AN ERROR OF THE ITERATION METHOD FOR A TIMOSHENKO NONHOMOGENEOUS EQUATION

J. Peradze*,**, Z. Tsiklauri**

Ivane Javakhishvili Tbilisi State University*
0186 University Street 2, Tbilisi, Georgia
Georgian Technical University**
77 Kostava Str., Tbilisi 0175, Tbilisi, Georgia

(Received: 19.02.10; accepted: 26.11.10)

Abstract

We consider the initial boundary value problem for an integro-differential equation describing the vibration of a beam. Using the Galerkin method and a symmetric difference scheme, the solution is approximates with respect to a spatial and a time variable. Thus the problem is reduced to a system of nonlinear discrete equations which is solved by the iteration method. The convergence of the method is proved.

Key words and phrases: Timoshenko beam equation, error of the iteration method.

AMS subject classification: 45K05, 65N06, 35K55.

1 Statement of Problem

Let us consider the following initial boundary value problem

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}-h\frac{\partial^4 u}{\partial x^2\partial t^2}-\left(\lambda+\frac{1}{2L}\int_0^L\left(\frac{\partial u}{\partial x}\right)^2dx\right)\frac{\partial^2 u}{\partial x^2}=f(x,t), \quad (1.1)$$
$$0<x<L,\quad 0<t\le T,$$

$$\begin{gathered} u(x,0)=u^0(x),\quad \frac{\partial u}{\partial t}(x,0)=u^1(x),\\ u(0,t)=u(L,t)=0,\quad \frac{\partial^2 u}{\partial x^2}(0,t)=\frac{\partial^2 u}{\partial x^2}(L,t)=0, \end{gathered} \quad (1.2)$$
$$0\le x\le L,\quad 0\le t\le T,$$

where h and λ are some non-negative constants, $f(x,t)$, $u^0(x)$ and $u^1(x)$ are the given functions, and $u(x,t)$ is the function to be defined. In the homogeneous case the equation (1.1) describing a dynamic beam was obtained by Henriques de Brito [1] and is a Timoshenko type equation [8].

Menzala and Zuazua [3], [4] arrived at the corresponding equation by making an additional assumption $\lambda = 0$ and passing to the limit in the system of von Karman equations [2]

$$v_{tt} - \left(v_x + \frac{1}{2}\,w_x^2\right)_x = 0,$$
$$w_{tt} + w_{xxxx} - hw_{xxtt} - \left[w_x\left(v_x + \frac{1}{2}\,w_x^2\right)\right]_x = 0$$

for a prismatic beam. In [5], the same authors write a generalized variant of the equation under discussion.

Note that the solvability of an operator equation, the particular case of which is the equation (1.1), is proved in [1].

In the present paper, we consider one numerical method of solution of the problem (1.1), (1.2). In [6], one can partly get acquainted with the bibliography on approximate algorithms for equations having nonlinearity analogous to that of (1.1).

2 The Algorithm

a. Galerkin method. A solution of the problem (1.1), (1.2) will be sought for as a finite sum

$$u_n(x,t) = \sum_{i=1}^{n} \frac{L}{i\pi}\,u_{ni}(t)\sin\frac{i\pi x}{L}\,, \tag{2.3}$$

where the coefficients $u_{ni}(t)$ are defined by the Galerkin method from the system of ordinary differential equations

$$\begin{aligned}&\left(h + \left(\frac{L}{i\pi}\right)^2\right)u_{ni}''(t) + \\ &\qquad + \left(\lambda + \left(\frac{i\pi}{L}\right)^2 + \frac{1}{4}\sum_{j=1}^{n} u_{nj}^2(t)\right)u_{ni}(t) = f_i(t), \\ &\qquad i = 1,2,\ldots,n,\end{aligned} \tag{2.4}$$

with the initial conditions

$$\begin{gathered}u_{ni}(0) = u_i^0, \quad u_{ni}'(0) = u_i^1, \\ i = 1,2,\ldots,n.\end{gathered} \tag{2.5}$$

We have used here the notation

$$f_i(t) = \frac{2}{i\pi}\int_0^L f(x,t)\sin\frac{i\pi x}{L}\,dx,$$
$$u_i^p = \frac{2i\pi}{L^2}\int_0^L u^p(x)\sin\frac{i\pi x}{L}\,dx, \quad p=0,1, \quad i=1,2,\dots,n.$$

The problem of accuracy of this part of the algorithm is studied in [7] for the case $f(x,t)=0$.

b. Difference scheme. On the time interval $[0,T]$ we introduce the net with constant step $\tau = \dfrac{T}{M}$ and nodes $t_m = m\tau$, $m=0,1,\dots,M$.

Denote by u_{ni}^m, $m=0,1,\dots,M$, a difference analogue of the function $u_{ni}(t)$ from the expansion (2.3). To the system (2.4) we put into correspondence the symmetric implicit scheme

$$\begin{gathered}
\left(h+\left(\frac{L}{i\pi}\right)^2\right)\frac{u_{ni}^{m+1}-2u_{ni}^m+u_{ni}^{m-1}}{\tau^2}+\frac{1}{4}\sum_{p=0}^{1}\left[\lambda+\left(\frac{i\pi}{L}\right)^2+\right.\\
\left.+\frac{1}{8}\sum_{j=1}^{n}\left((u_{nj}^{m+p})^2+(u_{nj}^{m+p-1})^2\right)\right]\left(u_{ni}^{m+p}+u_{ni}^{m+p-1}\right)=\\
=\frac{1}{4}\sum_{p=0}^{1}\left(f_i^{m+p}+f_i^{m+p-1}\right), \qquad (2.6)\\
m=1,2,\dots,M-1, \quad i=1,2,\dots,n,
\end{gathered}$$

and, using (2.4), replace the relations (2.5) by

$$\begin{aligned}
&u_{ni}^0 = u_i^0,\\
&u_{ni}^1 = u_i^0+\tau u_i^1+\frac{\tau^2}{2}\left(h+\left(\frac{L}{i\pi}\right)^2\right)^{-1}\left[-\left(\lambda+\left(\frac{i\pi}{L}\right)^2+\right.\right.\\
&\qquad\left.\left.+\frac{1}{4}\sum_{j=1}^{n}\frac{(u_{nj}^1)^2+(u_{nj}^0)^2}{2}\right)\frac{u_{ni}^1+u_{ni}^0}{2}+\frac{f_i^1+f_i^0}{2}\right], \qquad (2.7)\\
&\quad i=0,1,\dots,n\,.
\end{aligned}$$

Here we have used the notation $f_i^m = f_i(t_m)$, $m=0,1,\dots,M$, $i=1,2,\dots,n$.

c. Iteration method. Let us rewrite the system (2.6), (2.7) in the form

$$
\begin{aligned}
&u_{ni}^0 = u_i^0, \quad i = 1, 2, \ldots, n, \\
&\frac{4}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right) u_{ni}^m + \left[\lambda + \left(\frac{i\pi}{L}\right)^2 + \right. \\
&\qquad \left. + \frac{1}{8}\sum_{j=1}^{n}\left(\left(u_{nj}^m\right)^2 + \left(u_{nj}^{m-1}\right)^2\right)\right]\left(u_{ni}^m + u_{ni}^{m-1}\right) = \\
&\qquad = \frac{4}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right)\sum_{p=0}^{2}\tau^p a_{ni,p}^m, \\
&\qquad m = 1, 2, \ldots, M, \quad i = 1, 2, \ldots, n,
\end{aligned}
\tag{2.8}
$$

where

$$
a_{ni,0}^1 = u_i^0, \quad a_{ni,1}^1 = u_i^1, \quad a_{ni,2}^1 = \frac{1}{4}\left(h + \left(\frac{L}{i\pi}\right)^2\right)^{-1}\left(f_i^1 + f_i^0\right),
$$

$$
a_{ni,0}^m = 2u_{ni}^{m-1} - u_{ni}^{m-2}, \quad a_{ni,1}^m = 0,
$$

$$
a_{ni,2}^m = -\frac{1}{4}\left(h + \left(\frac{L}{i\pi}\right)^2\right)^{-1}\left[\lambda + \left(\frac{i\pi}{L}\right)^2 + \right.
$$

$$
\left. + \frac{1}{8}\sum_{j=1}^{n}\left(\left(u_{nj}^{m-1}\right)^2 + \left(u_{nj}^{m-2}\right)^2\right)\left(u_{ni}^{m-1} + u_{ni}^{m-2}\right) - \left(f_i^m + 2f_i^{m-1} + f_i^{m-2}\right)\right],
$$

$$
m = 2, 3, \ldots, M.
$$

We split the system (2.8) into subsystems corresponding to each $m = 1, 2, \ldots, M$ and will solve them individually by iteration

$$
\begin{aligned}
&\frac{4}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right) u_{ni,k+1}^m + \\
&\quad + \left[\lambda + \left(\frac{i\pi}{L}\right)^2 + \frac{1}{8}\left(\left(u_{ni,k+1}^m\right)^2 + \left(u_{ni}^{m-1}\right)^2\right) + \right. \\
&\quad \left. + \frac{1}{8}\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\left(u_{nj,k}^m\right)^2 + \left(u_{nj}^{m-1}\right)^2\right)\right]\left(u_{ni,k+1}^m + u_{ni}^{m-1}\right) = \\
&= \frac{4}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right)\sum_{p=0}^{2}\tau^p a_{ni,p}^m, \\
&\quad k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n.
\end{aligned}
\tag{2.9}
$$

Here $u^m_{ni,k+l}$ denotes the $(k+l)$-th approximation of u^m_{ni}, $l = 0, 1$.

Assume that we have already found u^o_{ni} for $m = 1$, and u^{m-2}_{ni} and u^{m-1}_{ni} for $m > 1$. For the sake of simplicity, we neglect the error corresponding to the values of these functions.

Since (2.9) is a cubic equation with respect to $u^m_{ni,k+1}$, the latter can be written in the explicit form

$$u^m_{ni,k+1} = -\frac{1}{3}\,u^{m-1}_{ni} + \sum_{l=1}^{2}(-1)^{l+1}\left[(-1)^l s_i + (s_i^2 + r_i^3)^{1/2}\right]^{1/3}, \qquad (2.10)$$
$$k = 0, 1, \ldots, \quad i = 1, 2, \ldots, n,$$

where

$$\begin{aligned} r_i = \frac{1}{3}\Bigg[&8\left(\lambda + \left(\frac{i\pi}{L}\right)^2\right) + \frac{2}{3}(u^{m-1}_{ni})^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\Big(\left(u^m_{nj,k}\right)^2 + \\ &+ \left(u^{m-1}_{nj}\right)^2\Big) + \frac{32}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right)\Bigg], \\ s_i = \frac{1}{3}u^{m-1}_{ni}\Bigg[&8\left(\lambda + \left(\frac{i\pi}{L}\right)^2\right) + \frac{10}{9}(u^{m-1}_{ni})^2 + \sum_{\substack{j=1\\ j\neq i}}^{n}\Big(\left(u^m_{nj,k}\right)^2 + \\ &+ \left(u^{m-1}_{nj}\right)^2\Big)\Bigg] - \frac{16}{\tau^2}\left(h + \left(\frac{L}{i\pi}\right)^2\right)\left(\frac{1}{3}u^{m-1}_{ni} + \sum_{p=0}^{2}\tau^p a^m_{ni,p}\right). \end{aligned} \qquad (2.11)$$

Thus the iteration method used here should be understood as counting by (2.10).

3 Error of the iteration method

Let us rewrite the system (2.10) as

$$u^m_{ni,k+1} = \varphi_i\left(u^m_{n1,k}, u^m_{n2,k}, \ldots, u^m_{nn,k}\right), \qquad (3.12)$$
$$k = 0, 1, \ldots, i = 1, 2, \ldots, n.$$

To estimate the error of the method (3.12) we need to consider the matrix-jacobian

$$J = \left(\frac{\partial\varphi_i}{\partial u^m_{nj,k}}\right)^n_{i,j=1}.$$

Taking into account (2.10)–(3.12), we conclude that the principal diagonal of the matrix J consists of zeros

$$\frac{\partial \varphi_i}{\partial u^m_{ni,k}} = 0,$$

as to the nondiagonal elements, for them we have

$$\frac{\partial \varphi_i}{\partial u^m_{nj,k}} = -\frac{1}{9} u^m_{nj,k} \sum_{l=1}^{2} \left[(-1)^l s_i + (s_i^2 + r_i^3)^{\frac{1}{2}} \right]^{-\frac{2}{3}} \times$$
$$\times \left[2u^{m-1}_{ni} + (-1)^l \left(2s_i u^{m-1}_{ni} + 3r_i^2 \right) (s_i^2 + r_i^3)^{-\frac{1}{2}} \right], \quad i \neq j.$$

Performing some transformations and using (2.11), we obtain

$$\begin{aligned} \left| \frac{\partial \varphi_i}{\partial u^m_{nj,k}} \right| &= \frac{4}{9r_i} |u^m_{nj,k}| \left(|u^{m-1}_{ni}| + \frac{|s_i|}{r_i} \right) \leq \\ &\leq \frac{\tau^2}{24} |u^m_{nj,k}| \left(h + \left(\frac{L}{i\pi} \right)^2 \right)^{-1} \times \\ &\times \left\{ \frac{\tau^2}{32} \left(h + \left(\frac{L}{i\pi} \right)^2 \right)^{-1} |u^{m-1}_{ni}| \left[8 \left(\lambda + \left(\frac{i\pi}{L} \right)^2 \right) + \right. \right. \\ &+ \left. \sum_{\substack{l=1 \\ l \neq i}}^{n} \left((u^m_{nl,k})^2 + (u^{m-1}_{nl})^2 \right) + \frac{10}{9} (u^{m-1}_{ni})^2 \right] + \\ &+ \left. \frac{7}{6} |u^{m-1}_{ni}| + \sum_{p=0}^{2} \tau^p |a^m_{ni,p}| \right\}. \end{aligned} \tag{3.13}$$

Let us apply the principle of compressed mappings. We define the vector and matrix norms by the expressions $\sum\limits_{i=1}^{n} |v_i|$ and $\max\limits_{1 \leq j \leq n} \sum\limits_{i=1}^{n} |m_{ij}|$, respectively, for $v = (v_i)_{i=1}^n$, and $M = (m_{ij})_{i,j=1}^n$.

Let in the vector domain

$$\left\{ (u_{ni})_{i=1}^n \in R^n \Big| \sum_{i=1}^{n} |u_{ni} - u^m_{ni,0}| \leq \frac{1}{1-q} \sum_{i=1}^{n} |u^m_{ni,1} - u^m_{ni,0}| \right\}, \tag{3.14}$$

the inequality

$$\max_{1 \leq j \leq n} \sum_{i=1}^{n} \left| \frac{\partial \varphi_i}{\partial u^m_{nj,k}} \right| < q$$

be fulfilled for q, $0 < q < 1$. As follows from (3.13) and (3.14), for this it suffices that the relation

$$\alpha\tau^4 + \beta\tau^2 - \gamma \leq 0 \tag{3.15}$$

holds, where the following notation is used

$$\alpha = \sum_{i=1}^{n} \left(h + \left(\frac{L}{i\pi} \right)^2 \right)^{-2} \left\{ \lambda + \left(\frac{i\pi}{L} \right)^2 + \frac{1}{8} \left[\sum_{j=1}^{n} \left(|u_{ni,0}^m| + \right.\right.\right.$$
$$\left.\left.\left. + \frac{1}{1-q} \left| u_{ni,1}^m - u_{ni,0}^m \right| \right) \right]^2 + \frac{5}{36} \sum_{j=1}^{n} \left(u_{nj}^{m-1} \right)^2 \right\} |u_{ni}^{m-1}| +$$
$$+ 4 \sum_{i=1}^{n} \left(h + \left(\frac{L}{i\pi} \right)^2 \right)^{-1} \left(\frac{\varepsilon}{4} \left| a_{ni,1}^m \right| + \left| a_{ni,2}^m \right| \right),$$
$$\beta = \sum_{i=1}^{n} \left(h + \left(\frac{L}{i\pi} \right)^2 \right)^{-1} \left(\frac{14}{3} \left| u_{ni}^{m-1} \right| + \left| a_{ni,0}^m \right| + \frac{1}{\varepsilon} \left| a_{ni,1}^m \right| \right),$$
$$\gamma = 96q \left[\sum_{i=1}^{n} \left(|u_{ni,0}^m| + \frac{1}{1-q} \left| u_{ni,1}^m - u_{ni,0}^m \right| \right) \right]^{-1}$$

and ε is an arbitrary positive number.

The relation (3.15) will be fulfilled if the set of the grid satisfies the inequality

$$\tau \leq \left[\frac{1}{2\alpha} \left(-\beta + (\beta^2 + 4\alpha\gamma)^{\frac{1}{2}} \right) \right]^{\frac{1}{2}}.$$

In that case, in the domain (3.14) there exists a unique vector $(u_{ni}^m)_{i=1}^n$ such that u_{ni}^m, $i = 1, 2, \ldots, n$, are a solution of the system (2.8), the sequence $u_{ni,k}^m$ of the process (2.10) tends to u_{ni}^m, $i = 1, 2, \ldots, n$, as $k \to \infty$, whereas the method error decreases at a geometrical progression rate

$$\sum_{i=1}^{n} \left| u_{ni,k}^m - u_{ni}^m \right| \leq \frac{q^k}{1-q} \sum_{i=1}^{n} \left| u_{ni,1}^m - u_{ni,0}^m \right|,$$
$$k = 0, 1, \ldots .$$

Acknowledgements

The first author J. Peradze acknowledges a support under the grant 7353 of Estonian Science Foundation.

References

1. Henriques de Brito E., A nonlinear hyperbolic equation. Internat. *J. Math. Math. Sci.* **3** (1980), no. 3, 505–520.
2. Lagnese J. E., Leugering G., Uniform stabilization of a nonlinear beam by nonlinear boundary feedback. *J. Differential Equations* **91** (1991), no. 2, 355–388.
3. Menzala G. P., Zuazua E., The beam equation as a limit of a 1-D nonlinear von Karman model. *Appl. Math. Lett.* **12** (1999), no. 1, 47–52.
4. Menzala G. P., Zuazua E., Timoshenko's beam equation as limit of a nonlinear one-dimensional von Karman system. *Proc. Roy. Soc. Edinburgh Sect. A* **130** (2000), no. 4, 855–875.
5. Menzala G.P., Zuazua E., On a one-dimensional version of the dynamical Marguerre-Vlasov system. Dedicated to Constantine Dafermos on his 60th birthday. *Bol. Soc. Brasil. Mat. (N.S.)* **32** (2001), no. 3, 303–319.
6. Peradze J., An approximate algorithm for a Kirchhoff wave equation. *SIAM J. Numer. Anal.* **47** (2009), no. 3, 2243–2268.
7. Peradze J., On the accuracy of the Galerkin method for onr nonlinear beam equation. *Math. Meth. Appl. Sci.* **34** (2011), 1725–1732.
8. Timoshenko S. H., Young D. H., Weaver W., Vibration problems on engineering. *John Willey & Sons,* 1974.