(f)|\,|\gamma_n| \leq \frac{1}{n^2} + \left|\frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])}\,a_n(f)\right|^2.$$

Corollary 3. If $f \in B^p(D)$ and $\alpha = 1 + [p^{-1}]$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f_{[\alpha]}(e^{it})dt}{(1 - ze^{-it})^{2+[p^{-1}]}}. \tag{15}$$

Indeed, if in formula (11) we use equality (**??**), then we obtain formula (15), where $f_{[\alpha]}\left(e^{it}\right) = \lim\limits_{z \widehat{\to} e^{it}} f_{[\alpha]}(z)$.

Theorem 4. *If $f \in B^p(D)$, then*

1) $F(z) = \sum\limits_{n=0}^{\infty} \dfrac{\Gamma(1+n)}{(n+1)\Gamma(n+2+[p^{-1}])}\,a_n(f)z^n \in A(D)$;

2) $F(e^{i\theta}) = \sum\limits_{n=0}^{\infty} \dfrac{\Gamma(1+n)}{(n+1)\Gamma(n+2+[p^{-1}])}\,a_n(f)e^{in\theta}$ *is an absolutely continuous function on the interval* $[0, 2\pi]$.

In other words, equality 1) means that $\left(\omega_n = \frac{\Gamma(1+n)}{(n+1)\Gamma(n+2+[p^{-1}])}\right)_{n\geq 0}$ is a multiplier of inclusion from the space $B^p(D)$ to the space $A(D)$.

Proof of Theorem 4. Since $f \in B^p(D)$, by formula (15) we have

$$f_{[\alpha]}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])}\,a_n(f)z^n \in H^2(D) \subset H^1(D),$$

therefore by Smirnov's theorem (see [6], Ch. II, §4), the primitive function of $f_{[\alpha]}$

$$F(z) = \int_0^z f_{[\alpha]}(t)dt \in A(D) \quad \text{and} \quad F(e^{i\theta}) = \int_0^{e^{i\theta}} f_{[\alpha]}(t)dt$$

is absolutely continuous on the interval $[0, 2\pi]$, but if we use term-by-term integration, we obtain

$$1) \qquad F(z) = \int_0^z f_{[\alpha]}(t)dt = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{\Gamma(n+2+[p^{-1}])}\,a_n(f)\int_0^z t^n dt$$
$$= \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{(n+1)\Gamma(n+2+[p^{-1}])}a_n(f)z^n;$$

$$2) \quad F(e^{i\theta}) = \sum_{n=0}^{\infty} \frac{\Gamma(1+n)}{(n+1)\Gamma(n+2+[p^{-1}])}a_n(f)e^{in\theta}.$$

The theorem is proved. □

For any value of the parameter α ($\alpha \geq 0$) let us consider the following functions

I. $C_\alpha(z,t) = \dfrac{\Gamma(1+\alpha)}{(1-\bar{t}z)^{\alpha+1}}, \quad t \in T, \quad z \in D,$

II. $H_\alpha(z,t) = 2C_\alpha(z,t) - C(0,t) = \Gamma(1+\alpha)\left[\dfrac{2}{(1-\bar{t}z)^{\alpha+1}} - 1\right],$

III. $P_\alpha(z,t) = \operatorname{Re} H_\alpha(z,t) = \Gamma(1+\alpha)\left[2\operatorname{Re}\dfrac{1}{(1-\bar{t}z)^{\alpha+1}} - 1\right].$

If $\alpha = 0$, then the functions

$$C(z,t) = C_0(z,t) = \frac{1}{1-\bar{t}z} = \frac{1}{1-ze^{-i\theta}}, \quad t = e^{i\theta},$$

$$H(z,t) = H_0(z,t) = \frac{2}{1-\bar{t}z} - 1 = \frac{1+\bar{t}z}{1-\bar{t}z} = \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}},$$

$$P(z,t) = P_0(z,t) = \frac{1-|z|^2}{|1-ze^{-i\theta}|^2} = \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}, \quad z = re^{i\varphi},$$

represent respectively the Cauchy, Schwartz and Poisson kernels for the unit disk. Using the representation of the functions $C(z,t)$, $H(z,t)$ and $P(z,t)$ in the form of series and the definition of a fractional integral of order α, it follows that

$$C(z,t) = \sum_{n=0}^{\infty} z^n e^{-in\theta},$$

$$H(z,t) = 2\sum_{n=0}^{\infty} z^n e^{-in\theta} - 1 = 1 + 2\sum_{n=1}^{\infty} z^n e^{-in\theta},$$

$$P(z,t) = 1 + 2\sum_{n=1}^{\infty} r^n \cos n\,(\theta-\varphi) = \sum_{n=-\infty}^{+\infty} r^{|n|} e^{in(\theta-\varphi)},$$

where from

$$C_\alpha(z,t) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, z^n e^{-in\theta}, \tag{16}$$

$$H_\alpha(z,t) = \Gamma(1+\alpha) + 2\sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, z^n e^{-in\theta}, \tag{17}$$

$$P_\alpha(z,t) = \sum_{n=-\infty}^{+\infty} \frac{\Gamma(|n|+\alpha+1)}{\Gamma(1+|n|)}\, z^{|n|} e^{in(\theta-\varphi)}, \tag{18}$$

These series are absolutely and uniformly convergent on every compact set of the disk D. They represent respectively fractional derivatives of order α of the Cauchy, Schwartz and Poisson kernels.

The following theorem is true.

Theorem 5. *If* $f(z) = \sum\limits_{n=0}^{\infty} a_n(f) z^n \in B^p(D)$, *then for* $\alpha = 1 + [p^{-1}]$ *the following integral representations are valid:*

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f_{[\alpha]}(e^{i\theta}) d\theta}{(1 - ze^{-i\theta})^{2+[p^{-1}]}} = \frac{1}{2\pi} \int_0^{2\pi} C_\alpha(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta, \tag{19}$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} H_\alpha(z, e^{i\theta}) u_\alpha(e^{i\theta}) d\theta + i \operatorname{Im} f(0), \tag{20}$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} P_\alpha(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta, \tag{21}$$

where $u_\alpha = \operatorname{Re} f_{[\alpha]}$.

Proof. Formula (19) will be proved by virtue of the proof of Theorem 3. Let us show that formulas (20) and (21) are valid. For $z = 0$, from formula (19) we obtain

$$f(0) = \frac{1}{2\pi} \int_0^{2\pi} f_{[\alpha]}(e^{i\theta}) d\theta, \tag{22}$$

which implies

$$\overline{f}(0) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f}_{[\alpha]}(e^{i\theta}) d\theta. \tag{23}$$

Using formulas (17) and (20) we have

$$\begin{aligned}
\frac{1}{2\pi} \int_0^{2\pi} H_\alpha(z, e^{i\theta}) u_\alpha(e^{i\theta}) d\theta &= \frac{1}{2\pi} \int_0^{2\pi} [2C_\alpha(z, e^{i\theta}) - 1] \\
\times \frac{f_{[\alpha]}(e^{i\theta}) + \overline{f}_{[\alpha]}(e^{i\theta})}{2} d\theta &= \frac{1}{2\pi} \int_0^{2\pi} C_\alpha(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta \\
+ \frac{1}{2\pi} \int_0^{2\pi} C_\alpha(z, e^{i\theta}) \overline{f}_{[\alpha]}(e^{i\theta}) d\theta &- \frac{1}{2\pi} \int_0^{2\pi} u_\alpha(e^{i\theta}) d\theta \\
= f(z) + \frac{1}{2\pi} \int_0^{2\pi} \overline{f}_{[\alpha]}(e^{i\theta}) d\theta - u_\alpha(0) &= f(z) + \overline{f}(0) - \operatorname{Re} f(0) \\
= f(z) + \operatorname{Re} f(0) - i \operatorname{Im} f(0) - \operatorname{Re} f(0) &= f(z) - i \operatorname{Im} f(0),
\end{aligned}$$

from which we obtain

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} H_\alpha(z, e^{i\theta}) u_\alpha(e^{i\theta}) d\theta + i \operatorname{Im} f(0).$$

Here we have used the equality

$$\overline{f}(0) = \frac{1}{2\pi}\int_0^{2\pi} \overline{f}_{[\alpha]} C_\alpha\left(z, e^{i\theta}\right) d\theta.$$

Indeed, if we use analyticity of the function $f_{[\alpha]}$, then by formula (16) and the Lebesgue theorem on bounded convergence, then we have

$$\frac{1}{2\pi}\int_0^{2\pi} \overline{f}_{[\alpha]}(e^{i\theta}) C_\alpha(z, e^{i\theta}) d\theta$$
$$= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi}\int_0^{2\pi} \overline{f}_{[\alpha]}(e^{i\theta}) e^{in\theta}\right]\frac{\Gamma(n+\alpha+1)}{\Gamma(1+n)} z^n = \frac{1}{2\pi}\int_0^{2\pi} \overline{f}_{[\alpha]}(e^{i\theta}) d\theta = \overline{f}(0).$$

Let us now show the validity of formula (21). If we use the definition of a fractional integral of order α and that of a fractional derivative, then it is clear that for each function $f \in H(D)$ we will have

$$(f_{[\alpha]})^{[\alpha]}(z) = (f^{[\alpha]})_{[\alpha]} = f(z). \tag{24}$$

If $f \in B^p(D)$, then, as we know, $f_{[\alpha]} \in H^2(D)$, where $\alpha = 1 + [p^{-1}]$, therefore, by Fichtenholz' theorem (see [6], ch. II, §5), it will be represented by the Poisson integral

$$f_{[\alpha]}(z) = \frac{1}{2\pi}\int_0^{2\pi} P(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta,$$

from which, using equality (24) and formula (17), we obtain

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} P_{[\alpha]}(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta = \frac{1}{2\pi}\int_0^{2\pi} P_\alpha(z, e^{i\theta}) f_{[\alpha]}(e^{i\theta}) d\theta.$$

Here we have used the equality

$$P_{[\alpha]}(z, e^{i\theta}) = \sum_{n=-\infty}^{+\infty} \frac{\Gamma(|n|+\alpha+1)}{\Gamma(n+1)} z^{|n|} e^{-in(\theta-\varphi)} = P_\alpha(z, e^{i\theta}), \;\; z = re^{i\varphi}.$$

The theorem is proved. □

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