AN INITIAL-CHARACTERISTIC PROBLEM FOR A QUASI-LINEAR EQUATION OF A PROBLEM OF NONLINEAR OSCILLATIONS

R. Bitsadze*, M. Menteshashvili**

Georgian Technical University*
77 Kostava Str., Tbilisi 0175, Georgia
Sokhumi State University**
12 Politkovskaja Str., Tbilisi 0186, Georgia

(Received: 15.03.2011; accepted: 16.11.2011)

Abstract

For the well-known quasi-linear equation of a problem of nonlinear oscillation the paper considers a nonlinear initial-characteristic problem which consists in the simultaneous definition of a solution together with the domain of its regular propagation. The problem as to the solvability of the stated problem is solved by the method of characteristics.

Key words and phrases: Characteristics, general integral, propagation domain.
AMS subject classification: 35L70.

1 Introduction

In this paper, an attempt is made to state correctly a mixed characteristic problem for a quasi-linear equation which has numerous practical and theoretical applications [1] and which arises in studying nonlinear oscillations

$$x^2(u_y^4 u_{xx} - u_{yy}) = cuu_y^4, \quad c = const. \tag{1}$$

Equation (1) is interesting because of the degeneration of order and, possibly, of hyperbolicity too. The former is well-defined and occurs on the ordinate axis. The latter, i.e. the parabolic degeneration [2] depends on the behavior of $u(x, y)$. Therefore the set of points of this degeneration is not a priori given and should be defined simultaneously with a solution.

In the papers dealing with non-strictly hyperbolic equations a great deal of attention is paid to equation (1) for $c = 0$. Its general integral is represented by means of a generator of groups of solutions based on contact transformations [3]. However the question as to the solvability of problems was not considered.

As is known, the characteristics of linear hyperbolic equations are completely defined by the main coefficients. In nonlinear cases these coefficients depend on the sought solution and its lowest derivatives. Since the characteristics also depend on them, the linear situation of a mixed characteristic problem cannot be automatically extended to the case of nonlinear equations. From the application standpoint, these are the equations that are especially interesting to investigate [4]. Therefore the formulations of problems for such equations should be revised taking into account their general characteristic invariants [5-6].

2 General Integral

Initial-characteristic problems are solved by different methods. These methods will yield even more numerous results if we enrich them with the method of characteristics. In this connection I. M. Gelfund expressed his opinion that the method of characteristics can be effective for both exact and approximate solutions of problems [7]. Based on this idea, we will need all the characteristic laws of equation (1).

One of the characteristic roots $\lambda_1 = u_y^{-2}$ of equation (1) must define everywhere the corresponding direction by the relation

$$u_y^2 dy - dx = 0,$$

which in turn, together with equation (1), yields

$$x^2\big(u_y^4\, d(u_x) - u_y^2\,(du_y)\big) = cu_y^4 u\, dx.$$

If to the equalities obtained above we add the consistency condition

$$du = u_x\, dx + u_y\, dy,$$

we come to a system of characteristic differential relations which though should not be treated as an ordinary system of differential equations, nevertheless enables us to turn to the question of construction of the first integral of this system [8].

For $c > -\frac{1}{4}$ there exist first integrals of the above mentioned differential relations

$$\begin{cases} \xi \equiv (u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1} \\ \xi_1 \equiv (u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha)ux^{-\alpha}, \quad \alpha = \dfrac{1}{2}\left(1 + \sqrt{4c+1}\right) \end{cases} \tag{2}$$

having constant values along any characteristic of the considered family [9].

The combinations ξ, ξ_1 containing the argument x, sought solution $u(x,y)$ and its first order derivatives are analogues of the well-known Riemann invariants [10]. It is commonly accepted to call them characteristic invariants.

In an analogous manner we define the characteristic invariants

$$\begin{cases} \eta \equiv (u_y^{-1} - u_x)x^{\alpha} + \alpha u x^{\alpha-1} \\ \eta_1 \equiv (u_y^{-1} - u_x)x^{1-\alpha} + (1-\alpha)ux^{-\alpha} \end{cases}, \tag{3}$$

of the other family corresponding to the root $\lambda_2 = -\lambda_1 = -u_y^{-2}$ [9].

The above two pairs of characteristic invariants make it possible to construct the intermediate integrals

$$\xi_1 = \varphi'(\xi), \quad \eta_1 = \psi'(\eta)$$

of equation (1) [11]. To provide the required smoothness of the solution $u(x,y)$, it is assumed that arbitrary functions φ, ψ are from the class $C^3(R^1)$.

The following statement is valid [9].

Theorem. *Equation* (1) *is equivalent to a triple of the following relations*

$$x = \Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1}{1-2\alpha}}, \tag{4}$$

$$y = \frac{1}{4(1-2\alpha)}\Big[(\xi+\eta)\big(\psi'(\eta) - \varphi'(\xi)\big) + 2\big(\varphi(\xi) - \psi(\eta)\big)\Big], \tag{5}$$

$$u = \frac{1}{1-2\alpha}\Big[\xi\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{1-\alpha}{1-2\alpha}} - \varphi'(\xi)\Big(\frac{\varphi'(\xi) + \psi'(\eta)}{\xi + \eta}\Big)^{\frac{\alpha}{1-2\alpha}}\Big]. \tag{6}$$

To relations (4)–(6) we have come without any additional conditions. Eliminating in them arbitrary parameters φ, ψ, we come back to equation (1). Therefore this triple of relations can be assumed to be a general integral of equation (1), and the invariants ξ, η to be characteristic variables.

However the above constructed general integral (4)–(6) defines in no way at least one characteristic of any family in order to take this characteristic as a data carrier of the mixed characteristic problem. Therefore this characteristic will be chosen arbitrarily at our discretion. Suppose it is some arc γ of continuously differentiable curvature given in the explicit form

$$\gamma: \ y = g(x), \quad 0 < a \le x \le b.$$

The function $g(x)$ is assumed to be strictly monotone, and the arc γ to be ascending. Without loss of generality it can be assumed that

$$g(a) = 0.$$

Let the function $h(x)$, given on some segment $[a,d], d < b$, be twice continuously differentiable and contracting it to the segment $[a,c], c < d$. Assume that the function $h(x)$ satisfies the condition $h(a) = a$, $h(d) = c$, $h' < 0$.

3 Initial-characteristic Problem

Find a regular hyperbolic solution $u(x,y)$ of equation (1) and, simultaneously, the domain of its propagation if the curve γ is characteristic along this solution and the solution itself satisfies the conditions

$$u(a,0) = \vartheta, \quad u_x(a,0) = \theta, \tag{7}$$

whereas each pair of points $(x,0)$, $(h(x), g(h(x)))$ connected with the mapping h belong to the respective general characteristic of the root λ_2.

According to the formulation of the problem, the curve γ actually belongs to the family of characteristics of the root λ_1, which is equivalent to the equality $g'(x) = u_y^{-2}(x, g(x))$ that enables us to define two variants of values of the derivative u_y along the curve γ

$$u_y = \frac{1}{\pm\sqrt{g'(x)}}. \tag{8}$$

Of them we choose an arithmetic value of the root. To solve the formulated problem, we need, along the arc γ with (8), also the values of the solution u and the derivative u_x. For this purpose we use the characteristic invariants ξ, ξ_1 of the family λ_1. The values of the sought u and u_x at the initial point $(a,0)$ of the arc γ are known. Using (7), (8), we calculate the values of the characteristic invariants ξ, ξ_1 at the $(a,0)$, for which we introduce the notation

$$\begin{aligned} \xi\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{\alpha} - \alpha\vartheta a^{\alpha-1} \equiv [\xi]_a, \\ \xi_1\big|_{(a,0)} &= \big(\sqrt{g'(a)} + \theta\big)a^{1-\alpha} - (1-\alpha)\vartheta a^{-\alpha} \equiv [\xi_1]_a. \end{aligned} \tag{9}$$

Since the characteristic invariants ξ, ξ_1 take constant values along the curve γ, we have

$$\big[(u_y^{-1} + u_x)x^{\alpha} - \alpha u x^{\alpha-1}\big]\Big|_{\gamma} = [\xi]_a,$$

$$\big[(u_y^{-1} + u_x)x^{1-\alpha} - (1-\alpha) u x^{-\alpha}\big]\Big|_{\gamma} = [\xi_1]_a.$$

Considering these two relations as a system relative to u and u_x, we define their values on γ as follows

$$u\big|_{\gamma} = \frac{1}{2-\alpha}\left[x^{1-\alpha}[\xi]_a - [\xi_1]_a x^{\alpha}\right],$$
$$u_x\big|_{\gamma} = \frac{1-\alpha}{2-\alpha}[\xi]_a x^{-\alpha} - \frac{\alpha}{1-2\alpha}[\xi_1]_a x^{\alpha-1} - \sqrt{g'(x)}\,.$$

Thus we have succeeded in defining the values of the sought solution and its first order derivatives all over the characteristic γ. Using these values we will define the solution and its first order derivatives outside γ and establish the limits of their propagation.

To define the values of $u(x,0)$, $u_x(x,0)$ and $u_y(x,0)$, from an arbitrary point $P(x,0)$, $a < x \leq d$, we draw the characteristic Γ of the family of the root λ_2, which by the conditions of problem (1), (7) intersect the characteristic γ at the point $N(h(x), g(h(x)))$. The invariants η and η_1 must be constant along the characteristic Γ.

Since the values of the invariants η, η_1 at the point N

$$\eta\big|_N = 2\sqrt{g'(h(x))}\,h^{\alpha}(x) - [\xi]_a,$$
$$\eta_1\big|_N = 2\sqrt{g'(h(x))}\,h^{1-\alpha}(x) - [\xi_1]_a,$$

remain unchanged all over the characteristic Γ, the point $(x,0)$ inclusive, the following equalities will be fulfilled:

$$\eta(x,0) = \eta\big|_N, \quad \eta_1(x,0) = \eta_1\big|_N.$$

These invariants can be written in the explicit form

$$[\eta]_x \equiv \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{\alpha} + \alpha u(x,0)x^{\alpha-1}$$
$$= 2\sqrt{g'(h(x))}\,h^{\alpha}(x) - [\xi]_a, \tag{10}$$
$$[\eta_1]_x \equiv \big(u_y^{-1}(x,0) - u_x(x,0)\big)x^{1-\alpha} + (1-\alpha)u(x,0)x^{-\alpha}$$
$$= 2\sqrt{g'(h(x))}\,h^{1-\alpha}(x) - [\xi_1]_a. \tag{11}$$

We take these equalities as a linear algebraic system and define the sought solution at an arbitrary point $(x,0)$ of the segment $[a,d]$

$$u(x,0) = \frac{2}{2\alpha-1}\sqrt{g'(h(x))}\,\big(h^{\alpha}(x)x^{1-\alpha} - h^{1-\alpha}(x)x^{\alpha}\big)$$
$$-\frac{1}{2\alpha-1}[\xi]_a x^{1-\alpha} + \frac{1}{2\alpha-1}[\xi_1]_a x^{\alpha}. \tag{12}$$

This is quite sufficient in order to define at the same points the first order derivatives $u_x(x,0)$ and $u_y(x,0)$ of the sought solution $u(x,y)$. The derivative u_x is obtained by direct differentiation of (12)

$$\begin{aligned} u_x(x,0) &= \frac{1}{2\alpha-1}\,\frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\left(h^{\alpha}(x)x^{1-\alpha}-h^{1-\alpha}(x)x^{\alpha}\right) \\ &+ \frac{2}{2\alpha-1}\sqrt{g'(h(x))}\Big(\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha}+(1-\alpha)h^{\alpha}(x)x^{-\alpha} \\ &- (1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha}-\alpha h^{1-\alpha}(x)x^{\alpha-1}\Big) \\ &- \frac{1-\alpha}{2\alpha-1}\,[\xi]_a x^{-\alpha}+\frac{\alpha}{2\alpha-1}\,[\xi_1]_a x^{\alpha-1}. \end{aligned} \tag{13}$$

The other derivative u_y is defined by substituting (12), (13) into (10) or (11)

$$\begin{aligned} u_y(x,0) = \Bigg\{ &\frac{2\sqrt{g'(h(x))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(x)x^{-\alpha}+(1-\alpha)h^{1-\alpha}(x)x^{\alpha-1} \\ &+\alpha h^{\alpha-1}(x)h'(x)x^{1-\alpha}-(1-\alpha)h^{-\alpha}(x)h'(x)x^{\alpha}\Big] \\ &+\frac{1}{2\alpha-1}\,\frac{g''(h(x))\cdot h'(x)}{\sqrt{g'(h(x))}}\left(h^{\alpha}(x)x^{1-\alpha}-h^{1-\alpha}(x)x^{\alpha}\right) \\ &+\frac{1-\alpha}{2\alpha-1}\,[\xi]_a x^{\alpha}\Bigg\}^{-1}. \end{aligned} \tag{14}$$

Let $(\rho, g(\rho))$ and $(\sigma, 0)$ be arbitrarily chosen points from the γ and segment $[a,d]$ respectively.

Using the values of $u(\sigma,0)$, $u_x(\sigma,0)$, $u_y(\sigma,0)$ we define the constants $[\xi]_\sigma$, $[\xi_1]_\sigma$, whose values must coincide with the invariants ξ, ξ_1 on the yet unknown characteristic Γ_1 of the family of the root λ_1 drawn from the point $A(\sigma,0)$. Assume that this characteristic is given by the formula $y=m(x)$ where the function m is to be defined. Then on this curve we have

$$\xi\big|_{\Gamma_1} = \Big(u_y^{-1}(x,m(x))+u_x(x,m(x))\Big)x^{\alpha}-\alpha u(x,m(x))x^{\alpha-1}=[\xi]_\sigma, \tag{15}$$

$$\begin{aligned} \xi_1\big|_{\Gamma_1} &= \Big(u_y^{-1}(x,m(x))+u_x(x,m(x))\Big)x^{1-\alpha} \\ &- (1-\alpha)u(x,m(x))x^{-\alpha}=[\xi_1]_\sigma, \end{aligned} \tag{16}$$

where

$$
\begin{aligned}
[\xi]_\sigma = \xi\big|_A = {} & \frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma)+(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} \\
& -\alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha}-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] \\
& +\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^\alpha(\sigma)\sigma-h^{1-\alpha}\sigma^{2\alpha}\right) \\
& +\frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha}-2\sqrt{g'(h(\sigma))}\,h^\alpha(\sigma)+[\xi]_a, \\
[\xi_1]_\sigma = \xi_1\big|_A = {} & \frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^\alpha(\sigma)\sigma^{-1}(1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2} \\
& +\alpha h^{\alpha-1}(\sigma)h'(\sigma)-(1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big] \\
& +\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^\alpha(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\right) \\
& +\frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha-1}-2\sqrt{g(h(\sigma))}+[\xi_1]_a.
\end{aligned}
$$

In an absolutely analogous manner we define the invariants η, η_1 on the characteristic Γ_2 of the other family drawn from the point $B(\rho, g(\rho))$. Assume that this characteristic is given by the equation $y=\ell(x)$, where ℓ is a yet unknown function. Thus we have

$$
\eta\big|_{\Gamma_2} = \Big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\Big)x^\alpha+\alpha u(x,\ell(x))x^{\alpha-1}=[\eta]_\rho, \tag{17}
$$

$$
\begin{aligned}
\eta_1\big|_{\Gamma_2} = {} & \Big(u_y^{-1}(x,\ell(x))-u_x(x,\ell(x))\Big)x^{1-\alpha} \\
& +(1-\alpha)u(x,\ell(x))x^{-\alpha}=[\eta_1]_\rho,
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
[\eta]_\rho &= 2\sqrt{g'(h(\rho))}\,h^\alpha(\rho)-[\xi]_a, \\
[\eta_1]_\rho &= 2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)-[\xi_1]_a.
\end{aligned}
$$

At the intersection point (x_1, y_1) of these characteristics, if such a point exists, conditions (15)–(18) and $\ell(x_1)=m(x_1)$ must be fulfilled simultaneously. Therefore in the left-hand parts of (15), (16) we can replace $m(x_1)$ by $\ell(x_1)$. As a result we obtain the following system for defining the values of x, u, u_x, u_y at the point $C(x_1, \ell(x_1))$

$$
\Big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\Big)x_1^\alpha-\alpha u(x_1,\ell(x_1))x_1^{\alpha-1}=[\xi]_\sigma, \tag{19}
$$

$$
\Big(u_y^{-1}(x_1,\ell(x_1))+u_x(x_1,\ell(x_1))\Big)x_1^{1-\alpha}
$$

$$- (1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha} = [\xi_1]_\sigma, \tag{20}$$

$$\Big(u_y^{-1}(x_1,\ell(x_1)) - u_x(x_1,\ell(x_1))\Big)x_1^{\alpha} + \alpha u(x_1,\ell(x_1))x_1^{\alpha-1} = [\eta]_\rho, \tag{21}$$

$$\Big(u_y^{-1}(x_1,\ell(x_1)) - u_x(x_1,\ell(x_1))\Big)x_1^{1-\alpha} + (1-\alpha)u(x_1,\ell(x_1))x_1^{-\alpha} = [\eta_1]_\rho. \tag{22}$$

Taking these equalities as a linear algebraic system we define the values of the abscissa x_1 of the intersection point C of the characteristics Γ_1 and Γ_2, and also of the sought solution $u(x_1,\ell(x_1))$ together with its first order derivative $u_x(x_1,\ell(x_1))$ and $u_y(x_1,\ell(x_1))$.

So far $\rho \in [a,b]$ and $\sigma \in [a,d]$ have been chosen arbitrarily on the segment $[a,d]$ and it has been through them that we have defined the coordinates (x_1,y_1) of the intersection point of the characteristics. Now, if we assume that they run through this segment, we obtain the set of intersection points of the characteristics drawn from all possible pairs of points $(\rho,0)$, $(\sigma,0)$. That is why in the notations of solutions of the algebraic system (19)–(22) we omit the indexes

$$x = X(\rho,\sigma), \tag{23}$$

$$u = U(\rho,\sigma), \tag{24}$$

$$u_x = P(\rho,\sigma), \tag{25}$$

$$u_y = Q(\rho,\sigma), \tag{26}$$

where

$$X(\rho,\sigma) = \Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1}{2\alpha-1}},$$

$$U(\rho,\sigma) = \frac{1}{1-2\alpha}\bigg[\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{1-\alpha}{2\alpha-1}}[\xi]_\sigma - [\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\bigg],$$

$$P(\rho,\sigma) = \Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{-\frac{\alpha}{2\alpha-1}}\Big(\frac{1}{2-4\alpha}[\xi]_\sigma - \frac{1}{2}[\eta]_\rho\Big) - \frac{\alpha}{1-2\alpha}[\xi_1]_\sigma\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha-1}{2\alpha-1}},$$

$$Q(\rho,\sigma) = 2\Big(\frac{[\xi]_\sigma + [\eta]_\rho}{[\xi_1]_\sigma + [\eta_1]_\rho}\Big)^{\frac{\alpha}{2\alpha-1}}\big([\xi]_\sigma + [\eta]_\rho\big)^{-1}.$$

To describe the structure of this set of points we must express the ordinate y as a function of arguments ρ, σ, in the same way as all other were represented by formulas (23)–(26). To construct the function $y = Y(\rho,\sigma)$, we need the explicit representations of X and Q in the form

$$X(\rho,\sigma) = \bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}$$

$$
\begin{aligned}
&- \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] \\
&+ \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right) + \frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha} \\
&- 2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma) + 2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\Bigg\}^{\frac{1}{2\alpha-1}} \\
&\times\Bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} \\
&- \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] \\
&+ \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right) + \frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha} \\
&- 2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma) + 2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\Bigg\}^{\frac{1}{1-2\alpha}}, \qquad (27)
\end{aligned}
$$

$$
\begin{aligned}
U(\rho,\sigma) = 2&\Bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} \\
&- \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] \\
&+ \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right) + \frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha} \\
&- 2\sqrt{g'(h(\sigma))}\,h^{\alpha}(\sigma) + 2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\Bigg\}^{\frac{\alpha}{2\alpha-1}} \\
&\times\Bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma) + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-1} \\
&- \alpha h^{\alpha-1}(\sigma)h'(\sigma)\sigma^{1-\alpha} - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha}\Big] \\
&+ \frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)\sigma - h^{1-\alpha}(\sigma)\sigma^{2\alpha}\right) + \frac{2-2\alpha}{2\alpha-1}\,[\xi]_a\sigma^{2\alpha} \\
&- 2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma) + 2\sqrt{g'(h(\rho))}\,h^{1-\alpha}(\rho)\Bigg\}^{\frac{\alpha}{1-2\alpha}} \\
&\times\Bigg\{\frac{4\sqrt{g'(h(\sigma))}}{2\alpha-1}\Big[(\alpha-1)h^{\alpha}(\sigma)\sigma^{-1} + (1-\alpha)h^{1-\alpha}(\sigma)\sigma^{2\alpha-2} \\
&+ \alpha h^{\alpha-1}(\sigma)h'(\sigma) - (1-\alpha)h^{-\alpha}(\sigma)h'(\sigma)\sigma^{2\alpha-1}\Big]
\end{aligned}
$$

$$+\frac{2}{2\alpha-1}\cdot\frac{g''(h(\sigma))h'(\sigma)}{\sqrt{g'(h(\sigma))}}\left(h^{\alpha}(\sigma)-h^{1-\alpha}(\sigma)\sigma^{2\alpha-1}\right)$$
$$+\frac{2-2\alpha}{2\alpha-1}[\xi]_a\sigma^{2\alpha-1}-2\sqrt{g'(h(\sigma))}\,h^{1-\alpha}(\sigma)$$
$$\left.+2\sqrt{g'(h(\rho))}\,h^{\alpha}(\rho)\right\}^{-1}. \tag{28}$$

By a straightforward check we establish that if $g\in C^2([a,b])$, $h\in C^1([a,d])$, then there exist constants $0<n<N$ for which we have

$$n<|Q|<N. \tag{29}$$

To define the function $m(x)$, the equation of the characteristic Γ_1 is formally written in the form

$$y=m(x)=m[X(\rho,\overline{\sigma})]\equiv M(\rho,\overline{\sigma}),$$

where the little line over the letter means this value is constant. The direction of this characteristic is defined by the root λ_1 or, in other words, by the values of the derivative $u_y=Q(\overline{\sigma},\rho)$. Therefore we have

$$\frac{dm(X(\rho,\overline{\sigma}))}{dX(\rho,\overline{\sigma})}=\frac{dm(X(\rho,\overline{\sigma}))}{X'_\rho(\rho,\overline{\sigma})\,d\rho}=\frac{1}{Q^2(\rho,\overline{\sigma})}$$

or, which is the same,

$$\frac{dM(\rho,\overline{\sigma})}{d\rho}=\frac{X'_\rho(\rho,\overline{\sigma})}{Q^2(\rho,\overline{\sigma})}\,. \tag{30}$$

Hence by integration we obtain

$$M(\rho,\overline{\sigma})=\int\limits_a^{\rho}\frac{X'_t(t,\overline{\sigma})}{Q^2(t,\overline{\sigma})}\,dt+M(a,\overline{\sigma}),\quad \rho\in[a,b],$$

where the value $M(a,\overline{\sigma})$ is yet unknown.

By an analogous reasoning, using the notation $y=\ell(x)=\ell\big[X(\overline{\rho},\sigma)\big]\equiv L(\overline{\rho},\sigma)$ and also taking into account the direction of the characteristic Γ_2 defined by the root λ_2, we obtain

$$\frac{dL(\overline{\rho},\sigma)}{d\sigma}=-\frac{X'_\sigma(\overline{\rho},\sigma)}{Q^2(\overline{\rho},\sigma)}$$

and

$$L(\overline{\rho},\sigma)=-\int\limits_a^{\sigma}\frac{X_z(\overline{\rho},z)}{Q^2(\overline{\rho},z)}\,dz+L(\overline{\rho},a),\quad \sigma[a,d],$$

where $L(\overline{\rho}, a)$ is not known either and has to be defined.

To define the unknown values, note that $L(\overline{\rho}, a)$ is the value of L at the intersection point of the characteristics γ Γ_2. Therefore

$$L(\overline{\rho}, a) = g(\overline{\rho}),$$

and

$$M(a, \overline{\sigma}) = L(a, \overline{\sigma}) = -\int\limits_a^{\overline{\sigma}} \frac{X'_z(a, z)}{Q^2(a, z)}\, dz + g(a),$$

where $g(a) = 0$.

In defining the characteristics of the families of the roots λ_1 and λ_2, the functions $M(\rho, \overline{\sigma})$ $L(\overline{\rho}, \sigma)$ are given by the equalities

$$M(\rho, \overline{\sigma}) = \int\limits_a^{\rho} \frac{X'_t(t, \overline{\sigma})}{Q^2(t, \overline{\sigma})}\, dt - \int\limits_a^{\overline{\sigma}} \frac{X'_z(a, z)}{Q^2(a, z)}\, dz \tag{31}$$

with an argument $\rho \in [a, b]$ and a parameter $\overline{\sigma} \in [a, d]$, and

$$L(\overline{\rho}, \sigma) = -\int\limits_a^{\sigma} \frac{X'_z(\overline{\rho}, z)}{Q^2(\overline{\rho}, z)}\, dz + g(\overline{\rho}), \tag{32}$$

with a variable $\sigma \in [a, d]$ and a parameter $\overline{\rho} \in [a, b]$.

Thus the integral of problem (1), (7) is given by formulas (23), (24) and

$$y = Y(\rho, \sigma), \tag{33}$$

where

$$Y(\rho, \sigma) = -\int\limits_a^{\sigma} \frac{X'_z(\rho, z)}{Q^2(\rho, z)}\, dz + g(h(\rho)),$$

and the variables $\sigma \in [a, d]$, $\rho \in [a, b]$.

The propagation domain D of the solution of problem (1), (7) is completely defined by relations (23), (33), where expressions of x, y depend on ρ, σ. The values of these functions are treated as the current coordinates describing the domain D.

The propagation domain of the solution of the considered problem is bounded by four characteristics. The first of them which is an arch of the curve γ is given by the condition of the problem. The other characteristics are represented parametrically. In our representations we take as parameters the values ρ, σ of the abscissa of the intersection points through which these characteristics pass:

$$\Gamma_3 : \; x = X(b, \sigma), \;\; y = L(b, \sigma),$$

$$\Gamma_4: \; x = X(\rho, d), \quad y = M(\rho, d),$$
$$\Gamma_5: \; x = X(a, \sigma), \quad y = L(a, \sigma),$$

where the functions X, M, L are given by (27), (31), (32).

Such is the structure of the propagation domain of the solution of the problem when the values of the derivative u_y on the arc γ in formula (8) are defined by the positive root. The domain has the same kind of structure when the root in (8) is negative. The latter case is investigated by analogy with the preceding case.

References

1. P. Oliver, *Applications of Lie groups to differential equations*, Mir, Moscow, 1989.

2. F. Tricomi, *Lectures on partial differential equations*, Moscow, 1975.

3. S. Kumei, G. W. Bluman, *When nonlinear differential equations are equivalent to linear differential equations*, SIAM J. Appl. Math. **42** (1982), No. 5, 1157–1173.

4. L. Bers, *Mathematical aspects of subsonic and transonic gas dynamics*, Surveys in Applied Mathematics, Vol. 3 John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London, 1958.

5. J. Gvazava, *On one nonlinear version of characteristic problem with a free support of data*, Proc. A. Razmadze Math. Inst. **140** (2006), 91–107.

6. J. Gvazava, *The mean value property for nonstrictly hyperbolic second order quasilinear equations and the nonlocal problems*, Proc. A. Razmadze Math. Inst. **135** (2004), 79–96.

7. I. M. Gelfund, *Some problems in the theory of quasilinear equations* (Russian), Uspekhi Mat. Nauk **14** (1959), No. 2(86), 87–158.

8. E. Goursat, Lecons sur l'intégration des équations aux dérivées partielles du second ordre à deux variables indépendantes. Tome II: La méthode de Laplace. Les systèmes en involutions. La methode de M. Darboux. Les équations de la première classe. Transformations des équations du second ordre. Généralisations diverses. 2 Notes. *A. Hermann, Paris*, 1898.

9. R. G. Bitsadze, *General representation of the solutions of a quasilinear equation of a problem of nonlinear oscillations* (Russian), Soobshch. Akad. Nauk Gruzin. SSR **128** (1987), No. 3, 493–496 (1988).

10. G. Darboux, Leçons sur la théorie générale des surfaces et les applications geometriques du calcul infinitésimal. *Gauthier-Villars et Fils, Paris* (1894 et 96) (1894).

11. E. Goursat, *Course of mathematical analysis. Vol. 3, part I.* Moscow, 1933 (translated into Russian).