SOLUTION OF A MIXED PROBLEM OF THE PLANE THEORY OF ELASTIC MIXTURE FOR A DOMAIN WITH A PARTIALLY UNKNOWN BOUNDARY

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(Received: 06.01.10; accepted: 12.05.10)

Abstract

In the present work we consider the problem statics of the plane elastic theory of mixture for a finite doubly-connected domain D whose external and internal boundaries are union of the given linear segments and unknown equally strong arcs.

Using the method of the theory of analytic functions, the portions of equally strong boundaries as well as the stressed state of the body are found.

Key words and phrases: Partially unknown boundary, Elastic mixture, Conformal mapping.

AMS subject classification: 73C35.

1 Introduction

The boundary value problems of the plane theory of elasticity and bending of a plate for an infinite plane weakened by unknown equal strong holes, have investigation in [6]–[8]. In [9], one can find the statement of the problem and the methods of solution of boundary value problems of the plane theory of elasticity for a convex polygon, when normal displacements on the external boundary are piecewise constant and tangential stresses are absent. The obtained results are generalized in [10] and [11].

In the work [1] by R. Bantsuri has been considered the problem of the problem of the plane theory of elasticity for a finite doubly-connected domain whose external and internal boundaries are a union of the given linear segments and unknown equal strong arcs when normal displacements on the segments are piecewise constant, while tangential stress are equal to zero.

In the present work, in the case of the plane theory of elastic mixtures we study the problem analogous to that solved in [1]. For the solution of the problem the use will be made of the generalized Kolosov-Muskhelishvili's formula [2,5] and the method developed in [1] and [5].

2 Some Auxiliary Formulas and Operators

The homogeneous equation of statics of the theory of elastic mixtures in a complex form looks as follows [2]

$$\frac{\partial^2 U}{\partial z \partial \overline{z}} + \mathcal{K} \frac{\partial^2 \overline{U}}{\partial \overline{z}^2} = 0, \tag{2.1}$$

where $z = x_1 + ix_2$, $\overline{z} = x_1 - ix_2$, $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} - i\frac{\partial}{x_2}\right)$ $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x_1} + i\frac{\partial}{x_2}\right)$, $U = (u_1 + iu_2, u_3 + iu_4)^T$, $u' = (u_1, u_2)^T$ and $u'' = (u_3, u_4)^T$ are partial displacements;

$$\mathcal{K} = -\frac{1}{2} e m^{-1}, \quad e = \begin{bmatrix} e_4 & e_5 \\ e_5 & e_6 \end{bmatrix}, \quad m^{-1} = \frac{1}{\Delta_0} \begin{bmatrix} m_3 & -m_2 \\ -m_2 & m_1 \end{bmatrix},$$

$$\Delta_0 = m_1 m_3 - m_2^2, \quad m_k = e_k + \frac{1}{2} e_{3+k}, \quad k = 1, 2, 3, \quad e_1 = a_2/d_2,$$

$$e_2 = -c/d_2, \quad e_3 = a_1/d_2, \quad d_2 = a_1 a_2 - c^2, \quad a_1 = \mu_1 - \lambda_5,$$

$$a_2 = \mu_2 - \lambda_5, \quad c = \mu_3 + \lambda_5, \quad e_1 + e_4 = b/d_1, \quad e_2 + e_5 = -c_0/d_1,$$

$$e_3 + e_6 = a/d_1, \quad a = a_1 + b_1, \quad b = a_2 + b_2, \quad c_0 = c + d, \quad d_1 = ab - c_0^2,$$

$$b_1 = \mu_1 + \lambda_1 + \lambda_5 - \alpha_2 \rho_2/\rho, \quad b_2 = \mu_2 + \lambda_2 + \lambda_5 + \alpha_2 \rho_1/\rho, \quad \alpha_2 = \lambda_3 - \lambda_4,$$

$$\rho = \rho_1 + \rho_2, \quad d = \mu_2 + \lambda_3 - \lambda_5 - \alpha_2 \rho_1/\rho = \mu_3 + \lambda_4 - \lambda_5 + \alpha_2 \rho_2/\rho.$$

Here μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$, are elastic modulus characterizing mechanical properties of the mixture, ρ_1 and ρ_2 partial densities of the mixture. It will be assumed that the elastic constants μ_1, μ_2, μ_3, λ_p, $p = \overline{1,5}$ and partial rigid densities ρ_1 and ρ_2 satisfy the conditions (inequalities) [3].

Let D^+ be a bounded two-dimensional domain (surrounded by the curve L) and D^- be the complement of $\overline{D^+} = D^+ \cup L$. A vector $u = (u_1, u_2, u_3, u_4)^T$ is said to be regular in D^+ $[D^-]$ if $u_k \in C^2(D^+) \cap C^1(\overline{D^+})$ $[u_k \in C^2(D^-) \cap C^1(\overline{D^-})]$ and the second order derivatives of u_k are sumable in D^+ D^-, in the case of the domain D^- we assume, in addition, the following conditions at infinity

$$u_k(x) = O(1), \quad |x|^2 \frac{\partial u_k}{\partial x_j} = O(1), \quad j = 1, 2, \quad k = \overline{1,4}$$

to be fulfilled with $|x|^2 = x_1^2 + x_2^2$.

In [2] and [5] M. Basheleishvili obtained the following representations

$$U = (u_1 + iu_2, u_3 + iu_4)^T = m\varphi(z) + \frac{1}{2} e z \overline{\varphi'(z)} + \overline{\psi(z)}, \tag{2.2}$$

$$TU = ((Tu)_2 - i(Tu)_1, (Tu)_4 - i(Tu)_3)^T =$$

$$= \frac{\partial}{\partial s(x)}[(A-2E)\varphi(z) + Bz\overline{\varphi'(z)} + 2\mu\overline{\psi(z)}], \tag{2.3}$$

where $\varphi(z) = (\varphi_1, \varphi_2)^T$ and $\psi(z) = (\psi_1, \psi_2)^T$ are arbitrary analytic vector-functions

$$A = 2\mu m, \quad \mu = \begin{bmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{bmatrix}, \quad B = \mu e, \quad E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad m = \begin{bmatrix} m_1 & m_2 \\ m_2 & m_3 \end{bmatrix},$$

$\frac{\partial}{\partial s(x)} = -n_2\frac{\partial}{\partial x_1} + n_1\frac{\partial}{\partial x_2}$, $n = (n_1, n_2)^T$, is the unit vector of the outer normal. $(Tu)_p$, $p = \overline{1,4}$, are the stress components [4]

$$(Tu)_1 = r'_{11}n_1 + r'_{21}n_2, \quad (Tu)_2 = r'_{12}n_1 + r'_{22}n_2,$$
$$(Tu)_3 = r''_{11}n_1 + r''_{21}n_2, \quad (Tu)_4 = r''_{12}n_1 + r''_{22}n_2,$$

r'_{kj} and r''_{kj}, $j,k = 1,2$, are stress tensor components [5].

$$r'_{11} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_2}(\mu_1 u_2 + \mu_3 u_4), \; r'_{21} = -a_1\omega' - c\omega'' + 2\frac{\partial}{\partial x_1}(\mu_1 u_2 + \mu_3 u_4),$$
$$r'_{12} = a_1\omega' + c\omega'' + 2\frac{\partial}{\partial x_2}(\mu_1 u_1 + \mu_3 u_3), \; r'_{22} = a\theta' + c_0\theta'' - 2\frac{\partial}{\partial x_1}(\mu_1 u_1 + \mu_3 u_3),$$
$$r''_{11} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_2}(\mu_3 u_2 + \mu_2 u_4), \; r''_{21} = -c\omega' - a_2\omega'' + 2\frac{\partial}{\partial x_1}(\mu_3 u_2 + \mu_2 u_4),$$
$$r''_{12} = c_0\omega' + a_2\omega'' + 2\frac{\partial}{\partial x_2}(\mu_3 u_1 + \mu_2 u_3), \; r''_{22} = c_0\theta' + b\theta'' - 2\frac{\partial}{\partial x_1}(\mu_3 u_1 + \mu_2 u_3),$$
$$\theta' = \operatorname{div} u' = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}, \quad \theta'' = \operatorname{div} u'' = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_4}{\partial x_2},$$
$$\omega' = \operatorname{rot} u' = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}, \quad \omega'' = \operatorname{rot} u'' = \frac{\partial u_4}{\partial x_1} - \frac{\partial u_3}{\partial x_2}.$$

Introduce the vectors [5]

$$\overset{(1)}{\tau} = (r'_{11}, r''_{11})^T, \quad \overset{(2)}{\tau} = (r'_{22}, r''_{22})^T, \quad \tau = \overset{(1)}{\tau} + \overset{(2)}{\tau}, \tag{2.4}$$

$$\overset{(1)}{\eta} = (r'_{21}, r''_{21})^T, \quad \overset{(2)}{\eta} = (r'_{12}, r''_{12})^T, \quad \eta = \overset{(1)}{\eta} + \overset{(2)}{\eta}, \quad \varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta}. \tag{2.5}$$

Elementary calculations result in [5]

$$\tau = \overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B)\operatorname{Re}\varphi'(z), \tag{2.6}$$

$$\varepsilon^* = \overset{(1)}{\eta} - \overset{(2)}{\eta} = 2(A - B - 2E)\operatorname{Im}\varphi'(z), \tag{2.7}$$

$$\overset{(1)}{\tau} - \overset{(2)}{\tau} - i\eta = 2(B\overline{z}\varphi''(z) + 2\mu\psi'(z)), \tag{2.8}$$

here $\det(2E - A - B) > 0$ [5].

Let us consider the rightangular system (**ns**), where s and n are respectively, the tangent and the normal t_0 the curve L. Let α be the angle of inclination of the normal n to the ox_1-axis, and $n = (n_1, n_2)^T = (\cos\alpha, \sin\alpha)^T$, $s^0 = (-n_2, n_1)^T = (-\sin\alpha, \cos\alpha)^T$ be the unit vector of the normal and tangent.

Introduce the vectors

$$\sigma_n = \begin{pmatrix} (Tu)_1 n_1 + (Tu)_2 n_2 \\ (Tu)_3 n_1 + (Tu)_4 n_2 \end{pmatrix} = \overset{(1)}{\tau}\cos^2\alpha + \overset{(2)}{\tau}\sin^2\alpha + \eta\sin\alpha\cos\alpha, \quad (2.9)$$

$$\sigma_s = \begin{pmatrix} (Tu)_2 n_1 - (Tu)_1 n_2 \\ (Tu)_4 n_1 - (Tu)_3 n_2 \end{pmatrix} = \frac{1}{2}(\overset{(2)}{\tau} - \overset{(1)}{\tau})\sin 2\alpha + \frac{1}{2}\eta\cos 2\alpha - \frac{1}{2}\varepsilon^*, \quad (2.10)$$

$$\sigma_s^* = \sigma_s + \frac{1}{2}\varepsilon^*,$$

$$\sigma_t = \begin{pmatrix} [r'_{21} n_1 - r'_{11} n_2, & r'_{22} n_1 - r'_{12} n_2]\xi \\ [r''_{21} n_1 - r''_{11} n_2, & r''_{22} n_1 - r''_{12} n_2]\xi \end{pmatrix} =$$

$$= \overset{(1)}{\tau}\sin^2\alpha + \overset{(2)}{\tau}\cos^2\alpha - \eta\sin\alpha\cos\alpha. \quad (2.11)$$

From (9)–(11) and (6)–(8) on L we obtain

$$\sigma_n + \sigma_t = \overset{(1)}{\tau} + \overset{(2)}{\tau} = 2(2E - A - B)\,\mathrm{Re}\,\varphi'(t), \quad (2.12)$$

$$\sigma_n - i\sigma_s = (2E - A)\overline{\varphi'(t)} - B\varphi'(t) + [B\bar{t}\varphi''(t) + 2\mu\psi'(t)]e^{2i\alpha}. \quad (2.13)$$

After elementary calculations we obtain

$$\sigma_n + 2\mu\Big(\frac{\partial U_s}{\partial s} + \frac{U_n}{\rho_0}\Big) + i\Big[\sigma_s - 2\mu\Big(\frac{\partial U_n}{\partial s} - \frac{U_s}{\rho_0}\Big)\Big] = 2\varphi'(t), \quad (2.14)$$

where $1/\rho_0$ is the curvature of the curve L at the point t

$$2\mu U_n = 2\mu\begin{pmatrix} u_1 n_1 + u_2 n_2 \\ u_3 n_1 + u_4 n_2 \end{pmatrix} = \mathrm{Re}\, e^{-i\alpha(t)}(A\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}), \quad (2.15)$$

$$2\mu U_s = 2\mu\begin{pmatrix} u_2 n_1 - u_1 n_2 \\ u_4 n_1 - u_3 n_2 \end{pmatrix}. \quad (2.16)$$

Direct calculations allow us to verify that on L

$$(\sigma_n + i\sigma_s)e^{i\alpha} = i\frac{\partial}{\partial s}[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}], \quad (2.17)$$

whence is follows that

$$[(A - 2E)\varphi(t) + Bt\overline{\varphi'(t)} + 2\mu\overline{\psi(t)}]_L = -i\int_L (\sigma_n + i\sigma_s)e^{i\alpha}ds. \quad (2.18)$$

Formulas (2), (3), (6), (8) and (12)–(14) are analogous to the Kolosov-Muskhelishvili's formulas for the linear theory of elastic mixtures.

3 Statement of the Problem and the Method of its Solving

In the present work we study an analogous problem which in the case of the plane theory of elasticity has been studied by R. Bantsuri ([1]). To solve the problem we use the formulas due to Kolocow-Muskelishvili and the method described in [1].

Let an isotropic elastic body of the elastic mixture on an a complex plane $z = x_1 + ix_2$ occupy a finite doubly-connected domain D whose external boundary consists of segments and unknown smooth arcs, while the internal boundary consists of unknown smooth arcs and segments joining them.

The point $z = 0$ is assumed to lie inside of the internal boundary of the domain D.

A union of segments of the external boundary we denote by L_1 and those of the internal boundary are denoted by L_2. A union of the smooth arcs of the external and internal boundary will be denoted by L_3 and L_4 respectively.

The ends of the segments contained in L_1 counted in the positive direction are denoted by A_k, $k = \overline{1, n}$, while those contained in L_2 are denoted by A_{n+k}, $k = \overline{1, m}$.

For the sake of simplicity, the geometrical point A_k and its affix will be denoted by the same symbol. The arc abscissa of the point $t \in L_1 \cup L_3$ counted from the point A_1 we denote by s_1 and the arc abscissas of the point A_k we denote by s_k, $k = \overline{1, n}$.

The arc abscissa of the point $t \in L_2 \cup L_4$ counted from the point A_{n+1} in the positive direction will also be denoted by s_{n+1} and that of the point A_{n+k} by s_{n+k}. Assume that A_1A_2 and $A_{n+1}A_{n+2}$ are the segments. A size of the angle θ_j at the point A_j, $j = \overline{1, m+n}$ is required to satisfy the condition $0 < \theta_j < \pi$. Suppose that the $\sigma_s = 0$ on the whole boundary and $\sigma_n = 0$ on the unknown arcs, also the vector U_n takes constant value on the segments.

We formulate the problems as follows: Find a stressed state of the body and unknown arcs under the condition that the σ_t vector takes on these arcs constant value $\sigma_t = -K^0 = -(k_1^0, k_2^0)^T = \text{const}$.

We consider two cases.

(1) The values of the constants U_n on the segments are known.

(2) The values of the principal vector of outer forces applied to every segment are known.

On the basis of analogous Kolosov-Muskelishvili's formulas (2), (3), (6), (12), (14) and (18), the above-posed problem is reduced to finding two analytic vector-functions $\varphi(z)$ and $\psi(z)$ in the domain D by boundary conditions

$$\operatorname{Re}([(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}]e^{-i\alpha(t)})=C(t),\quad t\in L_1\cup L_2, \tag{3.1}$$

$$(A-2E)\varphi(t)+Bt\overline{\varphi'(t)}+2\mu\overline{\psi(t)}=B^0(t),\quad t\in L_3\cup L_4, \tag{3.2}$$

$$\operatorname{Re}\varphi'(t)=H,\quad H=-\frac{1}{2}(2E-A-B)^{-1}K^0,\quad t\in L_3\cup L_4, \tag{3.3}$$

$$\operatorname{Im}\varphi'(t)=0,\quad t\in L_1\cup L_2, \tag{3.4}$$

where $\alpha(t)$ is the size of the angle made by the outer normal n and ox_1-axis; $K^0=(K_1^0,K_2^0)^T$ is to be defined when solving the problem.

$$C(t)=-\operatorname{Re}\left[i\int_0^s\sigma_n(t_0)\exp i(\alpha(t_0)-\alpha(t))ds_0\right],\quad t\in L_1\cup L_2, \tag{3.5}$$

$$B^0(t)=-i\int_0^s\sigma_n(t_0)\exp i\alpha(t_0)ds_0,\quad t\in L_3\cup L_4. \tag{3.6}$$

Taking into account that $\alpha(t)$ is the piecewise function $\alpha(t)=\alpha_k$ for $t\in L_1\cup L_2$, and $\sigma_n(t)=0$, for $t\in L_3\cup L_4$, we obtain

$$C(t)=C_k-\sum_{j=1}^{k}P_j\sin(\alpha_k-\alpha_j),\quad t\in A_kA_{k+1}\subset L_1,$$

where k and j take only those values under which A_kA_{k+1} are the segments.

It is easily seen from (15) and (19) that both cases reduce to the same problems of the theory of functions of complex variable.

We consider the second case, i. e. the case in which the values of the principal vector outer forces are given on the segments A_kA_{k+1}.

$$P_j=-\int_{s_i}^{s_{j+1}}\sigma_n(t)ds,\quad \text{for}\quad 1\le j\le n-1,$$

when A_j is the common point of the arc and of the segment.

If A_j is the common point of two segments, then

$$P_j=-\int_{s_j}^{s_{j+1}}\sigma_n(t)ds,\quad \text{for}\quad 1\le j\le n-1,$$

$P_n=-\int_{s_n}^{l}\sigma_n(t)ds$, where l is length of the external boundary.

$$B^0(t)=B_k^0=\sum_{j=1}^{k}P_je^{i\alpha_j},\quad t\in A_{k+1}A_{k+3}\in L_3,$$

$C(t) = C_{n+k} = \sum_{j=1}^{k} P_{n+j} sin(\alpha_{n+k} - \alpha_{n+j})$, $t \in A_{k+n}A_{k+n+1} \in L_2$, $k = 1, 3, 5, \ldots, m$. Here k and j take only odd values;

$$B^0(t) = B^0_{k+n} = \sum_{j=1}^{k-1} \int_{s_j}^{s_{j+1}} \sigma_n(t_0) e^{i\alpha_k} ds_0, \quad t \in A_{n+k}A_{n+k+1} \in L_4$$

k are even j are odd.

Assume that forces applied both to the external boundary $L \cup L_3$ and to the internal boundary $L_2 \cup L_4$ are self-balancing

$$\sum_{j=1}^{n} P_j e^{i\alpha_j} = 0, \quad \sum_{j=1}^{m} P_{j+n} e^{i\alpha_{j+n}} = 0. \tag{3.7}$$

Under the assumption $0 \leq \theta_j < \pi$, the vector $\varphi(z)$ are continuously extendable on the boundary of the domain D except possible for the points A_k, $k = \overline{1, m+n}$ in the neighborhood of which they admit the estimate of the type

$$|\varphi_j'(z)|, \quad |\psi_j(z)| < M|z - A_k|^{-\delta}, \quad j = 1, 2. \tag{3.8}$$

where $0 \leq \delta < 1$, if A_k is the common point of the two the segments, $0 \leq \delta < 1/2$, A_k is the common of the arc and the segment.

Equalities (21) and (22) is the Keldysh-Sedov problem in a vector form for the domain D. It is proved that the problem has a unique solution $\varphi'(z) = H$.

Leaving the constant summand vector out of account, we obtain

$$\varphi(z) = Hz, \quad H = -0,5(2E - A - B)^{-1} K^0. \tag{3.9}$$

Substituting the obtained value $\varphi(z)$ into the boundary conditions (19) and (20), we find that

$$\text{Re}\left(e^{-\alpha(t)}\left(\frac{1}{2}K^0 t + 2\mu\overline{\psi(t)}\right)\right) = c(t), \quad t \in L_1 \cup L_2, \tag{3.10}$$

$$\frac{1}{2}K^0 t + 2\mu\overline{\psi(t)} = B^0(t), \quad t \in L_3 \cup L_4. \tag{3.11}$$

If $t \in A_k A_{k+1} \in L_1 \cup L_2$, then $(t - A_k) = i|t - A_k|e^{i\alpha_k}$, whence

$$\text{Re}\, te^{-i\alpha(t)} = \text{Re}\, e^{-i\alpha(t)} A(t), \quad t \in L_1 \cup L_2, \tag{3.12}$$

where $A(t) = A_k$, for $t \in A_k A_{k+1}$, $k = \overline{1, n}$

Let the function $z = \omega(\varsigma)$ map conformally a circular ring $1 < |\varsigma| < R$ onto the domain D, where R is an unknown number, we have to define.

Assume that the circumference $|\varsigma| = R$ is mapped onto the curve $L_1 \cup L_3$. By a_k we denote the image A_k, i.e. $a_k = \omega^{-1}(A_k)$,

$$a_k = \mathrm{R}e^{i\delta_k}, \quad k = \overline{1,n}, \quad 0 = \delta_1 < \delta_2 < \delta_3 < \cdots \delta_n < 2\pi,$$
$$a_{n+k} = e^{i\delta_{n+k}}, \quad k = \overline{1,m}, \quad -\pi < \delta_{n+m} < \delta_{n+m-1} < \cdots < \delta_{n+2} < \delta_{n+1} \leq \pi$$

δ_k are unknown numbers for $k = \overline{2, m+n}$.

Denote by l_k the image L_k, i.e. $l_k = \omega^{-1}(L_k)$.

It follows from (28)–(30) that

$$\mathrm{Re}(e^{-i\alpha(\sigma)}\psi_0(\sigma)) = C(\sigma) - \frac{1}{2}\mathrm{Re}(K^0 A(\sigma)e^{-i\alpha(\sigma)}), \quad \sigma \in l_1 \cup l_2, \tag{3.13}$$
$$\mathrm{Re}(e^{-i\alpha(\sigma)}\omega(\sigma)) = \mathrm{Re}\, A(\sigma)e^{-i\alpha(\sigma)}, \quad \sigma \in l_1 \cup l_2, \tag{3.14}$$
$$\frac{1}{2}K^0\omega(\sigma) + \overline{\psi_0(\sigma)} = B^0(\sigma), \quad \sigma \in L_3 \cup l_4, \tag{3.15}$$

where $\psi_0(\zeta) = 2\mu\psi(\omega(\zeta))$, $1 < |\zeta| < R$.

Since $\alpha(\sigma)$, $A(\sigma)$, $C(\sigma)$ are piecewise constants, the $\alpha(\omega(\sigma))$, $A(\omega(\sigma))$, $C(\omega(\sigma))$ can be defined on the whole plane by means of the equalities

$$\alpha(r\sigma) = \alpha(\sigma), \quad A(r\sigma) = A(\sigma), \quad C(r\sigma) = C(\sigma), \quad 0 < r < \infty, \quad |\sigma| = 1.$$

Let $S = \{1 < |\zeta| < R \cup R < |\zeta| < R^2\}$, and introduce into consideration the arcs γ_k, $k = \overline{1,4}$, which are defined by the equalities

$$\gamma_1 = \{\sigma : R\sigma \in l_1\}, \quad \gamma_2 = \{\sigma : \sigma/R \in l_2\},$$
$$\gamma_3 = \{\sigma : R\sigma \in l_3\}, \quad \gamma_4 = \{\sigma : \frac{\sigma}{R} \in l_4\}.$$

It is seen that

$$\gamma_1 \in |\sigma| = 1, \quad \gamma_2 \in |\sigma| = R, \quad \gamma_3 \in |\sigma| = 1, \quad \gamma_4 \in |\sigma| = R.$$

Introduce the vector-function

$$W(\zeta) = F(\zeta) + \begin{cases} \frac{1}{2}K^0\overline{\omega(\frac{\zeta}{R})}, & R < |\zeta| < R^2, \\ -\overline{\psi_0(\frac{R}{\zeta})}, & 1 < |\zeta| < R, \end{cases} \tag{3.16}$$

where

$$F(\zeta) = \frac{1}{2\pi i}\int_{\gamma_2 \cup \gamma_4} \frac{f(\sigma)d\sigma}{\sigma - \zeta},$$

$$f(\sigma) = \begin{cases} B^0(\sigma), \quad \sigma \in (b_j, d_j) \in \gamma_4, \\ (b_{j+1} - d_j)^{-1}[(\sigma - d_j)B^0(b_{j+1}) - (\sigma - b_{j+1})B^0(d_j)], \\ \sigma \in (d_j, b_{j+1}) \in \gamma_2, \quad j = 1, 2, 3, \ldots, \frac{1}{2}m. \end{cases}$$

Clearly the vector-function f is of Hölder class.

It can be easily shown that $F^+(\sigma_0) - F^-(\sigma_0) = f(\sigma_0)$, $\sigma_0 \in \gamma_2 \cup \gamma_4$. $F^{\pm}(\sigma_0)$ are the boundary values of the vector-function $F(\zeta)$ from the left and from the right.

From (34) we have

$$\begin{aligned} \frac{1}{2}K^0\omega(R\sigma) &= W(R^2\sigma) - F(R^2\sigma), \quad \sigma \in \gamma_1 \cup \gamma_3, \\ \overline{\psi_0(R\sigma)} &= -W(\sigma) + F(\sigma), \quad \sigma \in \gamma_1 \cup \gamma_3, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \frac{1}{2}K^0\omega(\sigma) &= W^-(\sigma) - F^-(\sigma), \quad \sigma \in \gamma_2, \\ \overline{\psi_0(\sigma)} &= -W^+(\sigma) + F^+(\sigma), \quad \sigma \in \gamma_2. \end{aligned} \tag{3.18}$$

Moreover, taking into account (33) and $F^+(\sigma) - F^-(\sigma) = B^0(\sigma)$, $\sigma \in \gamma_4$ from (34) it follows that

$$W^+(\sigma) - W^-(\sigma) = 0, \quad \sigma \in \gamma_4,$$

e.i. the vector-function $W(\zeta)$ is analytic in the ring $1 < |\zeta| < R$, cut along γ_2.

Taking equalities (35) and (36) into account the condition (31)–(33) yield

$$\mathrm{Re}(e^{-i\alpha(\sigma)}W(R^2\sigma)) = \mathrm{Re}[(F(R^2\sigma) + \frac{1}{2}K^0A(\sigma))e^{-i\alpha(\sigma)}], \quad \sigma \in \gamma_1, \tag{3.19}$$

$$\mathrm{Re}(e^{-i\alpha(\sigma)}W(\sigma)) = \mathrm{Re}[(F(\sigma) + \frac{1}{2}K^0A(\sigma))e^{-i\alpha(\sigma)}] - C(\sigma), \quad \sigma \in \gamma_1, \tag{3.20}$$

$$\begin{aligned} \mathrm{Re}(e^{-i\alpha(\sigma)}W^+(\sigma)) &= \mathrm{Re}[(F^+(\sigma) + \frac{1}{2}K^0A(\sigma))e^{-i\alpha(\sigma)}] - C(\sigma), \\ \mathrm{Re}(e^{-i\alpha(\sigma)}W^-(\sigma)) &= \mathrm{Re}[(F^-(\sigma) + \frac{1}{2}K^0A(\sigma))e^{-i\alpha(\sigma)}], \quad \sigma \in \gamma_2, \end{aligned} \tag{3.21}$$

$$W(R^2\sigma) - W(\sigma) = F(R^2\sigma) - F(\sigma) + B^0(\sigma), \quad \sigma \in \gamma_3. \tag{3.22}$$

Thus we have obtained the boundary value problem of the theory of analytic functions with the condition of the Riemann-Hilbert problem on one part of the boundary and the condition of Carleman type problem on other part of the boundary.

In a general statement, the construction of the above problem is connected with insuperable difficulties. In some particular cases, one managed to obtain the solution of the problem under consideration in quadratures.

Assume that the segment composing $L_1 \cup L_2$ are parallel to one of the two arbitrarily given mutually perpendicular directions. We may assume that directions of the coordinate axes coincide with the those directions.

Denote by γ_{1k} a collection of those arcs of γ_k which correspond to the segments parallel to the ox_2-axis, and by γ_{2k} those which correspond to the segments parallel to the ox_1-axis.

Under these assumptions, the boundary value conditions (37)–(39) can be rewritten as follows:

$$\operatorname{Re} W(\sigma) = g_1(\sigma), \quad \operatorname{Re} W(R^2\sigma) = g_1^0(\sigma), \quad \sigma \in \gamma_{11}, \tag{3.23}$$

$$\operatorname{Im} W(\sigma) = g_2(\sigma), \quad \operatorname{Im} W(R^2\sigma) = g_2^0(\sigma), \quad \sigma \in \gamma_{21}, \tag{3.24}$$

$$\operatorname{Re} W^{\pm}(\sigma) = h_1^{\pm}(\sigma), \quad \sigma \in \gamma_{12}, \tag{3.25}$$

$$\operatorname{Im} W^{\pm}(\sigma) = h_2^{\pm}(\sigma), \quad \sigma \in \gamma_{22}, \tag{3.26}$$

$$W(R^2\sigma) - W(\sigma) = g_3(\sigma), \quad \sigma \in \gamma_3, \tag{3.27}$$

where

$$g_1(\sigma) = \operatorname{Re}[F(\sigma) + A_1(\sigma)] - \lambda C(\sigma), \quad g_1^0(\sigma) = \operatorname{Re}[F(R^2\sigma) + A_1(\sigma)],$$
$$g_2(\sigma) = \operatorname{Im}[F(\sigma) + A_1(\sigma)] - \delta C(\sigma), \quad g_2^0(\sigma) = \operatorname{Im}[F(R^2\sigma) + A_1(\sigma)],$$
$$h_1^+(\sigma) = \operatorname{Re}[F^+(\sigma) + A_1(\sigma)] - \lambda C(\sigma), \quad h_1^-(\sigma) = \operatorname{Re}[F^-(\sigma) + A_1(\sigma)],$$
$$h_2^+(\sigma) = \operatorname{Im}[F^+(\sigma) + A_1(\sigma)] - \delta C(\sigma), \quad h_2^-(\sigma) = \operatorname{Im}[F^-(\sigma) + A_1(\sigma)],$$
$$g_3(\sigma) = F(R^2\sigma) - F(\sigma) + B^0(\sigma), \quad A_1(\sigma) = \frac{1}{2}K^0 A(\sigma),$$
$$\lambda = \begin{cases} 1, & \text{for } \alpha(\sigma) = 0 \\ -1, & \text{for } \alpha(\sigma) = \pi, \end{cases} \quad \delta = \begin{cases} 1, & \text{for } \alpha(\sigma) = \frac{\pi}{2}, \\ -1, & \text{for } \alpha(\sigma) = \frac{3}{2}\pi. \end{cases}$$

The conditions (41)–(45) is the Keldysh-Sedok Carleman problem in a vector form.

To solve the problem we use method R. Bantsuri described in [1].

We rewrite boundary conditions (43) and (44) as follows:

$$\frac{W^+(\sigma) \pm \overline{W^-(\sigma)}}{2} \pm \frac{W^-(\sigma) \pm \overline{W^+(\sigma)}}{2} = h_1^+(\sigma) \pm h_1^-(\sigma), \quad \sigma \in \gamma_{12}, \tag{3.28}$$

$$\frac{W^+(\sigma) \pm \overline{W^-(\sigma)}}{2} \mp \frac{W^-(\sigma) \pm \overline{W^+(\sigma)}}{2} = i(h_2^+(\sigma) \pm h_2^-(\sigma)), \ \sigma \in \gamma_{22}. \tag{3.29}$$

The vector-function $\overline{W^-(\sigma)}$ $\overline{W^+(\sigma)}$ is the boundary value of the holomorphic in $1 < |\zeta| < R^2$ vector-function $\overline{\phi(R^2/\overline{\zeta})}$ on the left (right) contour of the cut γ_{k2}, $k = 1, 2$.

We introduce into consideration the vector-functions $H_1(\zeta)$ and $H_2(\zeta)$ which are defined by the equalities [1],

$$H_1(\zeta) = \frac{W(\zeta) + \overline{W(R^2/\overline{\zeta})}}{2}, \quad H_2(\zeta) = i\frac{W(\zeta) - \overline{W(R^2/\overline{\zeta})}}{2}. \tag{3.30}$$

It is seen that the vector-functions H_1 and H_2 must satisfy the conditions

$$H_k(\zeta) = \overline{H_k(R^2/\bar{\zeta})}, \quad k = 1, 2. \tag{3.31}$$

Boundary conditions (46) and (47) can be expressed in terms of the vector-function $H_k(\zeta)$, $k = 1, 2$, as follows:

$$\begin{aligned} H_1^+(\sigma) + H_1^-(\sigma) &= h_1^+(\sigma) + h_1^-(\sigma), \quad \sigma \in \gamma_{12}, \\ H_1^+(\sigma) - H_1^-(\sigma) &= i(h_2^+(\sigma) - h_2^-(\sigma)), \quad \sigma \in \gamma_{22}, \end{aligned} \tag{3.32}$$

$$\begin{aligned} H_2^+(\sigma) + H_2^-(\sigma) &= i(h_1^+(\sigma) - h_1^-(\sigma)), \quad \sigma \in \gamma_{12}, \\ H_2^+(\sigma) + H_2^-(\sigma) &= -(h_2^+(\sigma) + h_2^-(\sigma)), \quad \sigma \in \gamma_{22}. \end{aligned} \tag{3.33}$$

By means of analogous transformation from (41), (42) and (45), on the basic of (48), with regard for (49) we obtain

$$\begin{aligned} H_1(R^2\sigma) + H_1(\sigma) &= g_1(\sigma) + g_1^0(\sigma), \quad \sigma \in \gamma_{11}, \\ H_1(R^2\sigma) - H_1(\sigma) &= i(g_2^0(\sigma) - g_2(\sigma)), \quad \sigma \in \gamma_{21}, \\ H_1(R^2\sigma) - H_1(\sigma) &= i \operatorname{Im} g_3(\sigma), \quad \sigma \in \gamma_3, \end{aligned} \tag{3.34}$$

$$\begin{aligned} H_2(R^2\sigma) + H_2(\sigma) &= -(g_2(\sigma) + g_2^0(\sigma)), \quad \sigma \in \gamma_{21}, \\ H_2(R^2\sigma) - H_2(\sigma) &= i(g_1^0(\sigma) - g_1(\sigma)), \quad \sigma \in \gamma_{11}, \\ H_2(R^2\sigma) - H_2(\sigma) &= i \operatorname{Re} g_3(\sigma), \quad \sigma \in \gamma_3. \end{aligned} \tag{3.35}$$

Boundary conditions (50) and (51) can be rewritten as

$$\begin{aligned} H_k^+(\sigma) &= (-1)^{k+j-1} H_k^-(\sigma) + h_k^0(\sigma), \quad \sigma \in \gamma_{12} \cup \gamma_{22}, \\ h_k^0(\sigma) &= (h_j^+(\sigma) + (-1)^{k+j-2} h_j^-(\sigma)) i^{k+j-2}, \quad \sigma \in \gamma_{j2}, \end{aligned} \tag{3.36}$$

where $k, j = 1, 2$.

The boundary conditions (52) and (53) are reduced to the conditions:

$$\begin{aligned} H_k(R^2\sigma) &= G_k(\sigma) H_k(\sigma) + F_k^0(\sigma), \quad \sigma \in \gamma_{11} \cup \gamma_{21} \cup \gamma_3, \\ G_k(\sigma) &= -1, \quad \sigma \in \gamma_{k1}, \quad G_k(\sigma) = 1, \quad \sigma \in \gamma_{(3-k)1} \cup \gamma_3, \\ F_k^0(\sigma) &= \begin{cases} (g_j(\sigma) + (-1)^{k+j-2} g_j^0(\sigma)) i^{k+j-2}, & \sigma \in \gamma_{j1} \\ \frac{1}{2}(g_3(\sigma) + (-1)^k \overline{g_3(\sigma)}) i^{k-1}, & \sigma \in \gamma_3, \end{cases} \end{aligned} \tag{3.37}$$

where $k, j = 1, 2$.

The solution of the problem (54) can be represented as (see [1])

$$H_k(\zeta) = X_k(\zeta)[\phi_k(\zeta) + V_k(\zeta)], \quad k = 1, 2; \tag{3.38}$$

where $V_k(\zeta)$, $k=1,2$, is a new unknown holomorphic vector-function in the ring $1<|\zeta|<R^2$, and satisfying the conditions (see [1])

$$\overline{V_k(R^2/\overline{\zeta})} = \begin{cases} V_k(\zeta), & \text{for even} \quad p_k, \\ (R/\zeta)V_k(\zeta), & \text{for odd} \quad p_k, \end{cases} \tag{3.39}$$

p_k is a number of arcs contained in γ_{k2}, $k=1,2$,

$$\phi_k(\zeta) = \begin{cases} I_k(\zeta) - \frac{1}{2}I_k(0), & \text{for even} \quad p_k, \\ \\ I_k(\zeta) - \frac{1}{2}I_k(o) - \frac{1}{2}I_k'(o)\zeta, & \text{for odd} \quad p_k, \end{cases}$$

$$I_k(\zeta) = \frac{1}{2\pi i}\int_\gamma \frac{h_k^0(\sigma)d\sigma}{\chi_k^+(\sigma)(\sigma-\zeta)}, \quad \gamma = \gamma_{12}\cup\gamma_{22},$$

$$\chi_k(\zeta) = \begin{cases} e^{i\beta_k}\chi_{ok}(\zeta)\zeta^{-\frac{1}{2}P_k}, & \text{for even} \quad p_k, \\ e^{i\beta_k}\chi_{ok}(\zeta)\zeta^{-\frac{1+P_k}{2}}, & \text{for odd} \quad p_k, \end{cases}$$

$$\chi_{ok}(\zeta) = \prod_{j=1}^{p_k}[(\zeta-a_{kj})(\zeta-b_{kj})]^{\frac{1}{2}}, \quad R^{2p_k}\prod_{j=1}^{P_k}(a_{kj}b_{kj})^{-\frac{1}{2}} = e^{2\beta_k},$$

a_{kj}, b_{kj} are the ends of arcs contained in γ_{k2}, $k=1,2$; $j=\overline{1,p_k}$.

Substituting the values of the vector-functions $H_k(\zeta)$ defined by equalities (56) into (55), we obtain

$$\begin{gathered} V_k(R^2\sigma) = G_k^*(\sigma)V_k(\sigma) + Q_k(\sigma), \quad \sigma\in\gamma_{11}\cup\gamma_{21}\cup\gamma_3, \\ G_k^*(\sigma) = G_k(\sigma)\chi_k(\sigma)/\chi_k(R^2\sigma), \\ Q_k(\sigma) = (Q_{k1},Q_{k2})^T = C_k^*(\sigma)\phi_k(\sigma) - \phi_k(R^2\sigma) + F_k^0(\sigma)/\chi_k(R^2\sigma). \end{gathered} \tag{3.40}$$

Thus the problem under consideration is reduced to the Carleman type problem (58) with supplementary conditions (57), (see [1]).

The index of the problem (58) in the class of bounded vector-functions is equal to $-(p_k+q_k)$, $k=1,2$, where q_k is a number of arcs contained in γ_{k1}.

The solution of the problem (58) has the form ([1])

$$V_k(\zeta) = \frac{\chi_k^*(\zeta)}{2\pi i}\int_\gamma \frac{\mathcal{K}_{\nu k}(\zeta/\sigma)Q_k(\sigma)d\sigma}{\chi_k^*(R^2\sigma)\sigma}, \quad \gamma = \gamma_{21}\cup\gamma_{11}\cup\gamma_3, \tag{3.41}$$

satisfying the condition (57)

$$\mathcal{K}_{\nu k}(\zeta) = \frac{R^2}{R^2-\zeta} + \frac{1}{\nu_k(1-\zeta)} + \mathcal{K}_{\nu k}^0(\zeta), \tag{3.42}$$

where

$$\mathcal{K}^0_{\nu k}(\zeta) = \nu_k \sum_{n\geq 1} \frac{1}{R^{2n}-\nu_k}\Big(\frac{\zeta}{R^2}\Big)^n +$$

$$+\frac{1}{\nu_k}\sum_{n\leq -1}\frac{R^{2n}\zeta^n}{R^{2n}-\nu_k} + \begin{cases} \frac{\nu_k}{1-\nu_k}, & \nu_k \neq 1 \\ 0, & \nu_k = 1. \end{cases}$$

$\mathcal{K}^0_{\nu k}$ is holomorphic in the ring $1/R^2 < |\zeta| < R^4$

$$\chi^*_k(\zeta) = \omega_k(\zeta)\exp\left(\frac{1}{2\pi i}\int_\gamma \mathcal{K}_1\Big(\frac{\zeta}{\sigma}\Big)\ln\frac{G_{ok}(\sigma)}{\nu k}\frac{d\sigma}{\sigma}\right)$$

$$\nu_k = \exp\Big(\frac{1}{2\pi i}\int_\gamma \ln G_{0k}(\sigma)\frac{d\sigma}{\sigma}\Big),$$

$$G^{(\sigma)}_{0k} = G^*_k(\sigma)\omega_k(\sigma)/\omega_k(R^2\sigma)$$

$$\omega_k(\zeta) = (\zeta - Re^{i\beta})^{\varkappa_{0k}}\zeta^{-\delta(\varkappa_0 k)}e^{-i\beta\frac{\varkappa_{0k}}{2}}$$

β is a fixed number, $0<\beta<2\pi$, such that $\mathrm{Re}^{i\beta}$ does not coincide with the points d_j, b_j, $j=\overline{1,\ 0,5m}$

$$\delta(\varkappa_{0k}) = \begin{cases} 0,5\varkappa_{0k} & \text{for even} \quad \varkappa_{ok}, \\ \\ 0,5(\varkappa_{0k}-1), & \text{for odd} \quad \varkappa_{0k}, \end{cases}$$

where $\varkappa_{0k} = -(p_k+q_k), \quad k=1,2.$

For $\varkappa_{0k} = -(p_k+q_k)$ the function χ^*_k has at the point $\zeta = \mathrm{Re}^{i\beta}$ a pole order p_k+q_k, $k=1,2$. In this case the bounded solutions exist only under the condition

$$\int_\gamma \frac{d^q\mathcal{K}_{\nu_k}(\zeta/\sigma)}{d\zeta^q}\frac{Q_{k1}(\sigma)}{\sigma\chi^*_k(R^2\sigma)}d\sigma = 0, \tag{3.43}$$

$$\int_{\gamma_1} \frac{d^q\mathcal{K}_{\nu_j}(\zeta/\sigma)}{d\zeta^q}\frac{Q_{k2}(\sigma)}{\sigma\chi^*_k(R^2\sigma)}d\sigma = 0, \tag{3.44}$$

where $\zeta = e^{i\beta}$, $q=\overline{0,p_k+q_k-1}$, $k=1,2$, Q_{k1} and Q_{k2} are of the $Q_k = (Q_{k1},Q_{k2})^T$ vector components (see. [58]). (61), ((62)) represents a system of equations involving unknown parameters R, δ_{k0}, $k_0=\overline{2,m+n}$.

When the arcs $A_kA_{k+1}\subset L_3\cup L_4$ are unknown, they may have different positions on definite rays with vertices at the points A_{k-1}, A_{k+2}. Fixing the length of the arc A_kA_{k+1}, we obtain the condition which along with the conditions (61), ((62)), represent a system of $m+n$ equations with respect to R, δ_2, $\delta_3,\ldots,\delta_{m+n}$.

Thus for unknown parameters R, δ_2, $\delta_3, \dots, \delta_{m+n}$, we have system of $m+n+p_k+q_k$ equations, $(k=1,2)$. On the other hand note that, the obtained system may, in general, be unsolvable if the ends of unknown arcs and forces $P_l = (P_{l1,P_{l2}})^T$, $l = \overline{1, n' + \frac{1}{2}m}$, $(n' + \frac{1}{2}m$ is a number of segments), applied to the punch are fixed.

Leaving them in the obtained system unfixed, by the choice of parameters we can achieve is solvability.

Having known the vector-function $V_k(\zeta)$ we can define the unknown vector-functions $H_k(\zeta)$ and $W(\zeta)$, by formulas (56) and (48). Finally, from (34) we find $\omega(\zeta)$ and $\psi_0(\zeta)$. Hence we can define the stressed state of the body and of the unknown parts of the boundary domain D.

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