STATIONARY OSCILLATION BOUNDARY VALUE PROBLEMS OF THE THEORY OF TWO-TEMPERATURE ELASTIC MIXTURES FOR THE INFINITE SPACE WITH SPHERICAL CAVITY

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Abstract

We consider the stationary oscillation case of the theory of two-temperature elastic mixtures when partial displacements of the elastic components of which the mixture consists are equal to each other. The previously obtained general solution representation makes it possible to represent the displacement vector and the stress vector by Fourier-Laplace series with respect to a complete system of well-defined orthonormal vectors. Solutions are obtained in the form of absolutely and uniformly convergent series. A nev version of the proof of the uniqueness theorem of the considered the Dirichlet and Neumann problems is given.
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1 Introduction

A mathematical model of the linear theory of two-temperature elastic mixtures for composite materials of granular, fiber-like and layered structure was constructed by Khoroshun and Soltanov [11] in 1984.

Usually, the study of processes occurring in bodies reduces for the corresponding mathematical model described by a system of partial differential equations to the investigation of boundary value, mixed type and boundary-contact problems.

From the theoretical viewpoint, the well-posedness (the solution existence, smoothness, uniqueness and stability) of problems and the creation of adequate calculation algorithms for applied purposes are also the subject of great interest.

For diffusion and shear stress models of the thermoelastic theory of two-temperature elastic mixtures, the questions pertaining to stability and well-posedness, establishment of the asymptotic behavior of a problem solution, theorems of solution existence and uniqueness were studied by many scientists, including Alves, Munoz Rivera, and Quitamila [1], Basheleishvili

[2], Basheleishvili and Zazashvili [3], Burchuladze and Svanadze [4], Gales [5], Giorgashvili, Karseladze and Sadunishvili [9], Giorgashvili and Skhvutaridze [7], [8], Giorgashvili, Karseladze and Sadunisvili [10],Giorgashvili [6] Iesan [11], Nappa [13], Natroshvili, Jaghmaidze and Svanadze [14], Svanadze [17], Quintanilla [16], Pompei [15] and others.

2 Basic differential equations and Auxiliary Theorems

If it is assumed that the partial components of the mixture are equal to each other, then a homogeneous system of stationary oscillation differential equations of the theory of two-temperature elastic mixtures has the form [11]

$$\begin{aligned} \mu\Delta u + (\lambda+\mu)\operatorname{grad}\operatorname{div} u - \operatorname{grad}(\eta_1\vartheta_1 + \eta_2\vartheta_2) + \rho\sigma^2 u = 0,& \\ i\sigma\eta_1 \operatorname{div} u + (\varkappa_1\Delta + \alpha_1)\vartheta_1 + (\varkappa_2\Delta + \alpha)\vartheta_2 = 0,& \\ i\sigma\eta_2 \operatorname{div} u + (\varkappa_2\Delta + \alpha)\vartheta_1 + (\varkappa_3\Delta + \alpha_2)\vartheta_2 = 0,& \end{aligned} \tag{2.1}$$

where Δ is the three-dimensional Laplace operator, $u = (u_1, u_2, u_3)^\top$ is the displacement vector, ϑ_1, ϑ_2 are the temperatures of the variable components of the mixture, $\alpha_1 = -\alpha + i\sigma\varkappa'$, $\alpha_2 = -\alpha + i\sigma\varkappa''$ is the oscillation frequency, $\rho > 0$ is the density sum of the components; λ, μ, η_1, η_2, $\varkappa'$, $\varkappa''$, α, $\varkappa_j$, $j = 1,2,3$, are the positive constants characterizing the mechanical and thermal properties of the bodies contained in the elastic mixture which satisfy the conditions

$$\mu > 0, \ \ 3\lambda + 2\mu > 0, \ \ \varkappa_1\varkappa_3 - \varkappa_2^2 > 0.$$

$\top$ is the transposition symbol.

The following statement is true

Theorem 2.1. For the vector $U = (u, \vartheta_1, \vartheta_2)^\top$ to be a solution of system (2.1) in a domain $\Omega \subset R^3$ it is necessary and sufficient that it be represented in the form

$$\begin{aligned} u(x) &= \sum_{j=1}^{3} \operatorname{grad}\Phi_j(x) + \operatorname{rot}\operatorname{rot}(x\Phi_4(x)) + \operatorname{rot}(x\Phi_5(x)), \\ \vartheta_l(x) &= -\sum_{j=1}^{3} k_j^2 \beta_l^{(j)} \Phi_j(x), \ \ l = 1,2, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned}\beta_1^{(j)} &= \frac{i\sigma}{c_j}\Big[(\eta_1\varkappa_3 - \eta_2\varkappa_3)k_j^2 + \alpha\eta_2 - \alpha_2\eta_1\Big], \ \ j = 1,2,3,\\ \beta_2^{(j)} &= \frac{i\sigma}{c_j}\Big[(\eta_2\varkappa_1 - \eta_1\varkappa_2)k_j^2 + \alpha\eta_1 - \alpha_1\eta_2\Big], \ \ j = 1,2,3,\end{aligned} \tag{2.3}$$

$$c_j = (\varkappa_1\varkappa_3 - \varkappa_2^2)k_j^4 - (\alpha_1\varkappa_3 + \alpha_2\varkappa_1 - 2\alpha\varkappa_2)k_j^2 + \alpha_1\alpha_2 - \alpha^2 \neq 0, \ \ j = 1,2,3,$$
$$(\Delta + k_j^2)\Phi_j(x) = 0, \ \ j = 1,2,3, \quad (\Delta + k_4^2)\Phi_j(x) = 0, \ \ j = 4,5,$$

$k_4^2 = \frac{\rho\sigma^2}{\mu}$, k_j^2, $j = 1,2,3$ are the roots of the following equations

$$z^3 + a_1 z^2 + a_2 z + a_3 = 0, \tag{2.4}$$

$$\begin{aligned} a_1 =& -\frac{1}{a}\Big[\rho\sigma^2(\varkappa_1\varkappa_3 - \varkappa_2^2) + (\lambda + 2\mu)(\alpha_1\varkappa_3 + \alpha_2\varkappa_1 - 2\alpha\varkappa_2) + \\ & + i\sigma(\eta_1^2\varkappa_3 + \eta_2^2\varkappa_1 - 2\eta_1\eta_2\varkappa_2)\Big], \\ a_2 =& \frac{1}{a}\Big[(\lambda + 2\mu)(\alpha_1\alpha_2 - \alpha^2) + \rho\sigma^2(\alpha_1\varkappa_3 + \alpha_2\varkappa_1 - 2\alpha\varkappa_2) + \\ & + i\sigma(\eta_1^2\alpha_2 + \eta_2^2\alpha_1 - 2\eta_1\eta_2\alpha)\Big], \\ a_3 =& -\frac{1}{a}\,\rho\sigma^2(\alpha_1\alpha_2 - \alpha^2), \quad a = (\lambda + 2\mu)(\varkappa_1\varkappa_3 - \varkappa_2^2) > 0. \end{aligned}$$

For simplicity we assume that $\sigma > 0$, $k_j \neq k_p$ for $j \neq p = 1,2,3$, $\Im k_j \geq 0$, and $\Im k_j = 0$, then $k_j > 0$.

Theorem 2.2. A regular solution of equation (2.1) admits a representation of the form

$$u(x) = \sum_{j=1}^{4} u^{(j)}(x), \quad \vartheta_l(x) = \sum_{j=1}^{3} \beta_l^{(j)} \operatorname{div} u^{(j)}(x), \ \ l = 1,2,$$

where $\beta_l^{(j)}$, $l = 1,2$, $j = 1,2,3$ are given by (2.3) , and

$$\begin{aligned}&(\Delta + k_j^2)u^{(j)}(x) = 0, \ \ \operatorname{rot} u^{(j)}(x) = 0, \ \ j = 1,2,3,\\ &(\Delta + k_4^2)u^{(4)}(x) = 0, \ \ \operatorname{div} u^{(4)}(x) = 0.\end{aligned}$$

Let $\Omega^+ = B(0,R)$ be a ball in R^3 centered at the origin and radius R, and let $\sum_R = \partial\Omega$ be its boundary. Further, let $\Omega^- = R^3 \setminus \overline{\Omega^+}$.

Definition. A solution $U = (u, \vartheta_1, \vartheta_2)^\top$ of system (2.1) will called regular in the domain Ω^-, if $U \in C^2(\Omega^-) \cap C^{(1)}(\overline{\Omega^-})$, and will be satisfy the thermoelastic radiation condition

$$\begin{aligned} u^{(j)}(x) &= o(|x|^{-1}), \quad \frac{\partial u^{(j)}(x)}{\partial x_k} = O(|x|^{-2}), \quad k, j = 1, 2, 3, \\ u^{(4)}(x) &= O(|x|^{-1}), \quad \frac{\partial u^{(4)}(x)}{\partial |x|} - ik_4 u^{(4)}(x) = o(|x|^{-1}), \\ \vartheta_l(x) &= o(|x|^{-1}), \quad \frac{\partial \vartheta_l(x)}{\partial x_k} = O(|x|^{-2}), \quad l = 1, 2, \quad k = 1, 2, 3. \end{aligned}$$

Theorem 2.3. The vector $U = (u, \vartheta_1, \vartheta_2)^\top$ represented by equality (2.2) will be uniquely defined by the functions $\Phi_j(x)$, $j = 1, 2, ..., 5$ in the domain Ω^- if the folloving condition

$$\int\limits_{\Sigma_r} \Phi_j(x) d\Sigma_r = 0, \quad j = 4, 5, \quad r = |x| > R, \tag{2.5}$$

is fulfilled, i. e. to the zero value of the vector $U = (u, \vartheta_1, \vartheta_2)^\top$ there corresponds the zero value of the vector $(\Phi_1,\ \Phi_2, ..., \Phi_5)^\top$ and vice versa.

3 Statement of the Problems. The Uniqueness Theorem

Assume that the domain Ω^- is filled with an elastic two-component mixture.

Problem. Find, in the domain Ω^-, such a regular vector $U = (u, \vartheta_1, \vartheta_2)^\top$ that satisfies in this domain the system of differential equations (2.1) and, on the boundary $\partial\Omega$, satisfies one of the following boundary conditions:

$[\mathbf{I}^\sigma]^-$ (the Dirichlet problem)

$$\{u(z)\}^- = f(z), \quad \{\vartheta_1(z)\}^- = f_4(z), \quad \{\vartheta_2(z)\}^- = f_5(z), \quad z \in \partial\Omega; \tag{3.1}$$

$[\mathbf{II}^\sigma]^-$ (the Neumann problem)

$$\{T(\partial, n)U(z)\}^- = f(z), \quad \left\{\frac{\partial \vartheta_1(z)}{\partial n(z)}\right\}^- = f_4(z), \tag{3.2}$$
$$\left\{\frac{\partial \vartheta_2(z)}{\partial n(z)}\right\}^- = f_5(z), \quad z \in \partial\Omega,$$

where $f=(f_1,f_2,f_3)^\top$, f_j, $j=1,2,...,5$ are the functions given on the boundary, $n(z)$ is the external mormal unit passing at a point $z\in\partial\Omega$ with respect to the domain Ω^+, and

$$T(\partial,n)U=2\mu\frac{\partial u}{\partial n}+\lambda n\operatorname{div}u+\mu[n\times\operatorname{rot}u]-n(\eta_1\vartheta_1+\eta_2\vartheta_2).$$

A solution of system (2.1) will be sought for in form (2.2), where the function $\Phi_j(x)$, $j=1,2,...,5$, are represented as

$$\begin{aligned}\Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(k_jr)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\quad j=1,2,3,\\ \Phi_j(x)&=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}h_k(k_4r)Y_k^{(m)}(\vartheta,\varphi)A_{mk}^{(j)},\quad j=4,5,\end{aligned}\tag{3.3}$$

where $A_{mk}^{(j)}$, $j=1,2,...,5$ are unknown constants,

$$h_k(k_jr)=\sqrt{\frac{R}{r}}\,\frac{H_{k+1\backslash 2}^{(1)}(k_jr)}{H_{k+1\backslash 2}^{(1)}(k_jR)},$$

$$Y_k^{(m)}(\vartheta,\varphi)=\sqrt{\frac{2k+1}{4\pi}\cdot\frac{(k-m)!}{(k+m)!}}\,P_k^{(m)}(cos\vartheta)e^{im\varphi},$$

$H_{k+1\backslash 2}^{(1)}(x)$ are the first kind Hankel functions, $P_k^{(m)}(\cos\vartheta)$ is the associated Legendre polynomial of k-th kind and m-th order. (r,ϑ,φ) are the spherical coordinates of the point $x=(x_1,x_2,x_3)$.

The substitution of the values of the functions $\Phi_j(x)$, $j=4,5$ from (3.3) into (2.5) yields $A_{00}^{(j)}=0$, $j=4,5$.

Let us substitute the expressions (3.3) into (2.2) and apply the following identities [6]

$$\begin{aligned}\operatorname{grad}[a(r)Y_k^{(m)}(\vartheta,\varphi)]&=\frac{da(r)}{dr}\,X_{mk}(\vartheta,\varphi)+\frac{\sqrt{k(k+1)}}{r}a(r)Y_{mk}(\vartheta,\varphi),\\ \operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)]&=\sqrt{k(k+1)}a(r)\,Z_{mk}(\vartheta,\varphi)\\ \operatorname{rot}\operatorname{rot}[xa(r)Y_k^{(m)}(\vartheta,\varphi)]&=\frac{k(k+1)}{r}a(r)\,X_{mk}(\vartheta,\varphi)+\\ &\sqrt{k(k+1)}\left(\frac{d}{dr}+\frac{1}{r}\right)a(r)\,Y_{mk}(\vartheta,\varphi),\end{aligned}$$

here $a(r)$ is an arbitrary differentiable scalar function of r, we obtain

$$u(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\Big\{u_{mk}(r)X_{mk}(\vartheta,\varphi)+\sqrt{k(k+1)}\times$$
$$[v_{mk}(r)Y_{mk}(\vartheta,\varphi)+w_{mk}(r)Z_{mk}(\vartheta,\varphi)]\Big\}, \tag{3.4}$$
$$\vartheta_l(x)=\sum_{k=0}^{\infty}\sum_{m=-k}^{k}\omega_{mk}^{(l)}(r)Y_k^{(m)}(\vartheta,\varphi), \quad l=1,2,$$

where

$$u_{mk}(r)=\sum_{j=1}^{3}\frac{d}{dr}h_k(k_jr)A_{mk}^{(j)}+\frac{k(k+1)}{r}h_k(k_4r)A_{mk}^{(4)}, \quad k\geq 0,$$
$$v_{mk}(r)=\sum_{j=1}^{3}\frac{1}{r}\,h_k(k_jr)A_{mk}^{(j)}+\left(\frac{d}{dr}+\frac{1}{r}\right)h_k(k_4r)A_{mk}^{(4)}, \quad k\geq 1,$$
$$w_{mk}(r)=h_k(k_4r)A_{mk}^{(5)}, \quad k\geq 1,$$
$$\omega_{mk}^{(l)}(r)=-\sum_{j=1}^{3}k_j^2\,\beta_l^{(j)}h_k(k_jr)A_{mk}^{(j)}, \quad l=1,2, \quad k\geq 0,$$

$X_{mk}(\vartheta,\varphi)$, $Y_{mk}(\vartheta,\varphi)$, $Z_{mk}(\vartheta,\varphi)$ are orthonormal vectors system in the space $L_2(\Sigma_1)$ [6], [19]

$$X_{mk}(\vartheta,\varphi)=e_rY_k^{(m)}(\vartheta,\varphi), \quad k\geq 0,$$
$$Y_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\left(e_\vartheta\frac{\partial}{\partial\vartheta}+\frac{e_\varphi}{sin\vartheta}\,\frac{\partial}{\partial\varphi}\right)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 1, \tag{3.5}$$
$$Z_{mk}(\vartheta,\varphi)=\frac{1}{\sqrt{k(k+1)}}\left(\frac{e_\vartheta}{sin\vartheta}\frac{\partial}{\partial\varphi}-e_\varphi\,\frac{\partial}{\partial\vartheta}\right)Y_k^{(m)}(\vartheta,\varphi), \quad k\geq 1,$$

where $|m|\leq k$, e_r, e_ϑ, e_φ are unit orthogonal vectors in R^3,

$$e_r=(\cos\varphi\sin\vartheta,\ \sin\varphi\sin\vartheta,\ \cos\vartheta)^\top,$$
$$e_\vartheta=(\cos\varphi\cos\vartheta,\ \sin\varphi\cos\vartheta,\ -\sin\vartheta)^\top,$$
$$e_\varphi=(-\sin\varphi,\ \cos\varphi,\ 0)^\top.$$

Let us substitute the values of the vector $u(x)$, and the function $\vartheta_l(x)$, $l=1,2$ from (3.4) into value of a tress vectors and make use of the following equalities

$$
\begin{aligned}
& e_r \times X_{mk}(\vartheta,\varphi) = 0, \;\; e_r \times Y_{mk}(\vartheta,\varphi) = -Z_{mk}(\vartheta,\varphi), \\
& \qquad e_r \times Z_{mk}(\vartheta,\varphi) = Y_{mk}(\vartheta,\varphi) \\
& \operatorname{div}[a(r)X_{mk}(\vartheta,\varphi)] = \Big(\frac{d}{dr} + \frac{2}{r}\Big) a(r) Y_k^{(m)}(\vartheta,\varphi), \\
& \operatorname{div}[a(r)Y_{mk}(\vartheta,\varphi)] = -\frac{\sqrt{k(k+1)}}{r} a(r) Y_k^{(m)}(\vartheta,\varphi), \\
& \operatorname{div}[a(r)Z_{mk}(\vartheta,\varphi)] = 0, \\
& \operatorname{rot}\big[a(r)X_{mk}(\vartheta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}\, a(r) Z_{mk}(\vartheta,\varphi), \\
& \operatorname{rot}\big[a(r)Y_{mk}(\vartheta,\varphi)\big] = -\Big(\frac{d}{dr} + \frac{1}{r}\Big) a(r) Z_{mk}(\vartheta,\varphi), \\
& \operatorname{rot}\big[a(r)Z_{mk}(\vartheta,\varphi)\big] = \frac{\sqrt{k(k+1)}}{r}\, a(r) X_{mk}(\vartheta,\varphi) + \\
& \qquad + \Big(\frac{d}{dr} + \frac{1}{r}\Big) a(r) Y_{mk}(\vartheta,\varphi),
\end{aligned} \tag{3.6}
$$

we obtain

$$
\begin{aligned}
T(\partial,n)U(x) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ & a_{mk}(r) X_{mk}(\vartheta,\varphi) + \\
& + \sqrt{k(k+1)} \Big[b_{mk}(r) Y_{mk}(\vartheta,\varphi) + c_{mk}(r) Z_{mk}(\vartheta,\varphi) \Big] \Big\},
\end{aligned} \tag{3.7}
$$

where

$$
\begin{aligned}
a_{mk}(r) &= \sum_{j=1}^{3} \Big[2\mu \frac{d^2}{dr^2} + k_j^2 \big(\eta_1 \beta_1^{(j)} + \eta_2 \beta_2^{(j)} - \lambda\big) \Big] h_k(k_j r) A_{mk}^{(j)} + \\
& \quad + 2\mu \frac{k(k+1)}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) h_k(k_4 r) A_{mk}^{(4)}, \\
b_{mk}(r) &= 2\mu \sum_{j=1}^{3} \frac{1}{r} \Big(\frac{d}{dr} - \frac{1}{r}\Big) h_k(k_j r) A_{mk}^{(j)} + \\
& \quad + \mu \Big(\frac{d^2}{dr^2} + \frac{(k-1)(k+2)}{r^2} \Big) h_k(k_4 r) A_{mk}^{(4)}, \\
c_{mk}(r) &= \mu \Big(\frac{d}{dr} - \frac{1}{r} \Big) h_k(k_4 r) A_{mk}^{(5)}.
\end{aligned}
$$

Note that in formulas (3.4) and (3.7)and in the summand of analogous series below which contains the vectors $Y_{mk}(\vartheta,\varphi)$ and $Z_{mk}(\vartheta,\varphi)$, the summation index varies from 1 to $+\infty$.

From the second equality of (3.4) we obtain

$$\frac{\partial \vartheta_l(x)}{\partial n(x)} = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{d}{dr} \omega_{mk}^{(l)}(r) Y_k^{(m)}(\vartheta, \varphi), \quad l = 1, 2. \tag{3.8}$$

Theorem 3.1. Problems $[\mathrm{I}^\sigma]^-$ and $[\mathrm{II}^\sigma]^-$ have, in the domain Ω^-, a unique solution in the class of regular functions.

proof. The theorems will be proved if we show that the homogeneous problems ($f = 0,\ f_j = 0,\ j = 4, 5$) have the trivial solution only.

Let us assume that the vector $U = (u, \vartheta_1, \vartheta_2)^\top$ is a solution of system (2.1). We multiply both sides of the first equality (2.1) by the complex-conjugate $\overline{u}$, and the complex-conjugate functions of the second and third equalities (2.1) by the functions $\frac{1}{i\sigma}\vartheta_1$ and $\frac{1}{i\sigma}\vartheta_2$, respectively. The integration of these expressions over the domain Ω_r, which is bounded by the concentric spheres Σ_R and Σ_R ($r = |x| > R$). After summing the results, we obtained

$$\int\limits_{\Sigma_r} \left[\overline{u} \cdot T(\partial, n)U + (\varkappa_1 \vartheta_1 + \varkappa_2 \vartheta_2)\partial_n \overline{\vartheta_1} + (\varkappa_2 \vartheta_1 + \varkappa_3 \vartheta_2)\partial_n \overline{\vartheta_2}\right] ds -$$

$$- \int\limits_{\Sigma_R} \left[\overline{u} \cdot T(\partial, n)U + (\varkappa_1 \vartheta_1 + \varkappa_2 \vartheta_2)\partial_n \overline{\vartheta_1} + (\varkappa_2 \vartheta_1 + \varkappa_3 \vartheta_2)\partial_n \overline{\vartheta_2}\right]^- ds =$$

$$= \int\limits_{\Omega_r} \Big[E(\overline{u}, u) - \rho\sigma^2 |u|^2 + \frac{1}{i\sigma}(\varkappa_1 |\nabla \vartheta_1|^2 + \varkappa_2 (\nabla \vartheta_1 \cdot \nabla \overline{\vartheta_2} + \tag{3.9}$$

$$+ \nabla \overline{\vartheta_1} \cdot \nabla \vartheta_2) + \varkappa_3 |\nabla \vartheta_2|^2) + \frac{\alpha}{i\sigma}|\vartheta_1 - \vartheta_2|^2 - (\varkappa' |\vartheta_1|^2 + \varkappa'' |\vartheta_2|^2)\Big] dx = 0,$$

where

$$\nabla = (\partial_1,\, \partial_2,\, \partial_3)^\top, \ \ \partial_j = \frac{\partial}{\partial x_j}, \ \ j = 1, 2, 3, \quad \partial_n \vartheta_j = \frac{\partial \vartheta_j}{\partial n}, \ \ j = 1, 2,$$

$$E(\overline{u}, u) = \frac{3\lambda + 2\mu}{3} |\operatorname{div} u|^2 + \frac{\mu}{2} \sum_{k \neq j = 1}^{3} \left| \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} \right|^2 +$$

$$+ \frac{\mu}{3} \sum_{k,j=1}^{3} \left| \frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} \right|^2.$$

From the equality (3.9), if we use homogeneous boundary conditions, we obtain

$$\frac{2}{i\sigma\varkappa_3} \int\limits_{\Omega_r} \left[d_1 |\nabla \vartheta_1|^2 + |\varkappa_2 \nabla \vartheta_1 + \varkappa_3 \nabla \vartheta_2|^2 + \alpha \varkappa_3 |\vartheta_1 - \vartheta_2|^2\right] dx =$$

$$= \int\limits_{\Sigma_r} \left[\overline{u} \cdot T(\partial,n)U - u \cdot T(\partial,n)\overline{U} + \varkappa_1(\vartheta_1 \partial_n \overline{\vartheta}_1 - \overline{\vartheta}_1 \partial_n \vartheta_1) + \right. \tag{3.10}$$

$$\left. + \varkappa_2(\vartheta_2 \partial_n \overline{\vartheta}_1 - \overline{\vartheta}_2 \partial_n \vartheta_1 + \vartheta_1 \partial_n \overline{\vartheta}_2 - \overline{\vartheta}_1 \partial_n \vartheta_2) + \varkappa_3(\vartheta_2 \partial_n \overline{\vartheta}_2 - \overline{\vartheta}_2 \partial_n \vartheta_2) \right] ds,$$

where $d_1 = \varkappa_1 \varkappa_3 - \varkappa_2^2$.

Substituting the expressions for u, $T(\partial,n)U$, ϑ_j and $\partial_n \vartheta_j$, $j=1,2$ from (3.4),(3.7) and (3.8) into (3.10) and apply the following formulas [18]

$$H^{(1)}_{k+1/2}(k_4 r) \frac{d}{dr} H^{(2)}_{k+1/2}(k_4 r) - H^{(2)}_{k+1/2}(k_4 r) \frac{d}{dr} H^{(1)}_{k+1/2}(k_4 r) = \frac{4}{i\pi r},$$

$$H^{(l)}_{k+1/2}(k_4 r) = O(r^{-1/2}), \;\; l=1,2, \;\; H^{(l)}_{k+1/2}(k_j r) = o(e^{-\Im k_j r}), \;\; j=1,2,3,$$

and the fact that the vectors X_{mk}, Y_{mk}, Z_{mk} are normalized, we have

$$\frac{2}{i\sigma\varkappa_3} \lim_{r\to\infty} \int\limits_{\Omega_r} \Big[d_1 |\nabla\vartheta_1|^2 + |\varkappa_2 \nabla\vartheta_1 + \varkappa_3 \nabla\vartheta_2|^2 + \alpha\varkappa_3 |\vartheta_1 - \vartheta_2|^2 \Big] dx +$$

$$+ \frac{4\mu R}{\pi i} \sum_{k=1}^{\infty} \sum_{m=-k}^{k} \frac{k(k+1)}{\left|H^{(1)}_{k+1/2}(k_4 r)\right|^2} \Big[k_4^2 |A^{(4)}_{mk}|^2 + |A^{(5)}_{mk}|^2 \Big] = 0.$$

Hence it follows that

$$\vartheta_1(x) = \vartheta_2(x) = c = const, \;\; x \in \Omega^-, \;\; A^{(j)}_{mk} = 0, \;\; j=4,5, \;\; k \geq 1. \tag{3.11}$$

Taking into account the behavior of $\vartheta_j(x)$, $j=1,2$ at infinity and expansion (3.3), from equality (3.11) we obtain

$$\vartheta_j(x) = 0, \;\; j=1,2, \;\; \Phi_j(x) = 0, \;\; j=4,5, \;\; x \in \Omega^-. \tag{3.12}$$

(2.2) and (3.12) imply

$$\sum_{j=1}^{3} k_j^2 \beta_l^{(j)} \Phi_j(x) = 0, \;\; l=1,2, \;\; x \in \Omega^-,$$

from which, we obtain $\Phi_j(x)=0$, $j=1,2,3$, $x\in\Omega^-$. Thus we have shown that $\Phi_j(x)=0$, $j=1,2,..,5$, $x\in\Omega^-$. By virtue of these equalities we conclude that $u(x)=0$, $\vartheta_l(x)=0$, $l=1,2$, $x\in\Omega^-$.

4 Solution of the Dirichlet Problem

Assume that the boundary vector functions $f(z)$ and functions $f_j(z)$, $j = 4, 5$, satisfy the sufficient conditions of smoothness which enable us to represent the following Fourier-Laplace series

$$f(z) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \Big\{ \alpha_{mk} X_{mk}(\vartheta, \varphi) + \sqrt{k(k+1)} \times \big[\beta_{mk} Y_{mk}(\vartheta, \varphi) + \gamma_{mk} Z_{mk}(\vartheta, \varphi)\big] \Big\}, \tag{4.1}$$

$$f_j(z) = \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \alpha_{mk}^{(j)} Y_k^{(m)}(\vartheta, \varphi), \ \ j = 4, 5,$$

where α_{mk}, β_{mk}, γ_{mk} and $\alpha_{mk}^{(j)}$, $j = 4, 5$ are the Fourier coefficients.

Possing to the limit on both sides of equalities (3.4) $x \to z \in \partial\Omega$ $(r \to R)$ and taking into account the boundary conditions (3.1) and also equality(4.1), for the sought for constants $A_{mk}^{(j)}$, $j = 1, 2, ..., 5$, we obtain the following system of linear algebraic equations

$$\begin{gathered}
\sum_{j=1}^{3} \frac{d}{dR} h_k(k_j R) A_{mk}^{(j)} + \frac{k(k+1)}{R} A_{mk}^{(4)} = \alpha_{mk}, \ \ k \geq 0, \\
\sum_{j=1}^{3} \frac{1}{R} A_{mk}^{(j)} + \left(\frac{d}{dR} + \frac{1}{R}\right) h_k(k_4 R) A_{mk}^{(4)} = \beta_{mk}, \ \ k \geq 1, \\
\sum_{j=1}^{3} k_j^2 \beta_1^{(j)} A_{mk}^{(j)} = -\alpha_{mk}^{(4)}, \ \ \sum_{j=1}^{3} k_j^2 \beta_2^{(j)} A_{mk}^{(j)} = -\alpha_{mk}^{(5)}, \ \ k \geq 0, \\
A_{mk}^{(5)} = \gamma_{mk}, \quad k \geq 1.
\end{gathered} \tag{4.2}$$

Nov we formulate several technical lemmas [6].

lemma 4.2. Let $f \in C^l(\Sigma_1)$, $f_j \in C^{(l)}(\Sigma_1)$, $l \geq 1$, $j = 4, 5$. Then the coefficients α_{mk}, β_{mk}, γ_{mk} and $\alpha_{mk}^{(j)}$, $j = 4, 5$ have the proporties

$$\alpha_{mk} = O(k^{-l}), \ \ \beta_{mk} = O(k^{-l-1}), \quad \gamma_{mk} = O(k^{-l-1}), \tag{4.3}$$

$$\alpha_{mk}^{(j)} = O(k^{-l}), \ \ j = 4, 5.$$

lemma 4.3. For the vectors $X_{mk}(\vartheta, \varphi)$, $Y_{mk}(\vartheta, \varphi)$ and $Z_{mk}(\vartheta, \varphi)$ given by (3.5) the following inequalities hold

$$\begin{gathered}
\big|X_{mk}(\vartheta,\varphi)\big| \leq \sqrt{\frac{2k+1}{4\pi}}, \ \ k \geq 0, \\
\big|Y_{mk}(\vartheta,\varphi)\big| < \sqrt{\frac{k(k+1)}{2k+1}}, \ \ \big|Z_{mk}(\vartheta,\varphi)\big| < \sqrt{\frac{k(k+1)}{2k+1}}, \ \ k \geq 1.
\end{gathered} \tag{4.4}$$

Moreover, as is known [18]

$$\big|Y_k^{(m)}(\vartheta,\varphi)\big| \leq \sqrt{\frac{2k+1}{4\pi}}, \ \ k \geq 0. \tag{4.5}$$

System (4.2) is compatible according to Theorem 3.1 and Theorem 2.3 If the solutions of these system are substituted into (3.4), then we obtain a formal solution of the Dirichlet problem. We need to show that series (3.4), (3.7) and (3.8) are absolutely and uniformly convergent tn the domain $\overline{\Omega}^-$.

The following asymptotic representations are true as $k \to +\infty$ [18]

$$h_k(k_j r) \sim \left(\frac{R}{r}\right)^{k+1}, \quad h'_k(k_j r) \sim -\frac{k}{r}\left(\frac{R}{r}\right)^{k+1} \tag{4.6}$$

If $x \in \Omega^-$ $(r < R)$, then by the asymptotic representation (4.6) the above-mentationed series are convergent.

If $x \in \partial\Omega$ $(r = R)$, then by lemma 4.3 and asymptotic representation (4.6), series (3.4), (3.7) and (3.8) are absolutely and uniformly convergent provided that the majorant series

$$\sum_{k=k_0}^{\infty} k^{3/2}\Big[|\alpha_{mk}| + k\big(|\beta_{mk}| + |\alpha_{mk}^{(4)}| + |\alpha_{mk}^{(5)}|\big) + |\gamma_{mk}|\Big].$$

is convergent.

The obtained majorant series will be convergent if the Fourier coefficients admit the estimates

$$\begin{gathered}
\alpha_{mk} = O(k^{-3}), \quad \beta_{mk} = O(k^{-4}), \quad \gamma_{mk} = O(k^{-3}), \\
\alpha_{mk}^{(j)} = O(k^{-4}), \ \ j = 4, 5.
\end{gathered} \tag{4.7}$$

By Lemma 4.2, estimates (4.7) hold if the boundary vector functions are assumed to satisfy the following smoothness conditions

$$f(y) \in C^3(\partial\Omega), \quad f_j(y) \in C^4(\partial\Omega), \ \ j = 4, 5. \tag{4.8}$$

Therefore if the vector functions $f(y)$ and $f_j(y)$, $j = 4, 5$, satisfy the smoothness conditions (4.8), then the vector $U(u, \vartheta_1, \vartheta_2)^\top$ represented in form (3.4) is a regular solution of Problem $[\mathrm{I}^\sigma]^-$ in the domain Ω^-.

5 Solution of the Neumann Problem

Passing to the limit on both sides of equalities (3.7)–(3.8) as $x \to z \in \partial\Omega$ ($r \to R$), for the sought constants $A^{(j)}_{mk}$, $j = 1, 2, \ldots, 5$, we obtain the following systems of algebraic equations:

$$\begin{gathered}
\sum_{j=1}^{3}\Big[2\mu\,\frac{d^2}{dR^2} + k_j^2\big(\eta_1\beta_1^{(j)} + \eta_2\beta_2^{(j)} - \lambda\big)\Big]h_k(k_jR)A^{(j)}_{mk} + \\
+2\mu\,\frac{k(k+1)}{R}\Big(\frac{d}{dR} - \frac{1}{R}\Big)h_k(k_4R)A^{(4)}_{mk} = \alpha_{mk}, \quad k \geq 0, \\
\sum_{j=1}^{3}\frac{2\mu}{R}\Big(\frac{d}{dR} - \frac{1}{R}\Big)h_k(k_jR)A^{(j)}_{mk} + \\
+\mu\Big(\frac{d^2}{dR^2} + \frac{(k-1)(k+2)}{R^2}\Big)h_k(k_4R)A^{(4)}_{mk} = \beta_{mk}, \quad k \geq 1, \\
\sum_{j=1}^{3}k_j^2\beta_1^{(j)}\,\frac{d}{dR}\,h_k(k_jR)A^{(j)}_{mk} = -\alpha^{(4)}_{mk}, \\
\sum_{j=1}^{3}k_j^2\beta_2^{(j)}\,\frac{d}{dR}\,h_k(k_jR)A^{(j)}_{mk} = -\alpha^{(5)}_{mk}, \quad k \geq 0, \\
\mu\Big(\frac{d}{dR} - \frac{1}{R}\Big)h_k(k_4R)A^{(5)}_{mk} = \gamma_{mk}, \quad k \geq 1.
\end{gathered} \tag{5.1}$$

According to Theorem 3.1 and Theorem 2.3, systems (5.1) is compatible. If we repeat the reasoning of Section 4, then we obtain that if the vector functions $f(y)$ and $f_j(y)$, $j = 4, 5$, satisfy the smoothness conditions

$$f(y) \in C^2(\partial\Omega), \quad f_j(y) \in C^3(\partial\Omega), \quad j = 4, 5,$$

then the vector $U = (u, \vartheta_1, \vartheta_2)^\top$ represented in form (4.2) is a regular solution of Problem $[\mathrm{II}^\sigma]^-$ in the domain Ω^-.

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