HIGH ORDER ACCURACY SPLITTING FORMULAS FOR COSINE OPERATOR FUNCTION AND THEIR APPLICATIONS

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Abstract

In the present work the high order accuracy rational splitting for cosine operator function is constructed. On the basis of this formula, the fourth order of accuracy decomposition scheme for homogeneous abstract hyperbolic equation with operator A is constructed. This operator is a self-adjoint, positive definite operator and is represented as a sum of the same type operators. Error of approximate solution is estimated. In the work a method for constructing any order accuracy splitting formula for cosine operator function is also introduced.

Key words and phrases: Cosine Operator Function; Decomposition Scheme; Operator Split; Abstract Hyperbolic Equation; Rational Approximation.

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1 Introduction

Let A is a self-adjoint, positive definite operator and be represented as a finite sum of the same kind of operators. In the present work the high order accuracy rational splitting for $\cos\left(\tau A^{1/2}\right)$ operator function (obviously, accuracy of the splitting formula is understand with respect to parameter τ) is constructed. Our goal is also to introduce the method for constructing $2p+2$ order accuracy splitting formula on the basis of $2p$ order accuracy formulas. Finally, on the basis of these formulas, we aim to construct high order accuracy decomposition schemes for abstract hyperbolic equation.

As is known, the solution of Cauchy problem for an abstract hyperbolic equation can be given by means of sine and cosine operator functions, where square root from the main operator is included in the argument. Using this formula, for the equally distanced values of time variable, the precise three-layer semi-discrete scheme can be constructed whose transition operator is a cosine operator function. On the basis of this relation we can obtain decomposition scheme for an abstract hyperbolic equation. Of course, for this purpose it is necessary to replace cosine operator function by splitting formula. In the present work, using the fourth order of accuracy splitting formula for cosine operator function, decomposition scheme is constructed and error for approximate solution is estimated.

Qin Sheng, Voss David A. and Khaliq Abdul Q. M. in the work [10] constructed the second order accuracy decomposition scheme for sin-Gordon equation. It should be pointed out that these authors have constructed the schemes using exponential splitting and only then they have obtained the corresponding rational splitting using Pade approximation.

Let us note that, using the above-mentioned precise three-layer scheme, Baker G. A., Dougalis V. A. and Serbin S.M. (see [1]) for the first time constructed high order accuracy unsplitted scheme for an abstract hyperbolic equation for solution of Cauchy problem.

In [8] we have constructed the fourth order accuracy decomposition scheme for a homogeneous hyperbolic equation in two-dimensional case. Let us note that the scheme constructed in [8] does not represent a particular case of the scheme given in the present paper.

Let us note that high order precision decomposition schemes for parabolic equations constructed in the works [2],[3],[4] and [9] are also based on the splitting of the solving operator (semigroup).

2 Rational splitting for cosine operator function

Let A be a self-adjoint, positive definite (generally unbounded) operator in the Hilbert space H, with domain $D(A)$ everywhere dense in H. Let $A = A_1 + A_2 + ... + A_m$, where A_j $(j = 1, ..., m)$ are self-adjoint positive definite operators. Our aim is to construct high order accuracy rational splitting for the operator function $\cos\left(\tau A^{1/2}\right)$, $\tau > 0$.

As it is known $\cos\left(tA^{1/2}\right)$ operator function is defined by Euler generalization formulas:

$$\cos\left(tA^{1/2}\right) = \frac{1}{2}\left(e^{-it\sqrt{A}} + e^{it\sqrt{A}}\right), \tag{1}$$

where $\left\{e^{\pm it\sqrt{A}}\right\}$ is a unitary group of operators generated by operators $\left(\pm iA^{1/2}\right)$.

It is proved that there exists limit $\lim\limits_{n\to\infty}\left(I \pm \frac{t}{n}iA^{1/2}\right)^{-n}\varphi$ (I is a unit operator), for any $\varphi \in H$-and this limit is defined as $e^{\pm it\sqrt{A}}\varphi$ (see [5], Chapter IX).

Let us consider the following rational splitting:

$$\begin{aligned} V(\tau) &= \frac{1}{m+2}\left[V_0(\tau;A_1,...,A_m)+V_0(\tau;A_m,...,A_1)\right. \\ &\quad \left.+\sum_{j=1}^{m}\left(I+\lambda\tau^2A_j\right)^{-1}\right], \qquad (2) \\ V_0(\tau;A_1,...,A_m) &= \left(I+\alpha\tau^2A_1\right)^{-1}...\left(I+\alpha\tau^2A_m\right)^{-1} \\ &\quad \times\left(I+\overline{\alpha}\tau^2A_m\right)^{-1}...\left(I+\overline{\alpha}\tau^2A_1\right)^{-1}, \end{aligned}$$

where $\lambda=\frac{m+2}{2}-\frac{\sqrt{m+2}}{\sqrt{6}}$, $\alpha=\frac{\sqrt{m+2}}{4\sqrt{6}}\pm i\sqrt{\frac{m+2}{96}+\frac{\lambda^2}{2}}$, $\overline{\alpha}$ is a conjugate of α.

Let us show that (2) formula gives splitting of cosine operator function with locally sixth order of accuracy.

Let us introduce the following notation:

$$\begin{aligned} \|\varphi\|_A &= \|A_1\varphi\|+...+\|A_m\varphi\|, \quad \varphi\in D(A), \\ \|\varphi\|_{A^2} &= \sum_{i,j=1}^{m}\|A_iA_j\varphi\|, \quad \varphi\in D\left(A^2\right). \end{aligned}$$

Analogously is defined $\|\varphi\|_{A^k}$ $(k>2)$.

According to formula (1) the following expansion is valid for cosine operator function:

$$\cos\left(\tau A^{1/2}\right)=\sum_{i=0}^{k}(-1)^i\frac{\tau^{2i}}{(2i)!}A^i+R_k(\tau,A), \qquad (3)$$

where $R_k(\tau,A)$ is a residual member, for which the following estimate holds:

$$\|R_k(\tau,A)\varphi\|\leq\frac{1}{(2k+2)!}\tau^{2k+2}\|\varphi\|_{A^{k+1}}, \quad \varphi\in D\left(A^{k+1}\right). \qquad (4)$$

Using the method of induction, the following expansion can be obtained:

$$\left(I+\tau^2A\right)^{-1}=\sum_{i=0}^{k}(-1)^i\tau^{2i}A^i+\widetilde{R}_k(\tau,A), \qquad (5)$$

where

$$\widetilde{R}_k(\tau,A)=(-1)^{k+1}\tau^{2k+2}\left(I+\tau^2A\right)^{-1}A^{k+1}. \qquad (6)$$

It is obvious that for the residual member of $\widetilde{R}_k(\tau,A)$ the following estimate holds:

$$\left\|\widetilde{R}_k(\tau,A)\varphi\right\|\leq\tau^{2k+2}\|\varphi\|_{A^{k+1}}, \quad \varphi\in D\left(A^{k+1}\right). \qquad (7)$$

We expand the operator $V(\tau)$ from the right to the left using the formula (5) in the way that each residual member be of sixth order with respect to τ. Therefore we obtain:

$$\begin{aligned} V(\tau) &= \frac{1}{m+2}\left[V_0(\tau; A_1, ..., A_m) + V_0(\tau; A_m, ..., A_1) \right. \\ &\left. + \sum_{j=1}^{m} \left(I + \lambda\tau^2 A_j\right)^{-1}\right] \\ &= \frac{1}{m+2}\left[(m+2) I + \tau^2 \sum_{i=1}^{m} (2(\alpha+\overline{\alpha}) + \lambda) A_i \right. \\ &+ \tau^4 \left(\sum_{i=1}^{m} \left(2\left(\alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2\right) + \lambda^2\right) A_i^2 \right. \\ &\left.\left. + \sum_{i,j=1, i\neq j}^{m} (\alpha+\overline{\alpha})^2 A_i A_j\right)\right] + \widetilde{R}(\tau), \end{aligned} \tag{8}$$

where for $\widetilde{R}(\tau)$, according to (7), the following estimate holds:

$$\left\|\widetilde{R}(\tau)\varphi\right\| \leq c\tau^6 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{9}$$

Parameters $\alpha, \overline{\alpha}$ and λ satisfy the following equalities:

$$\begin{aligned} 2(\alpha+\overline{\alpha}) + \lambda &= \frac{m+2}{2}, \\ 2\left(\alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2\right) + \lambda^2 &= \frac{m+2}{24}, \\ (\alpha+\overline{\alpha})^2 &= \frac{m+2}{24}. \end{aligned}$$

Taking into account these equalities, from (8) we obtain:

$$V(\tau) = I - \frac{\tau^2}{2}A + \frac{\tau^4}{24}A^2 + \widetilde{R}(\tau). \tag{10}$$

Due to (3), we have:

$$\cos\left(\tau A^{1/2}\right) = I - \frac{\tau^2}{2}A + \frac{\tau^4}{24}A^2 + R_2(\tau, A). \tag{11}$$

From (10) and (11), taking into account inequalities (4) and (9), we obtain:

$$\left\|\left(\cos\left(\tau A^{1/2}\right) - V(\tau)\right)\varphi\right\| \leq c\tau^6 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{12}$$

3 Decomposition scheme for homogeneous abstract hyperbolic equation

Let us consider the Cauchy problem for an abstract hyperbolic equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + Au(t) = 0, \quad t \in [0,T], \tag{1}$$

$$u(0) = \varphi_0, \quad \frac{du(0)}{dt} = \varphi_1, \tag{2}$$

where A is a self-adjoint (A does not depend on t), positive definite (generally unbounded) operator with domain $D(A)$ everywhere dense in H, $\overline{D(A)} = H$, $A = A^*$ and

$$(Au, u) \geq a\|u\|^2, \quad \forall u \in D(A), \quad a = const > 0,$$

where $\|\cdot\|$ and $(\cdot,\cdot)$ define the norm and scalar product in H, respectively; φ_0 and φ_1 are the given vectors from H; $u(t)$ is a continuous, twice continuously differentiable, sought function with values in H.

It is well-known that if $\varphi_0 \in D(A)$, $\varphi_1 \in D\left(A^{1/2}\right)$ then there exists such twice continuously differentiable function $u(t)$, that satisfies equation (1) and initial conditions (2) (see [6], Chapter III, ♮1). In this case the solution is given by the following formula:

$$u(t) = \cos\left(tA^{1/2}\right)\varphi_0 + A^{-1/2}\sin\left(tA^{1/2}\right)\varphi_1, \tag{3}$$

where operator functions $\cos\left(tA^{1/2}\right)$ and $\sin\left(tA^{1/2}\right)$ are defined by Euler generalization formulas.

Let $A = A_1 + A_2 + ... + A_m$, where A_j $(j = 1, ..., m)$ are self-adjoint positive definite operators.

Let us introduce the grid:

$$\omega_\tau = \left\{t_k = k\tau, \quad k = 0, 1, ...n, \quad n > 1, \quad \tau = \frac{T}{n}\right\}.$$

From the formula (3), the following three-point recurrent relation can be easily obtained:

$$u(t_{k+1}) = 2\cos\left(\tau A^{1/2}\right)u(t_k) - u(t_{k-1}). \tag{4}$$

If in this formula we replace cosine operator function by rational splitting obtained in the previous section, we receive the following decomposition scheme

$$u_{k+1} = 2V(\tau)u_k - u_{k-1}, \quad k = 1, ..., n-1, \tag{5}$$

where

$$u_0 = \varphi_0, \quad u_1 = V(\tau)\varphi_0 + \tau V\left(\frac{\tau}{\sqrt{3}}\right)\varphi_1. \tag{6}$$

We declare function u_k as an approximation of $u(t)$ in the node $t = t_k$.

4 Theorem on error estimate for approximate solution

The following theorem takes place (everywhere below c denotes a positive constant).

Theorem 4.1 Let the following conditions be fulfilled:

(a) $\lambda = \frac{m+2}{2} - \frac{\sqrt{m+2}}{\sqrt{6}}, \quad \alpha = \frac{\sqrt{m+2}}{4\sqrt{6}} \pm i\sqrt{\frac{m+2}{96} + \frac{\lambda^2}{2}}$;

(b) A, A_j $(j = 1, ..., m)$ are self-adjoint positive definite (generally unbounded) operators;

(c) $\varphi_0 \in D\left(A^3\right), \ \varphi_1 \in D\left(A^{2+1/2}\right)$;

Then for error of the approximate solution obtained by scheme (5)-(6), the following estimate holds:

$$\|u(t_k) - u_k\| \leq c\nu\tau^4 \left(\|\varphi_1\|_{A^2} + \tau\|\varphi_0\|_{A^3} + t_k \max_{1 \leq i \leq k} \|u(t_i)\|_{A^3} \right),$$

where $\nu = \left(1 + \tau^2\nu_0\right)/\sqrt{\nu_0}$, ν_0 is the minimum lower boundary of operators A_j $(j = 1, ..., m)$.

Proof. Let us note that if $\varphi_0 \in D\left(A^3\right)$ and $\varphi_1 \in D\left(A^{2+1/2}\right)$, then from formula (3) it automatically follows that $u(t) \in D\left(A^3\right)$ for every $t \in [0, T]$.

We denote an error of the approximate solution at $t = t_k$ by z_k, $z_k = u(t_k) - u_k$. Due to formulas (4) and (5), we have:

$$z_{k+1} = 2V(\tau) z_k - z_{k-1} + 2R(\tau) u(t_k), \tag{7}$$

where

$$R(\tau) = \cos\left(\tau A^{1/2}\right) - V(\tau). \tag{8}$$

From (7), using induction we obtain:

$$\begin{aligned} z_{k+1} &= \widetilde{U}_k(L) z_1 - \widetilde{U}_{k-1}(L) z_0 + \sum_{i=1}^{k} \widetilde{U}_{k-i}(L) R(\tau) u(t_i) \\ &= \widetilde{U}_{k-i}(L) z_1 + \sum_{i=1}^{k} \widetilde{U}_{k-i}(L) R(\tau) u(t_i), \quad L = 2V(\tau), \end{aligned} \tag{9}$$

where $\widetilde{U}_k(L)$ operator-polynomials satisfy the following recurrent relation:

$$\begin{aligned}\widetilde{U}_k(L) &= L\widetilde{U}_{k-1}(L)-\widetilde{U}_{k-2}(L), \\ \widetilde{U}_0(L) &= I, \quad \widetilde{U}_{-1}(L)=0.\end{aligned} \tag{10}$$

Let us consider the scalar polynomial $\widetilde{U}_k(x)$, corresponding to the operator-polynomial $\widetilde{U}_k(L)$. It is important that $U_k(x)=\widetilde{U}_k(2x)$ represent Chebyshev polynomials of second kind, for which the following well-known representation is valid (see, e. g., [11]):

$$U_k(x)=\frac{\sin\left((k+1)\arccos x\right)}{\sin\left(\arccos x\right)}, \quad x\in]-1,1[.$$

From here it follows:

$$\widetilde{U}_k(x)=\frac{\sin\left((k+1)\arccos\frac{x}{2}\right)}{\sin\left(\arccos\frac{x}{2}\right)}, \quad x\in]-2,2[. \tag{11}$$

Finally on the interval $]-1,1[$ we obtain the estimate, which is analogous to the well-known estimate for classical Chebyshev polynomial:

$$\left|\widetilde{U}_k(x)\right|\le\frac{2}{\sqrt{4-x^2}}, \quad x\in]-2,2[. \tag{12}$$

Let us estimate the norm of the operator $\left(I+\alpha\tau^2A_i\right)^{-1}$. As, A_i is self-adjoint and positive definite operator, we have:

$$\begin{aligned}\left\|\left(I+\alpha\tau^2A_i\right)^{-1}\right\| &= \sup_{x\in[\nu_0,+\infty)}\frac{1}{|1+\alpha\tau^2x|} \\ &\le \left(1+\frac{\sqrt{m+2}}{4\sqrt{6}}\tau^2\nu_0\right)^{-1}.\end{aligned} \tag{13}$$

Analogously, we obtain:

$$\left\|\left(I+\overline{\alpha}\tau^2A_i\right)^{-1}\right\| \le \left(1+\frac{\sqrt{m+2}}{4\sqrt{6}}\tau^2\nu_0\right)^{-1}, \tag{14}$$

$$\left\|\left(I+\lambda\tau^2A_i\right)^{-1}\right\| \le \frac{1}{1+\lambda\tau^2\nu_0}\le\frac{1}{1+\tau^2\nu_0}. \tag{15}$$

From the estimates (13) and (14) it follows:

$$\begin{aligned}\left\|V_0\left(\tau;A_1,...,A_m\right)\right\| &\le \left(1+\frac{\sqrt{m+2}}{4\sqrt{6}}\tau^2\nu_0\right)^{-2m} \\ &\le \left(1+\frac{m\sqrt{m+2}}{2\sqrt{6}}\tau^2\nu_0\right)^{-1} \\ &\le \left(1+\frac{\sqrt{6}}{3}\tau^2\nu_0\right)^{-1}.\end{aligned} \tag{16}$$

Similarly we obtain:

$$\left\| V_0\left(\tau; A_m, ..., A_1\right)\right\| \leq \left(1+\frac{\sqrt{6}}{3}\tau^2\nu_0\right)^{-1}. \tag{17}$$

From (2), taking into account (16) and (17), we obtain:

$$\|L\| \leq \frac{2}{m+2}\left(\frac{2}{1+\frac{\sqrt{6}}{3}\tau^2\nu_0}+\frac{m}{1+\tau^2\nu_0}\right) \leq \frac{2}{1+\frac{\sqrt{6}}{3}\tau^2\nu_0}. \tag{18}$$

As L is self-adjoint operator, from (18) it follows:

$$Sp\left(L\right) \subset \left[-\nu_1, \nu_1\right], \tag{19}$$

where $\nu_1 = 2/\left(1+\frac{\sqrt{6}}{3}\tau^2\nu_0\right)$.

Let us estimate the norm of the operator $\tau\widetilde{U}_k\left(L\right)$. As it is known, when the argument is a self-adjoint bounded operator, the norm of the operator polynomial is equal to the C-norm of the corresponding scalar polynomial on the spectrum (see, e.g., [7] Chapter VII). Due to this fact, from (12), taking into account (19), we obtain

$$\begin{aligned}\tau\left\|\widetilde{U}_k\left(L\right)\right\| &= \tau\max_{x\in Sp(L)}\left|\widetilde{U}_k\left(x\right)\right| \leq \tau\max_{x\in[-\nu_1,\nu_1]}\frac{2}{\sqrt{4-x^2}} \\ &= \frac{2\tau}{\sqrt{4-\nu_1^2}} \leq \nu. \end{aligned} \tag{20}$$

For z_1 we have:

$$z_1 = u\left(t_1\right)-u_1 = R\left(\tau\right)\varphi_0+\left(A^{-1/2}\sin\left(\tau A^{1/2}\right)-\tau V\left(\frac{\tau}{\sqrt{3}}\right)\right)\varphi_1. \tag{21}$$

Analogously to the estimate (12), we obtain:

$$\left\|\left(A^{-1/2}\sin\left(\tau A^{1/2}\right)-\tau V\left(\frac{\tau}{\sqrt{3}}\right)\right)\varphi_1\right\| \leq c\tau^5\left\|\varphi_1\right\|_{A^2}, \quad \varphi\in D\left(A^2\right). \tag{22}$$

From (21), taking into account (12) and (22) the following estimate can be obtained:

$$\left\|z_1\right\| \leq c\tau^5\left(\left\|\varphi_1\right\|_{A^2}+\tau\left\|\varphi_0\right\|_{A^3}\right). \tag{23}$$

From the formula (9), taking into account inequalities (12), (23) and (20) we obtain the sought estimate.

5 Arbitrary order accuracy splitting for cosine operator function

In this section we introduce an algorithm, which allows us to construct any order accuracy splitting formula for cosine operator function. Main idea of the algorithm consists in the following: we take any $2p$ ($p > 1$ is natural power) order accuracy splitting formula, and using this formula we construct new $2p + 2$ order accuracy splitting formula. This idea, for exponential operator function, for the first time was used by us to obtain the third and fourth order accuracy splitting formulas ((see [4])), but for the general case this idea was developed by Castella F., Chartier P., Descombes S., and Vilmart G. (see [2]). For cosine operator function we will use the algorithm which is similar to the algorithm used by these authors.

Let $U(\tau)$ represent $2p$ order of accuracy splitting formula for cosine operator function. In the expansion of this formula even powers of τ will be included. Thus, the following representation is valid:

$$\cos\left(\tau A^{1/2}\right) = U(\tau) + \tau^{2p} F(A_1, A_2..., A_m) + R(\tau), \tag{24}$$

where $F(A_1, A_2..., A_m)$ is a homogeneous operator function of its arguments; Residual member $R(\tau)$ is of $O\left(\tau^{2p+2}\right)$ order, more precisely $\|R(\tau)\varphi\| = O\left(\tau^{2p+2}\right), \quad \varphi \in D\left(A^{p+1}\right)$.

It is clear that the summary formula for cosine operator function is valid. Using this formula we obtain:

$$\begin{aligned}
\frac{1}{2}\cos\left(\tau A^{1/2}\right) &= \frac{1}{2}\cos\left((\gamma_1+\gamma_2)\tau A^{1/2}\right) \\
&= \cos\left(\gamma_1 \tau A^{1/2}\right)\cos\left(\gamma_2 \tau A^{1/2}\right) \\
&\quad -\frac{1}{2}\cos\left((\gamma_1-\gamma_2)\tau A^{1/2}\right),
\end{aligned} \tag{25}$$

where $\gamma_1 + \gamma_2 = 1$.

From the formula (25), using (24) we obtain:

$$\begin{aligned}
\frac{1}{2}\cos\left(\tau A^{1/2}\right) &= \frac{1}{2}\cos\left((\gamma_1+\gamma_2)\tau A^{1/2}\right) \\
&= \left(U(\gamma_1\tau) + (\tau^{2p}\gamma_1^{2p})F\right)\left(U(\gamma_2\tau) + (\tau^{2p}\gamma_2^{2p})F\right) \\
&\quad -\frac{1}{2}\left(U((\gamma_1-\gamma_2)\tau) + \tau^{2p}(\gamma_1-\gamma_2)^{2p}F\right) + R_1(\tau) \\
&= U(\gamma_1\tau)U(\gamma_2\tau) - \frac{1}{2}U((\gamma_1-\gamma_2)\tau) \\
&\quad +\tau^{2p}\left(\gamma_2^{2p}U(\gamma_1\tau)F + \gamma_1^{2p}FU(\gamma_2\tau) - \frac{1}{2}(\gamma_1-\gamma_2)^{2p}F\right) \\
&\quad +R_1(\tau),
\end{aligned} \tag{26}$$

where the residual member $R_1(\tau)$ is of order $O\left(\tau^{2p+2}\right)$.

As $U(\tau)$ represents an expansion of cosine operator function, it is clear that the first term will be identity operator, and other terms will have even powers of τ as multipliers. According to this, we obtain:

$$\begin{aligned} & \gamma_2^{2p} U(\gamma_1 \tau) F + \gamma_1^{2p} F U(\gamma_2 \tau) - \frac{1}{2}(\gamma_1 - \gamma_2)^{2p} F \\ = & \left(\gamma_2^{2p} + \gamma_1^{2p} - \frac{1}{2}(\gamma_1 - \gamma_2)^{2p}\right) F + R_2(\tau), \end{aligned} \tag{27}$$

where the residual member $R_2(\tau)$ is of order $O\left(\tau^2\right)$.

It is clear that, if the parameter $\gamma_1$da γ_2 satisfies condition:

$$\gamma_2^{2p} + \gamma_1^{2p} - \frac{1}{2}(\gamma_1 - \gamma_2)^{2p} = 0,$$

then from (26) according to (27) we obtain:

$$\frac{1}{2}\cos\left(\tau A^{1/2}\right) = U(\gamma_1 \tau) U(\gamma_2 \tau) - \frac{1}{2} U((\gamma_1 - \gamma_2)\tau) + \widetilde{R}(\tau),$$

where the residual member $\widetilde{R}(\tau)$ is of order $O\left(\tau^{2p+2}\right)$.

Finally we obtain that the formula

$$\widetilde{U}(\tau) = U(\gamma_1 \tau) U(\gamma_2 \tau) - \frac{1}{2} U((\gamma_1 - \gamma_2)\tau),$$

where parameters $\gamma_1$da γ_2 satisfy conditions:

$$\begin{aligned} \gamma_1 + \gamma_2 &= 1, \\ \gamma_2^{2p} + \gamma_1^{2p} - \frac{1}{2}(\gamma_1 - \gamma_2)^{2p} &= 0, \end{aligned} \tag{28}$$

represents $2p+2$ order of accuracy splitting formulas for cosine operator function.

If we insert $\gamma_2 = 1 - \alpha$, $\alpha = \gamma_1$ in equation (27), we obtain:

$$(1-\alpha)^{2p} + \alpha^{2p} - \frac{1}{2}(2\alpha - 1)^{2p} = 0. \tag{29}$$

Easily we can show that the equation (29) ($p > 1$) has a real solution. This fact is very important for the case, when A is self-adjoint positive definite operator. As it is known, for this case Euler's generalized formula (1) is valid. It is clear that this formula is not valid for complex t.

After this remark let us return to equation (29). Let us define the right hand side of this equation by $\psi(\alpha)$. Using simple transformations we obtain:

$$\begin{aligned}\psi(1) &= \frac{1}{2} > 0,\\ \psi(2) &= 1 + 2^{2p} - \frac{1}{2}3^{2p}\\ &= 1 + \left(1 - \frac{1}{2}\left(\frac{3}{2}\right)^{2p}\right)2^{2p} \leq -1, \quad p > 1.\end{aligned}$$

Hence it follows that equation (29) has at least one real solution on the interval [1, 2].

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