PROPERTIES OF PARTIAL SUMS OF FOURIER SERIES OF FUNCTIONS WITH A FINITE VARIATION

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Abstract

In the paper the necessary and sufficient conditions are found, which should be satisfied by orthonormal system so that the partial sums of Fourier series of functions with finite variation were uniformly bounded.

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1 Statement of the problem

S. Banach [2] has proved that for any function $f(x) \in L_2(I)$ $(I = [0,1]$, $f(x) \not\equiv 0)$ there exists an orthonormal on I system (ONS) $(\varphi_n(x))$, such that $\overline{\lim\limits_{n\to\infty}} |S_n(f,x)| = +\infty$ almost everywhere on I, where $S_n(f,x)$ is a partial sum of the Fourier series of function $f(x)$ with respect to the system $(\varphi_n(x)) = \Phi$.

Denote as usual by $V(I)$ the space of functions with finite variation with the norm

$$\|f\|_V = \int_0^1 |f'(x)|\,dx + \|f\|_C.$$

Let $f \in L(I)$,

$$\sum_{n=1}^{\infty} \widehat{\varphi}_n(f)\varphi_n(x) \tag{1}$$

be its Fourier series with respect to ONS $(\varphi_n(x))$ and

$$\widehat{\varphi}_n(f) = \int_0^1 f(x)\varphi_n(x)\,dx$$

be the Fourier coefficients;

$$S_N(f,x) = \sum_{n=1}^{N} \widehat{\varphi}_n(f)\varphi_n(x)$$

is the partial sum of series (1).

Let

$$D_N(t,x)=\sum_{n=1}^{N}\varphi_n(t)\varphi_n(x)$$

be the Dirichlet kernel. Assume

$$D_N(x)=\max_{1\le i\le N}\left|\int_0^{i/N}D_N(t,x)\,dt\right|. \tag{2}$$

Definition 1. We say that ONS Φ has the property A, if there exists a positive constant $C(\Phi)$ depending only on the system Φ such that

$$\sup_{x\in I}N^{-1}\sum_{n=1}^{N}\varphi_n^2(x)\le C(\Phi). \tag{3}$$

Lemma 1. Let $f(x)$, $g(x)$ and $f(x)g(x)\in L(I)$ and $f(x)$ take the finite values at every point of the segment I. Then we have the equality

$$\int_0^1 f(t)g(t)\,dt=\sum_{k=1}^{N-1}\left(f\left(\frac{k}{N}\right)-f\left(\frac{k+1}{N}\right)\right)\int_0^{k/N}g(t)\,dt$$
$$+\sum_{k=1}^{N}\int_{\frac{k-1}{N}}^{\frac{k}{N}}\left(f(t)-f\left(\frac{k}{N}\right)\right)g(t)\,dt+f(1)\int_0^1 g(t)\,dt. \tag{4}$$

Proof. Using the Abel transformation we have

$$0=\sum_{k=1}^{N-1}\left(f\left(\frac{k}{N}\right)-f\left(\frac{k+1}{N}\right)\right)\int_0^{k/N}g(t)\,dt$$
$$-\sum_{k=1}^{N}f\left(\frac{k}{N}\right)\int_{\frac{k-1}{N}}^{\frac{k}{N}}g(t)\,dt+f(1)\int_0^1 g(t)\,dt.$$

Since

$$\int_0^1 f(t)g(t)\,dt=\sum_{k=1}^{N}\int_{\frac{k-1}{N}}^{\frac{k}{N}}f(t)g(t)\,dt,$$

summing these two equalities we get (4). □

Definition 2. The partial sums of the Fourier series of the function $f(x)$ with respect to the system Φ are said to be uniformly bounded, if there exists a positive constant $C(f,\Phi)$ depending only on the function f and the system Φ such that

$$|S_N(f,x)|<C(f,\Phi) \tag{5}$$

for any $x\in I$ and natural N.

2 Main results

Theorem 1. Let ONS Φ have the property A. Then in order that for any function $f(x) \in V(I)$ the partial sums of the Fourier series of $f(x)$ with respect to Φ were uniformly bounded, it is necessary and sufficient that there exists a positive constant $M(\Phi) > 0$ depending only on the system Φ such that

$$\sup_{N\geq 1} \sup_{x\in I} D_N(x) < M(\Phi). \tag{6}$$

Proof. Since for any $x \in I$

$$S_N(f,x) = \int_0^1 f(t) D_N(t,x)\, dt,$$

assuming in equality (4) $g(t) = D_N(t,x)$ we get

$$\begin{aligned} S_N(f,x) &= \int_0^1 f(t) D_N(t,x)\, dt \\ &= \sum_{k=1}^{N-1} \left(f\left(\frac{k}{N}\right) - f\left(\frac{k+1}{N}\right)\right) \int_0^{k/N} D_N(t,x)\, dt \\ &+ \sum_{k=1}^{N} \int_{\frac{k-1}{N}}^{\frac{k}{N}} \left(f(t) - f\left(\frac{k}{N}\right)\right) D_N(t,x)\, dt \\ &+ f(1) \int_0^1 D_N(t,x)\, dt. \end{aligned} \tag{7}$$

Hence, taking into account that $f(x) \in V(I)$, (2) and (6), we have

$$\begin{aligned} &\left| \sum_{k=1}^{N-1} \left(f\left(\frac{k}{N}\right) - f\left(\frac{k+1}{N}\right)\right) \int_0^{k/N} D_N(t,x)\, dt \right| \\ &\leq \sum_{k=1}^{N-1} \left| f\left(\frac{k}{N}\right) - f\left(\frac{k+1}{N}\right)\right| \left| \int_0^{k/N} D_N(t,x)\, dt \right| \\ &\leq \|f\|_V \max_{1\leq k<N} \left| \int_0^{k/N} D_N(t,x)\, dt \right| \leq \|f\|_V \cdot M(\Phi). \end{aligned} \tag{8}$$

Further, in view of the conditions of Theorem 1 and $f(x) \in V(I)$ we have (see (3))

$$\begin{aligned}
&\left|\sum_{k=1}^{N}\int_{\frac{k-1}{N}}^{\frac{k}{N}}\left(f(t)-f\left(\frac{k}{N}\right)\right)D_N(t,x)\,dt\right| \qquad (9)\\
&\leq \sum_{k=1}^{N}\sup_{x\in[\frac{k-1}{N},\frac{k}{N}]}\left|f(x)-f\left(\frac{k}{N}\right)\right|\int_{\frac{k-1}{N}}^{\frac{k}{N}}|D_N(t,x)|\,dt\\
&\leq \|f\|_V\cdot\frac{1}{\sqrt{N}}\left(\int_0^1 D_N^2(t,x)\,dt\right)^{\frac{1}{2}}=\frac{\|f\|_V}{\sqrt{N}}\left(\sum_{k=1}^{N}\varphi_k^2(x)\right)^{\frac{1}{2}}\\
&\leq \|f\|_V\cdot\sqrt{C(\Phi)}\,.
\end{aligned}$$

Since (see (2))

$$|f(1)|\left|\int_0^1 D_N(t,x)\,dt\right|\leq\|f\|_C\cdot D_N(x)\leq\|f\|_C\cdot M(\Phi),$$

from the latter inequality on account of (9), (8) and (7) we get

$$|S_N(f,x)|\leq\left(M(\Phi)+\sqrt{C(\Phi)}\right)\|f\|_V+\|f\|_C M(\Phi).$$

The sufficiency of Theorem 1 is proved.

Necessity. It is given that for any function $f(x)\in V(I)$ inequality (5) is fulfilled and the validity of inequality (6) should be proved. Assume the opposite, then for some sequences of natural numbers (N_m) and points $x_m\in I$ we have

$$\lim_{m\to\infty}D_{N_m}(x_m)=+\infty. \qquad (10)$$

Assume that

$$D_{N_m}(x_m)=\max_{1\leq k\leq N_m}\left|\int_0^{k/N}D_{N_m}(t,x_m)\,dt\right|=\left|\int_0^{k_m/N_m}D_{N_m}(t,x_m)\,dt\right|.$$

Define the sequence of functions $(f_m(t))$ in the following way

$$f_m(t)=\begin{cases}0 \quad \text{when } x\in\left[0,\frac{k_m}{N_m}\right],\\ 1 \quad \text{when } x\in\left[\frac{k_m+1}{N_m},1\right],\\ \text{linear and continuous on } \left[\frac{k_m}{N_m},\frac{k_m+1}{N_m}\right].\end{cases}$$

Assuming in equality (7) that $f(t)=f_m(t)$, $N=N_m$ and $D_N(t,x)=$

$D_{N_m}(t, x_m)$, we get

$$\begin{aligned}
&\int_0^1 f_m(t) D_{N_m}(t, x_m)\, dt \qquad (11)\\
&= \sum_{k=1}^{N_m-1} \left(f_m\left(\frac{k}{N_m}\right) - f_m\left(\frac{k+1}{N_m}\right)\right) \int_0^{k/N_m} D_{N_m}(t, x_m)\, dt \\
&\quad + \sum_{k=1}^{N_m} \int_{\frac{k-1}{N_m}}^{\frac{k}{N_m}} \left(f_m(t) - f_m\left(\frac{k}{N_m}\right)\right) D_{N_m}(t, x_m)\, dt \\
&\quad + f_m(1) \int_0^1 D_{N_m}(t, x_m)\, dt.
\end{aligned}$$

From the definition of function $f_m(t)$ for the first summand of (11) we have

$$\begin{aligned}
&\left| \sum_{k=1}^{N_m-1} \left(f_m\left(\frac{k}{N_m}\right) - f_m\left(\frac{k+1}{N_m}\right)\right) \int_0^{k/N_m} D_{N_m}(t, x_m)\, dt \right| \\
&\qquad = \left| \int_0^{k_m/N_m} D_{N_m}(t, x_m)\, dt \right| = D_{N_m}(x_m). \qquad (12)
\end{aligned}$$

Since $\left| f_m(t) - f_m\left(\frac{k}{N_m}\right)\right| = 0$ when $t \in \left[\frac{k-1}{N_m}, \frac{k}{N_m}\right]$ and $k \neq k_m + 1$, for the second summand of equality (11) we get

$$\begin{aligned}
&\left| \sum_{k=1}^{N_m} \int_{\frac{k-1}{N_m}}^{\frac{k}{N_m}} \left(f_m(t) - f_m\left(\frac{k}{N_m}\right)\right) D_{N_m}(t, x_m)\, dt \right| \\
&\qquad \leq \int_{\frac{k_m}{N_m}}^{\frac{k_m+1}{N_m}} |D_{N_m}(t, x_m)|\, dt \leq \frac{1}{\sqrt{N_m}} \left(\sum_{k=1}^{N_m} \varphi_k^2(x_m) \right)^{\frac{1}{2}} \leq \sqrt{C(\Phi)}\,. \qquad (13)
\end{aligned}$$

Sine $1 \in V(I)$ and inequality (5) is fulfilled for any $f(x) \in V(I)$, we have

$$\left| \int_0^1 D_{N_m}(t, x_m)\, dt \right| \leq C(1, \Phi),$$

where $C(1, \Phi)$ is a positive constant depending only on the system Φ.

Consequently, using (11), (12) and (13) we get

$$\left| \int_0^1 f_m(t) D_{N_m}(t, x_m)\, dt \right| \geq D_{N_m}(x_m) - \sqrt{C(\Phi)} - C(1, \Phi).$$

Hence in view of (10)

$$\lim_{m\to\infty} \left| \int_0^1 f_m(t) D_{N_m}(t, x_m)\, dt \right| = +\infty. \qquad (14)$$

Now since

$$\|f_m\|_V = \int_0^1 |f'(t)|\,dt + \|f_m\|_C = 2,$$

from (14) applying the Banach-Steinhaus theorem there exists the function $f_0(t) \in V(I)$ such that

$$\overline{\lim_{m\to\infty}} \left| \int_0^1 f_0(t) D_{N_m}(t, x_m)\,dt \right| = +\infty.$$

Hence

$$\overline{\lim_{m\to\infty}} |S_{N_m}(f_0, x_m)| = +\infty.$$

This contradicts inequality (5). Theorem 1 is completely proved. □

The similar problem may be posed for the summability of (C, α) in the Cèsaro sense for the functions $f(x) \in V(I)$.

Assume, in fact, that (see [1 p. 77])

$$K_N^\alpha(t, x) = \frac{1}{A_N^\alpha} \sum_{k=0}^{N} A_{N-k}^\alpha \varphi_k(t)\varphi_k(x),$$

where $A_N^\alpha = \binom{N+\alpha}{N}$ and $\alpha > 0$. Hence

$$\sigma_N^\alpha = \frac{1}{A_N^\alpha} \int_0^1 f(t) K_N^\alpha(t, x)\,dt.$$

Introduce the notation

$$H_N^\alpha(x) = \max_{1\le i\le N} \left| \int_0^{i/N} K_N^\alpha(t, x)\,dt \right|.$$

Theorem 2. Let ONS Φ have the property A. Then the condition

$$|\sigma_N^\alpha(f, x)| < M < +\infty$$

for any function $f(x) \in V(I)$ is satisfied if and only if when

$$\sup_{x\in I} H_N^\alpha(x) < M_1(\Phi) < +\infty,$$

where $M > 0$ does not depend on N and x, but $M_1(\Phi) > 0$ depends only on Φ; $\alpha > 0$ is any fixed number.

Proof. Theorem 2 is proved in a similar way as Theorem 1. Indeed, in equality (6) let us substitute $g(t) = K_n^\alpha(x, t)$ and repeating the above reasoning (see (6), (7) and (8)) we get

$$|\sigma_N^\alpha(f, x)| \le M_1(\Phi)\|f\|_V + M_1(\Phi)\|f\|_C + \|f\|_V \sqrt{M_1(\Phi)}\,.$$

This proves the sufficiency of Theorem 2.

For proving the necessity assume that for some sequences of natural numbers N_m and points $x_m \in I$

$$\lim_{m\to\infty} H_{N_m}(x_m) = +\infty.$$

For this case consider the sequence of functions $(f_m(t))$ which was constructed in the proof of Theorem 2. By repeating the similar reasoning, it is easily obtained that

$$\overline{\lim_{m\to\infty}} |\sigma_{N_m}^{\alpha}(f_0, x_m)| = +\infty,$$

where $f_0(x) \in V(I)$ is a certain function. Thus Theorem 2 is completely proved. $\square$

References

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