ON UNIFORM APPROXIMATION OF CAUCHY TYPE INTEGRALS ON CLOSED CONTOURS OF INTEGRATION

J. Sanikidze*, K. Ninidze**

N. Muskhelishvili Institute of Computational Mathematics*
CAkuri str. 8a, 0171, Tbilisi, Georgia
Sokhumi State University**
9, Politkovskaya, Tbilisi, Georgia

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Abstract

Certain quadrature processes are considered for integrals with kernels $(t-z)^{-1}$, $(t-z)^{-2}$ along piece-wise smooth closed contours, bounding finite or infinite domain D involving z. Uniform estimates are given for the corresponding remainder terms namely for the case of arbitrary closeness of z to the boundary of the domain.

Key words and phrases: Cauchy type integral, quadrature formulas, uniform approximation, error estimates.

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1 Introduction

The necessity of calculation of Cauchy type integrals mostly arises at numerical solution of a number of problems, usually connected with boundary problems of the theory of analytical functions of complex variable, both in the case of finite and infinite domains. To such problems belong, e.g. classical and modified problems (see [1,2]) of the theory of harmonic functions, problems of the plane theory of elasticity and other similar problems, which are reduced to boundary integral equations both with regular and singular kernels. Usually the final step at solving such problems, assuming that the solution to the corresponding boundary integral equation is already found, consists in computation of Cauchy type integrals and their derivatives (whose densities are connected in a known way with the solution of the mentioned integral equations). Such situation occurs e.g. at solving the basic problems of elasticity theory by the method of boundary integral equations (see, e.g. [2,3]). Namely, the basic characteristics of the solution to the problem – the components of vectors of stress and displacement are expressed via certain Cauchy type integrals (so-called complex potentials), connected with the solution of the corresponding boundary integral equation in a known way.

The main considerations of the present paper concern the question of uniform approximation of integrals of the form

$$I(\varphi; z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t)dt}{t - z} \quad (z \in D)$$

and also the derivatives of such integrals for values z from some finite or infinite domain D of complex plane with boundary L, representing a smooth (generally piece-wise smooth) closed oriented contour. Concerning the derivatives of such integrals it should be noted, that though the derivatives of the Cauchy type integrals may be reduced to integrals with the same kernel $(t - z)^{-1}$, using elementary transformation, however, from the viewpoint of obtaining the provided quadrature formulas and estimates, it's more preferable to consider the corresponding derivatives in the form of Cauchy type integral with the initial density and differentiated kernel at this. Namely, using such consideration, it is not necessary to apply formulas of numerical differentiation with approximate values of function φ found by means of solution of this or that boundary problem. By that, in the present paper along with $I(\varphi; x)$ to the similar extent we will also consider the integral

$$I'(\varphi; z) = \frac{1}{2\pi i} \int_L \frac{\varphi(t)dt}{(t - z)^2} \quad (z \in D).$$

2 On construction of approximation formula

First of all, before proceeding to the question of construction of supposed calculation schemes (quadrature formulas) for the indicated integrals, we note that for the fixed values of z in domain D the corresponding integrals may be calculated applying directly ordinary quadrature formulas. However, it is clear that in the general case the accuracy of such formulas decreases perceptibly as z tends to boundary points of the domain (to points of contour L). Meanwhile such situations represent the biggest interest from the viewpoint of applications. In the present paper, an approximate calculation scheme for Cauchy type integrals, based on a certain special process of approximation of their densities, is offered. The mentioned schemes bring to relatively conveniently realizable calculation processes and at this enable us to get uniform (with respect to z) error estimates in the whole domain D, among them, for arbitrarily closeness of point z to the boundary L. Realization of such estimates is carried out by appropriate variation of certain variable parameters, generated by the very structure of approximating expressions constructed here.

We introduce a system of points $\{\tau_j\}_{j=1}^{n+1}$ $(n > 1,\ \tau_{n+1} = \tau_1)$ on L partitioning the given contour into arcs $\tau_\sigma\tau_{\sigma+1}$; $\sigma = \overline{1,n}$ (it is meant that increase of index j corresponds to positive direction on L). Depending on the way of definition of L by this or that equation or graphically (which usually may take place in applied problems) we will consider the given partitioning possibly close to an uniform one. in other words, if s_σ is a length of arc $\tau_\sigma\tau_{\sigma+1}$, then for some positive (independent of n) constants q_1, q_2 $(q_1 < q_2)$ independently of the values σ $(1 \le \sigma \le n)$ condition $(q_1 h \le s_\sigma \le q_2 h)$ for $h = \dfrac{l}{n}$ is fulfilled, where l is a length of contour L. Below everywhere, speaking about concrete schemes of approximation of integrals $I(\varphi; z)$ and $I'(\varphi; z)$, we will mean that the indicated condition is fulfilled.

Now, assuming that the corresponding division of contour L by points $\{\tau_\sigma\}$ is realized and

$$l_{\sigma 0}(t) = \frac{t - \tau_{\sigma+1}}{\tau_\sigma - \tau_{\sigma+1}}, \quad l_{\sigma 1}(t) = \frac{t - \tau_\sigma}{\tau_{\sigma+1} - \tau_\sigma} \quad (t \in \tau_\sigma\tau_{\sigma+1},\ \sigma = \overline{1,n}),$$

we will denote by $L_\sigma(\varphi; t)$ the Lagrange linear interpolating polynomial:

$$L_\sigma(\varphi; t) = l_{\sigma 0}(t)\varphi(\tau_\sigma) + l_{\sigma 1}(t)\varphi(\tau_{\sigma+1}) \quad (t \in \tau_\sigma\tau_{\sigma+1}).$$

Further, assuming that as yet t_0 is an arbitrary point of contour L, and ν $(1 \le \nu \le n)$ is a number for which $t_0 \in \tau_\nu\tau_{\nu+1}$, we will approximate function $\varphi(t)$ $(t \in \tau_\sigma\tau_{\sigma+1})$ for each $\sigma = \overline{1,n}$ on arcs $\tau_\sigma\tau_{\sigma+1}$ by expression of the form

$$\varphi(t) \approx L_\nu(\varphi; t_0) + (t - t_0)\sum_{k=0}^{1} l_{\sigma k}(t)\frac{\varphi(\tau_{\sigma+k}) - L_{\sigma k}(\varphi; t_0)}{\tau_{\sigma+k} - t_0} \qquad (1)$$

$$(t \in \tau_\sigma\tau_{\sigma+1};\ 1 \le \sigma \le n),$$

where (assuming that k takes on values 0 and 1)

$$L_{\sigma k}(\varphi; t_0) = \begin{cases} \varphi(t_0),\ \sigma + k \ne \nu, \nu + 1; \\ L_\nu(\varphi; t_0),\ \sigma + k = \nu, \nu + 1 \quad (t_0 \in \tau_\nu\tau_{\nu+1};\ \nu = \overline{1,n}). \end{cases}$$

Here, in accordance with the indicated above,

$$L_\nu(\varphi; t_0) = l_{\nu 0}(t_0)\varphi(\tau_\nu) + l_{\nu 1}(t_0)\varphi(\tau_{\nu+1}) \quad (t_0 \in \tau_\nu\tau_{\nu+1}).$$

In further considerations the indicated conditions will be mostly considered in the form $\sigma \ne \nu, \nu \pm 1$ and $\sigma = \nu, \nu \pm 1$ respectively.

From the viewpoint of more detailed representation of approximate expressions based on formula (1) we will write separately formulas, corresponding to values $\sigma = \nu, \nu \pm 1$:

$$\varphi(t) \approx L_\nu(\varphi; t_0) + (t - t_0)\left\{ l_{\nu-10}(t)\frac{\varphi(\tau_{\nu-1}) - \varphi(t_0)}{\tau_{\nu-1} - t_0} \right.$$

$$\left. + l_{\nu-11}(t)\frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_0)}{\tau_\nu \ - t_0} \right\}$$

$$(t_0 \neq \tau_\nu), \ \ \sigma = \nu - 1;$$

$$\varphi(t) \approx L_\nu(\varphi; t_0) + (t - t_0)\sum_{k=0}^{1} l_{\nu k}(t)\frac{\varphi(\tau_{\nu+k}) - L_\nu(\varphi; t_0)}{\tau_{\nu+k} - t_0} \quad (t_0 \neq \tau_\nu, \tau_{\nu+1}),$$

$$\sigma = \nu;$$

$$\varphi(t) \approx L_\nu(\varphi; t_0) + (t - t_0)\left\{ l_{\nu+10}(t)\frac{\varphi(\tau_{\nu+1}) - L_\nu(\varphi; t_0)}{\tau_{\nu+1} - t_0} + \right.$$

$$\left. + l_{\nu+11}(t)\frac{\varphi(\tau_{\nu+2}) - \varphi(t_0)}{\tau_{\nu+2} - t_0} \right\} \ (t_0 \neq \tau_{\nu+1}), \quad \sigma = \nu + 1.$$

The last formulas show that for the considered values of σ certain additional consideration of cases $t_0 \to \tau_\nu, \tau_{\nu+1}$ is required. Analogous situation must be taken into account in the quadrature formula itself for $I(\varphi; z)$, constructed further on the base of formula (1), also, in constructed further quadrature formula for differentiated integral $I'(\varphi; z)$.

Considering firstly integral $I(\varphi; z)$, we will change in it the function $\varphi(t)$ by approximate formula (1). Denoting at this

$$p_{\sigma k}(t_0, z) = \frac{1}{2\pi i} \int\limits_{\tau_\sigma \tau_{\sigma+1}} \frac{(t - t_0) l_{\sigma k}(t)}{t - z} dt \ \ (\sigma = \overline{1, n}),$$

we will have

$$I(\varphi; z) = I(1; z) L_\nu(\varphi; t_0) + \sum_{\sigma=1}^{n}\sum_{k=0}^{1} p_{\sigma k}(t_0, z)\frac{\varphi(\tau_{\sigma+k}) - L_{\sigma k}(\varphi; t_0)}{\tau_{\sigma+k} - t_0} + \quad (2)$$

$$+ R_n(\varphi; z, t_0),$$

$$R_n(\varphi; z, t_0) = \frac{1}{2\pi i}\sum_{\sigma=1}^{n} \int\limits_{\tau_\sigma \tau_{\sigma+1}} \frac{R_{n\sigma}(\varphi; t, t_0)}{t - z} dt \ \ (t_0 \in \tau_\nu \tau_{\nu+1}),$$

where $R_{n\sigma}(\varphi;t,t_0)$ is a remainder of approximation of $\varphi(t)$ by formula (1) on the corresponding arc $\tau_\sigma\tau_{\sigma+1}$. we note that coefficients $p_{\sigma k}$ may be written in a form more convenient for calculations:

$$p_{\sigma k}(t_0,z)=\frac{1}{2\pi i}p_\sigma+\frac{z-t_0}{2\pi i}\int\limits_{\tau_\sigma\tau_{\sigma+1}}\frac{l_{\sigma k}(t)dt}{t-z},\ \ p_\sigma=\frac{\tau_{\sigma+1}-\tau_\sigma}{2}.$$

Now we will consider a question of approximation of the derivative of integral $I(\varphi;z)$:

$$I'(\varphi;z)=\frac{1}{2\pi i}\int\limits_L\frac{\varphi(t)}{(t-z)^2}dt.$$

Assuming again $t\in\tau_\sigma\tau_{\sigma+1}$ $(\sigma=\overline{1,n})$, we will start from approximation of $\varphi(t)$ by the scheme

$$\varphi(t)\approx\varphi(t_0)+(t-t_0)\varphi(t_0,t_1)+(t-t_0)(t-t_1) \tag{3}$$

$$\times\sum_{k=0}^{1}l_{\sigma k}(t)\frac{\varphi(\tau_{\sigma+k},t_1)-\varphi(t_0,t_1)}{\tau_{\sigma+k}-t_0},$$

where the point $t_1\in L$ $(t_1\neq t_0)$ is arbitrary for a while, $\varphi(t_0,t_1)$, $\varphi(\tau_{\sigma+k},t_1)$ represent the first kind difference quotients. Further, the difference quotient

$$\varphi(\tau_{\sigma+k},t_1)=\frac{\varphi(\tau_{\sigma+k})-\varphi(t_1)}{\tau_{\sigma+k}-t_1}\ \ (\tau_{\sigma+k}\neq t_1)$$

in the expression (3) will be changed by the ratio

$$\frac{\varphi(\tau_{\sigma+k})-L_{\sigma k}(\varphi;t_1)}{\tau_{\sigma+k}-t_1},$$

and $\varphi(t_0,t_1)$ will be changed by the expression

$$\frac{L_\nu(\varphi;t_1)-L_\nu(\varphi;t_0)}{t_1-t_0}\ \ (t_0\neq t_1).$$

as a result starting from (3), we will have approximation of $\varphi(t)$ by the scheme

$$\varphi(t)\approx\varphi(t_0)+(t-t_0)\frac{L_\nu(\varphi;t_1)-L_\nu(\varphi;t_0)}{t_1-t_0}+(t-t_0)\times$$

$$(t-t_1)\sum_{k=0}^{1}\frac{l_{\sigma k}(t)}{\tau_{\sigma+k}-t_0}\left\{\frac{\varphi(\tau_{\sigma+k})-L_{\sigma k}(\varphi;t_1)}{\tau_{\sigma+k}-t_1}-\frac{L_\nu(\varphi;t_1)-L_\nu(\varphi;t_0)}{t_1-t_0}\right\}, \tag{4}$$

where it is assumed that, $t \in \tau_\sigma\tau_{\sigma+1}$, $t_0, t_1 \in \tau_\nu\tau_{\nu+1}$ $(\nu, \sigma = \overline{1,n})$ and , as yet, $t_0 \neq t_1$. Besides, $t_0, t_1 \neq \tau_{\sigma+k}$. After introducing (4) in the expression of derivative $I'(\varphi; z)$ we will have

$$I'(\varphi; z) = I'(t; z)\big\{L_\nu(\varphi; t_1) - L_\nu(\varphi; t_0)\big\}\frac{1}{t_1 - t_0} + \sum_{\sigma=1}^{n}\sum_{k=0}^{1} p_{\sigma k}^{(1)}(t_0, t_1, z)$$

$$\times\frac{1}{\tau_{\sigma+k} - t_0}\left\{\frac{\varphi(\tau_{\sigma+k}) - L_{\sigma k}(\varphi; t_1)}{\tau_{\sigma+k} - t_1} - \frac{L_\nu(\varphi; t_1) - L_\nu(\varphi; t_0)}{t_1 - t_0}\right\} \quad (5)$$

$$+R_n^{(1)}(\varphi; z, t_0, t_1),$$

where it is clear that

$$p_{\sigma k}^{(1)}(t_0, t_1, z) = \frac{1}{2\pi i}\int\limits_{\tau_\sigma\tau_{\sigma+1}}\frac{(t - t_0)(t - t_1)}{(t - z)^2} l_{\sigma k}(t)dt$$

(and $R_n^{(1)}(\varphi; z, t_0, t_1)$ is the corresponding remainder). Here also, $t \in \tau_\sigma\tau_{\sigma+1}$; $t_0, t_1 \in \tau_\nu\tau_{\nu+1}$ $(t_0 \neq t_1)$. Further in expressions (2) and (5) transform the terms, corresponding to $\tau_{\sigma+k} - t_0$, $\tau_{\sigma+k} - t_1 = 0$. Firstly we use the relation

$$\frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_0)}{\tau_\nu - t_0} = \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}$$

and, similarly,

$$\frac{\varphi(\tau_{\nu+1}) - L_\nu(\varphi; t_0)}{\tau_{\nu+1} - t_0} = \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}.$$

Taking into account such transformations, from (2) we can write

$$I(\varphi; z) = I(1; z)L_\nu(\varphi; t_0) + p_{\nu-10}(t_0, z)\frac{\varphi(\tau_{\nu-1}) - \varphi(t_0)}{\tau_{\nu-1} - t_0} + \Big\{p_{\nu-11}(t_0, z) +$$

$$+p_{\nu 0}(t_0, z) + p_{\nu 1}(t_0, z) + p_{\nu+10}(t_0, z)\Big\}\frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} + p_{\nu+11}(t_0, z)\times$$

$$\frac{\varphi(\tau_{\nu+2}) - \varphi(t_0)}{\tau_{\nu+2} - t_0} + \sum_{\sigma\neq\nu\pm1,\nu}^{n-1}\sum_{k=0}^{1} p_{\sigma k}(t_0, z)\frac{\varphi(\tau_{\sigma+k}) - \varphi(t_0)}{\tau_{\sigma+k} - t_0} + R_n(\varphi; z, t_0). \quad (2_1)$$

It is clear that the right hand side of (2_1) is defined also for $t_0 = \tau_\nu$, $t_0 = \tau_{\nu+1}$ already explicitly. As for the similar transformations of (5) note that for arbitrary t_0, t_1 (assuming again as yet $t_0 \neq t_1$) the following is true

$$\frac{L_\nu(\varphi; t_0) - L_\nu(\varphi; t_1)}{t_0 - t_1} = \left\{\frac{t_0 - t_1}{\tau_\nu - \tau_{\nu+1}}\varphi(\tau_\nu) + \frac{t_0 - t_1}{\tau_{\nu+1} - \tau_\nu}\varphi(\tau_{\nu+1})\right\}\frac{1}{t_0 - t_1} =$$

$$= \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}.$$

Besides, in accordance with the considered above,

$$\frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_1)}{\tau_\nu - t_1} = \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}.$$

Along with that, for each from the mentioned above $t_0,\ t_1$ we have

$$\left\{ \frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_1)}{\tau_\nu - t_1} - \frac{L_\nu(\varphi; t_0) - L_\nu(\varphi; t_1)}{t_0 - t_1} \right\} \frac{1}{\tau_\nu - t_0} = 0.$$

Similarly,

$$\left\{ \frac{\varphi(\tau_{\nu+1}) - L_\nu(\varphi; t_1)}{\tau_{\nu+1} - t_1} - \frac{L_\nu(\varphi; t_1) - L_\nu(\varphi; t_0)}{t_1 - t_0} \right\} \frac{1}{\tau_{\nu+1} - t_0} = 0.$$

By that, the approximate formula corresponding to (5) obtains the form

$$I'(\varphi; z) = I'(t; z) \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} +$$

$$\frac{p^{(1)}_{\nu-10}(t_0, t_1, z)}{\tau_{\nu-1} - t_0} \left\{ \frac{\varphi(\tau_{\nu-1}) - L_\nu(\varphi; t_1)}{\tau_{\nu-1} - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right\} +$$

$$\frac{p^{(1)}_{\nu+11}(t_0, t_1, z)}{\tau_{\nu+2} - t_0} \left\{ \frac{\varphi(\tau_{\nu+2}) - L_\nu(\varphi; t_1)}{\tau_{\nu+2} - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right\} +$$

$$\sum_{\sigma \neq \nu \pm 1, \nu} \sum_{k=0}^{1} \frac{p^{(1)}_{\sigma k}(t_0, t_1, z)}{\tau_{\sigma+2} - t_0} \left\{ \frac{\varphi(\tau_{\sigma+k}) - L_\nu(\varphi; t_1)}{\tau_{\nu+k} - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right\} +$$

$$R^{(1)}_n(\varphi; z, t_0, t_1). \tag{5_1}$$

In the last expression $t_0, t_1 \in \tau_\nu\tau_{\nu+1}$ (at that their equality with knots $\tau_\nu,\ \tau_{\nu+1}$) may be meant as well; in addition to that equality $t_0 = t_1$ may take place also, and this fact will be taken into account where necessary. In accordance with this, namely, coefficients $p^{(1)}_{\sigma k}(t_0, t_1, z)$ in (5_1) obtain the following form

$$\frac{1}{2\pi i} \int\limits_{\tau_\sigma \tau_{\sigma+1}} \frac{(t - t_0)^2 l_{\sigma k}(t) dt}{(t - z)^2} \ \ (\sigma = \overline{1, n}). \tag{6}$$

Our further considerations concern the question of accuracy estimate (convergence rate) of the above mentioned quadrature formulas, what was agreed at the beginning. For this the corresponding formulas excluding a number of details, will be considered more frequently in the form (2), (5). But, from the viewpoint of practical realization, it must be noted that formulas (2_1), (5_1) are more convenient (noting at this that expressions (6) can be reduced to relatively easily computable integrals in an evident way).

3 On error estimates of quadrature formulas

Everywhere in further considerations concerning the functions $\varphi(t)$ we will mean that on the considered contour L they have bounded second order derivatives $\varphi''(t)$. Now we proceed to estimation of error, generated in the issue of changing $I(\varphi; z)$ and $I'(\varphi; z)$ by their approximate expressions. For this purpose in this section, similarly to theory of ordinary quadrature formulas (see, e.g.,[4],[5]), we will indicate integral representations of the corresponding remainder terms, using the assumed above statement concerning $\varphi(t)$. Considering, firstly, the case of approximation of integral $I(\varphi; z)$, we will start from representation of remainder $R_{n\sigma}(\varphi; t, t_0)$ of approximation of function $\varphi(t)$ involved in (2):

$$I(1; z)\Big[\varphi(t_0) - L_\nu(\varphi; t_0)\Big] + (t - t_0) r_{n\sigma}(\varphi; t, t_0)$$

$$(t \in \tau_\sigma\tau_{\sigma+1},\ t_0 \in \tau_\nu\tau_{\nu+1};\ \ \sigma, \nu = \overline{1, n}),$$

where the expression $r_{n\sigma}$, connected with $R_{n\sigma}$ by relation $R_{n\sigma} = (t - t_0) r_{n\sigma}$, has the form

$$r_{n\sigma}(\varphi; t, t_0) = \frac{\varphi(t) - \varphi(t_0)}{t - t_0} - \sum_{k=0}^{1} l_{\sigma k}(t) \frac{\varphi(\tau_{\sigma+k}) - L_{\sigma k}(\varphi; t_0)}{\tau_{\sigma+k} - t_0} \quad (t \neq t_0).$$

The difference $\varphi(t_0) - L_\nu(\varphi; t_0)$, evidently, represents remainder of approximation of function $\varphi(t_0)$ on arc $\tau_\nu\tau_{\nu+1}$ by interpolating polynomial $L_\nu(\varphi; t_0)$. Taking this into account, under the agreed assumption on function φ we can easily get an estimate:

$$\varphi(t_0) - L_\nu(\varphi; t_0) = O(h^2) \sup_{t_0 \in L} |\varphi''(t_0)|. \tag{7}$$

By that, error of the first term in approximating sum (2) is $O(h^2) I(1; z)$.

Now let us find the estimate of $r_{n\sigma}(\varphi; t, t_0)$. For this we will use its certain integral representation, based in turn on the following representation of function $\varphi(t)$:

$$\varphi(t) = \lambda_{\sigma\nu}(\varphi; t, t_0) + (t - t_0) \int\limits_{\tau_\sigma t} \left[\frac{2(t-\tau)}{(t_0-\tau)^3} \int\limits_{\tau t_0} (t_0 - u)\varphi''(u)du - \right.$$

$$\left. - \frac{t-\tau}{t_0-\tau}\varphi''(\tau) \right] d\tau, \quad t \in \tau_\sigma\tau_{\sigma+1},\ t_0 \in \tau_\nu\tau_{\nu+1}\,(\sigma \neq \nu \pm 1, \nu), \tag{8}$$

$$\lambda_{\sigma\nu}(\varphi; t, t_0) = \varphi(t_0) + \frac{t - t_0}{\tau_\sigma - t_0}[\varphi(\tau_\sigma) - \varphi(t_0)] +$$

$$\frac{(t-t_0)(t-\tau_\sigma)}{(t_0-\tau_\sigma)^2} \int\limits_{\tau_\sigma t_0} (t_0 - u)\varphi''(u)du$$

(at this integration everywhere in (8) is considered along positive direction on L). In order to prove (8), we may see that the expression in square brackets in (8) can be substituted by the integral

$$\int\limits_{\tau_\sigma t} (t-\tau) \left\{ \frac{d}{d\tau} \frac{1}{(t_0-\tau)^2} \int\limits_{\tau t_0} (t_0 - u)\varphi''(u)du \right\} d\tau.$$

After this, applying integration by parts to it, we come to (8). Evidently, $\lambda_{\sigma\nu}(\varphi; t, t_0)$ represents a polynomial of order ≤ 2 with respect to t ($t \in \tau_\sigma\tau_{\sigma+1}$), taking on values $\varphi(t_0)$ at $t = t_0$ ($t_0 \in \tau_\nu\tau_{\nu+1}$, $0 \leq \nu \leq n$). Via the fact that for $\sigma \neq \nu, \nu\pm 1$ the equality $L_{\sigma k}(\varphi; t_0) = \varphi(t_0)$ is valid, substituting $\varphi(t)$ by formula (8) in the above indicated expression of $r_{n\sigma}(\varphi; t, t_0)$, we get

$$r_{n\sigma}(\varphi, t, t_0) = \int\limits_{\tau_\sigma t} \left[\frac{2(t-\tau)}{(t_0-\tau)^3} \int\limits_{\tau t_0} (t_0 - u)\varphi''(u)du - \frac{t-\tau}{t_0-\tau}\varphi''(\tau) \right] d\tau -$$

$$-l_{\sigma 1}(t) \int\limits_{\tau_\sigma \tau_{\sigma+1}} \left[\frac{2(\tau_{\sigma+1}-\tau)}{(t_0-\tau)^3} \int\limits_{\tau t_0} (t_0 - u)\varphi''(u)du - \frac{\tau_{\sigma+1}-\tau}{t_0-\tau}\varphi''(\tau) \right] d\tau. \tag{9}$$

The expression in the right hand side of (9), can evidently be considered as remainder of linear interpolation by t of function (vanishing at $t = \tau_\sigma$)

$$\int\limits_{\tau_\sigma t} \left[\frac{2(t-\tau)}{(t_0-\tau)^3} \int\limits_{\tau t_0} (t_0 - u)\varphi''(u)du - \frac{t-\tau}{t_0-\tau}\varphi''(\tau) \right] d\tau$$

by its values at the knots τ_σ, $\tau_{\sigma+1}$ ($\sigma \neq \nu \pm 1, \nu$). We will use this in order to estimate the corresponding $r_{n\sigma}$, starting from the known estimates of remainder of complicated (according to terminology introduced in [4]) quadrature formulas under condition of boundedness of the second order derivative of the integrand. Along with this we note that representation of $r_{n\sigma}$ for the values $\sigma = \nu, \nu \pm 1$ can be obtained from the above indicated general formula for $r_{n\sigma}$. Thus, e.g. at $\sigma = \nu - 1$ we have:

$$r_{n\nu-1}(\varphi; t, t_0) = \varphi(t, t_0) - l_{\nu-10}(t)\frac{\varphi(\tau_{\nu-1}) - \varphi(t_0)}{\tau_{\nu-1} - t_0} -$$

$$l_{\nu-11}(t)\frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}, \quad t \in \tau_{\nu-1}\tau_\nu.$$

In order to estimate the corresponding values in the given case we will use the representation of $\varphi(t)$ by the Taylor formula with the remainder in the integral form

$$\varphi(t) = \varphi(\tau_{\nu-1}) + (t - \tau_{\nu-1})\varphi'(\tau_{\nu-1}) + \int\limits_{\tau_{\nu-1}t} (t-u)\varphi''(u)du.$$

Due to application of such formulas, via the fact that the expressions $r_{n\sigma}$ vanish on linear functions, we can see the validity $r_{n\sigma}(\varphi; t, t_0) = O(h)M$ $(\sigma = \nu, \nu \pm 1)$, $M = \sup\limits_L |\varphi''(t)|$. Before proceeding to estimation of $r_{n\sigma}(\varphi; t, t_0)$ for the rest of the values σ, we note that according to the above indicated representation of $R_n(\varphi; t, t_0)$ the general estimation of the remainder of approximation of $I(\varphi; z)$ is reduced mainly to estimation of the sum

$$\sum_{\sigma=1}^{n-1} \int\limits_{\tau_\sigma\tau_{\sigma+1}} \frac{(t-t_0)r_{n\sigma}(\varphi; t, t_0)}{t - z}dt =$$

$$= \sum_{\sigma=1}^{n-1}\left\{ \int\limits_{\tau_\sigma\tau_{\sigma+1}} r_{n\sigma}(\varphi; t, t_0)dt + (z - t_0) \int\limits_{\tau_\sigma\tau_{\sigma+1}} \frac{r_{n\sigma}(\varphi; t, t_0)}{t - z}dt\right\}.$$

Estimation for $\sum\limits_{\sigma=1}^{n-1} \int\limits_{\tau_\sigma\tau_{\sigma+1}} r_{n\sigma}(\varphi; t, t_0)dt$ can be obtained using (9) and the indicated above estimates of $r_{n\sigma}(\varphi; t, t_0)$ for $\sigma = \nu, \nu \pm 1$. Finally we get convinced in the validity of estimate $O(h^2)M$ $(n > 1)$, which fulfills uniformly with respect to t_0. Concerning the integral of form

$$(z - t_0) \int\limits_{\tau_\sigma\tau_{\sigma+1}} \frac{r_{n\sigma}(\varphi; t, t_0)}{t - z}dt,$$

we will mean that t_0, which was assumed to be an arbitrarily chosen point on L, is selected in such a way that for the given z the distance between z

and t_0 does not exceed the minimal distance between z and the points of the boundary L. Then for each considered σ we can write

$$(z-t_0)\int\limits_{\tau_\sigma\tau_{\sigma+1}}\frac{r_{n\sigma}(\varphi;t,t_0)}{t-z}dt=O(1)\int\limits_{s_\sigma}^{s_{\sigma+1}}|r_{n\sigma}(\varphi;t,t_0)|ds.$$

Via this, we get for the sum

$$\sum_\sigma(z-t_0)\int\limits_{\tau_\sigma\tau_{\sigma+1}}\frac{r_{n\sigma}(\varphi;t,t_0)}{t-z}dt$$

also analogous to the above indicated estimate. Finally, taking into account estimation (7), we may say that for any $z\in D$ (taking into account also the points, lying arbitrarily close to the boundary) by appropriate selection of t_0 we can obtain estimation $O(h^2\ln n)$ $(n\to\infty)$ for the remainder $R_n(\varphi;z,t_0)$ on the above indicated class of functions. The constant involved in the estimation can be meant to be independent of z and t_0.

Now we proceed to estimate error $R_n^{(1)}(\varphi;z,t_0,t_1)$ of approximation $I'(\varphi;z)$ (assuming meanwhile, as in the beginning, $t_1\neq t_0$). Similarly to the above, the corresponding estimate bases on the estimation of expressions of the form

$$I'(t;z)r_\nu(\varphi;t_0,t_1)+(t-t_0)(t-t_1)r_{n\sigma}^{(1)}(\varphi;t,t_0,t_1),$$

where

$$r_\nu(\varphi;t_0,t_1)=\frac{\varphi(t_1)-L_\nu(\varphi;t_1)-[\varphi(t_0)-L_\nu(\varphi;t_0)]}{t_1-t_0}$$
$$(t_0,t_1\in\tau_\nu\tau_{\nu+1},\ \ t_1\neq t_0),$$

and $r_{n\sigma}^{(1)}(\varphi;t,t_0,t_1)$ represents an error of approximation of expression

$$\frac{\varphi(t,t_1)-\varphi(t,t_0)}{t_1-t_0}$$

on the basis of application of the approximating formula (4). Firstly we find estimation of expression $r_\nu(\varphi;t_0,t_1)$. For this in each of differences in the numerator we will use the Taylor formula on arcs $\tau_\nu t_0$, $\tau_\nu t_1$ in the neighborhood of point τ_ν:

$$\varphi(t_j)=\varphi(\tau_\nu)+(t_j-\tau_\nu)\varphi'(\tau_\nu)+\int\limits_{\tau_\nu t_j}(t_j-u)\varphi''(u)du\ \ (j=0,j=1).$$

Introducing this into the indicated differences for the corresponding j via a number of transformations in the numerator of $r_\nu(\varphi;t_0,t_1)$ brings us to estimate $O(h)|t_1 - t_0|M$ $(M = \sup_u |\varphi''(u)|)$, which is valid uniformly with respect to t_0, t_1 ν. By that we have

$$r_\nu(\varphi; t_0, t_1) = O(h)M. \tag{10}$$

Further, according to obtained in the beginning notations, for values $\sigma = \nu, \nu \pm 1$ the following is true

$$r_{n\sigma}^{(1)}(\varphi; t, t_0, t_1) = \frac{\varphi(t, t_1) - \varphi(t_0, t_1)}{t - t_0} -$$

$$\frac{1}{t - t_0} \sum_{k=0}^{1} \frac{l_{\sigma k}(t)}{\tau_{\sigma+k} - t_0} \left\{ \frac{\varphi(\tau_{\sigma+k}) - L_{\sigma k}(\varphi; t_1)}{\tau_{\sigma+k} - t_1} - \frac{L_\nu(\varphi; t_1) - L_\nu(\varphi; t_0)}{t_1 - t_0} \right\}, \tag{11}$$

$$(t_0, t_1 \in \tau_\nu \tau_{\nu+1}),$$

where, as above, $\varphi(t, t_1)$, $\varphi(t_0, t_1)$ denote the divided differences with the corresponding arguments. At this, as it was noted already while obtaining (5_1), the following is true:

$$\frac{L_\nu(\varphi; t_1) - L_\nu(\varphi; t_0)}{t_1 - t_0} = \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu}.$$

Similarly to approximation of integral $I(\varphi; z)$, here we will use formula (11) in order to represent $r_{n\sigma}^{(1)}$ for the values $\sigma = \nu, \nu \pm 1$. For the sake of clearness we will write, e.g. the formula:

$$r_{n\nu-1}^{(1)}(\varphi; t, t_0, t_1) = \frac{\varphi(t, t_1) - \varphi(t_0, t_1)}{t - t_0}$$

$$-\frac{1}{t - t_0} \left\{ \frac{l_{\nu-1\,0}(t)}{\tau_{\nu-1} - t_0} \left[\frac{\varphi(\tau_{\nu-1}) - \varphi(t_1)}{\tau_{\nu-1} - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right] \right\}$$

$$-\frac{l_{\nu-1\,1}(t)}{\tau_\nu - t_0} \left[\frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_1)}{\tau_\nu - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right] \quad (t \in \tau_{\nu-1}\tau_\nu).$$

which corresponds to $\sigma = \nu - 1$. At this, remembering that

$$\frac{\varphi(\tau_\nu) - L_\nu(\varphi; t_1)}{\tau_\nu - t_1} = \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu},$$

we get

$$r_{n\nu-1}^{(1)}(\varphi; t, t_0, t_1) = \frac{\varphi(t, t_1) - \varphi(t_0, t_1)}{t - t_0}$$

$$-\frac{1}{t - t_0} \frac{l_{\nu-1\,0}(t)}{\tau_{\nu-1} - t_0} \left\{ \frac{\varphi(\tau_{\nu-1}) - \varphi(t_1)}{\tau_{\nu-1} - t_1} - \frac{\varphi(\tau_{\nu+1}) - \varphi(\tau_\nu)}{\tau_{\nu+1} - \tau_\nu} \right\}.$$

Formulas, corresponding to $\sigma = \nu, \nu+1$ in (11) can be obtained analogously. Now, taking into account that $\sigma = \nu - 1$ corresponds to $t \in \tau_{\nu-1}\tau_\nu$, we will apply the Taylor formula to function $\varphi(t)$ on the arc $t_1 t$ (considering, that $t_1 t$ is the shortest arc on L) in the neighborhood of t_1 with remainder in an integral form. Writing the corresponding formula in the form

$$\varphi(t, t_1) = \varphi'(t_1) + \frac{1}{t - t_1} \int\limits_{t_1 t} (t-u)\varphi''(u)du \quad (t \neq t_1), \tag{12}$$

we will change in it t by t_0 and combine the obtained one with (12). Finally we get

$$\frac{\varphi(t,t_1) - \varphi(t_0,t_1)}{t - t_0} = \frac{1}{t - t_0}\Bigg\{ \frac{1}{t - t_1} \int\limits_{t_1 t} (t-u)\varphi''(u)du$$

$$- \frac{1}{t_0 - t_1} \int\limits_{t_1 t_0} (t_0 - u)\varphi''(u)du \Bigg\}.$$

Further, we denote $\mu(\varphi; t, t_1) = \dfrac{1}{t - t_1} \displaystyle\int\limits_{t_1 t} (t-u)\varphi''(u)du$. Differentiating the last by t we get $\mu'_t(\varphi; t, t_1) = O(1)M$ ($M = \sup\limits_{u \in L} |\varphi''(u)|$), uniformly with respect to t, t_1. On this basis for expression $\dfrac{\varphi(t,t_1) - \varphi(t_0,t_1)}{t - t_0}$ we obtain similar estimate on the set of the considered values of t, t_0, t_1. Further, noticing that the ratio $\dfrac{l_{\nu-1\,0}(t)}{(t-t_0)(\tau_{\nu-1} - t_0)}$ is bounded with respect to t, t_0 $(t \in \tau_{\nu-1}\tau_\nu)$, $(t_0 \in \tau_\nu\tau_{\nu+1})$, for $r^{(1)}_{n\nu-1}(\varphi; t, t_0, t_1)$ we have the estimation $O(1)M$. Similar estimations are valid also for values $\sigma = \nu, \nu+1$ for $r^{(1)}_{n\sigma}(\varphi; t, t_0, t_1)$.

Now let us proceed to the question of estimation of $r^{(1)}_{n\sigma}(\varphi; t, t_0, t_1)$ for values $\sigma \neq \nu, \nu \pm 1$. Taking into account the previous considerations, we may start from the formula

$$\frac{\varphi(t,t_1) - \varphi(t,t_0)}{t_1 - t_0} \approx \frac{1}{t_1 - t_0} \sum_{k=0}^{1} \frac{l_{\sigma k}(t)}{\tau_{\sigma+k} - t_0} \times \tag{13}$$

$$\left\{ \frac{\varphi(\tau_{\sigma+k}) - \varphi(t_1)}{\tau_{\sigma+k} - t_1} - \frac{\varphi(\tau_{\sigma+k}) - \varphi(t_0)}{\tau_{\sigma+k} - t_0} \right\}.$$

In the given case on the basis of this formula we have to obtain an integral representation of $r^{(1)}_{n\sigma}$ (similar to (9)) for the mentioned values of σ. For this purpose, meaning $t \in \tau_\sigma\tau_{\sigma+1}$, we will rewrite formula (8) in the following form

$$\varphi(t,t_1)-\varphi(t,t_0)=\lambda_\sigma^*(\varphi;t,t_1)-\lambda_\sigma^*(\varphi;t,t_0)+$$
$$\int\limits_{\tau_\sigma t}\left[\frac{2(t-\tau)}{(t_1-\tau)^3}\int\limits_{\tau t_1}(t_1-u)\varphi''(u)du-\frac{t-\tau}{t_1-\tau}\varphi''(\tau)\right]d\tau-$$
$$\int\limits_{\tau_\sigma t}\left[\frac{2(t-\tau)}{(t_0-\tau)^3}\int\limits_{\tau t_0}(t_0-u)\varphi''(u)du-\frac{t-\tau}{t_0-\tau}\varphi''(\tau)\right]d\tau. \tag{14}$$

Taking into account that on arcs $\tau_\sigma\tau_{\sigma+1}$ the expressions

$$\frac{\lambda_\sigma^*(\varphi;t,t_1)-\lambda_\sigma^*(\varphi;t,t_0)}{t-t_0}\quad(t_0\neq t_1)$$

are linear functions of variable t, on the basis of formulas (13), (14) we get

$$r_{n\sigma}^{(1)}(\varphi;t,t_0,t_1)=\frac{1}{t_1-t_0}\Big\{G_\sigma(\varphi;t,t_1)-G_\sigma(\varphi;t,t_0)-$$
$$l_{\sigma1}(t)[G_\sigma(\varphi;\tau_{\sigma+1},t_1)-G_\sigma(\varphi;\tau_{\sigma+1},t_0)]\Big\},$$

where

$$G_\sigma(\varphi;t,t_j)=\int\limits_{\tau_\sigma t}\left[\frac{2(t-\tau)}{(\tau_j-\tau)^3}\int\limits_{\tau t_j}(t_j-u)\varphi''(u)du-\frac{t-\tau}{t_j-\tau}\varphi''(\tau)\right]d\tau$$
$$(j=0,j=1).$$

at this in the indicated formula we take into account (compare with (9)) that $G_\sigma(\varphi;\tau_\sigma,t_j)=0$. Writing the obtained formula in the form

$$r_{n\sigma}^{(1)}(\varphi;t,t_0,t_1)=\frac{1}{t_1-t_0}\int\limits_{t_0t_1}\Big[(G_\sigma)'_\xi(\varphi;t,\xi)-l_{\sigma1}(t)(G_\sigma)'_\xi(\varphi;\tau_{\sigma+1},\xi)\Big]d\xi$$
$$(\xi=t_0,t_1;\ t_1\neq t_0),$$

we will base the further estimation of $r_{n\sigma}^{(1)}(\varphi;t,\xi)$ on the estimation of the difference involved in the indicated integral. For this we notice that similar to $G_\sigma(\varphi;\tau_\sigma,\xi)=0$ the following $(G_\sigma)'_\xi(\varphi;\tau_\sigma,\xi)=0$ is valid too. By that the corresponding integrand difference for each considered σ represents a remainder of linear interpolation (by variable t) of the function $(G_\sigma)'_\xi(\varphi;t,\xi)$ differentiated by ξ. On the basis of integral representation of the remainder of the indicated difference for $r_{n\sigma}^{(1)}(\varphi; t, t_0, t_1)$ we get an estimate of form $O(h^2)|t_1 - t_0| \max\limits_t |\dfrac{\partial^2 (G)'_\xi}{\partial t^2}|$ for the considered σ, at this the constant in $O(h^2)$ does not depend on t_0, t_1. Observing the corresponding details in estimate of $\dfrac{\partial^2 (G_\xi)'}{\partial t^2}$, via (10) and estimates obtained above for $\sigma = \nu, \nu \pm 1$ we get convinced in validity of estimations of type

$$\sum_{\sigma=1}^{n} \int\limits_{\tau_\sigma \tau_{\sigma+1}} |r_{n\sigma}(\varphi; t, t_0, t_1)| dt = O(h) M \quad (M = \sup_{t \in L} |\varphi''(t)|). \tag{15}$$

Now, using the estimates obtained above, we can estimate the remainder $R_n^{(1)}(\varphi; z, t_0, t_1)$ of the considered quadrature process, which finally consists in estimation of expression

$$\frac{1}{2\pi i} \sum_{\sigma=1}^{n-1} \int\limits_{\tau_\sigma \tau_{\sigma+1}} (t - t_0)(t - t_1) \frac{r_{n\sigma}^{(1)}(\varphi; t, t_0, t_1)}{(t - z)^2} dt. \tag{16}$$

As it was noted above, the approximating expression for integral $I'(\varphi; z)$ (involving arbitrarily given parameters $t_0, t_1 \in L$) considered here has a sense for $t_1 = t_0$ also. Assuming this in (16), the general value of these parameters can be related with z, in a certain way (as it was done at estimation of $R_n(\varphi; z, t_0)$). This finally brings us to estimation with right hand side of form (15) with a constant independent of z.

Concerning the results stated here we note, that the accuracy rate of the indicated quadrature processes generally is conditioned by accuracy of the approximation applied to density $\varphi(t)$ of the given integrals (in combination with selection of used at this correcting parameters). From this viewpoint construction of more accurate quadrature formulas can be realized with the help of interpolating (piece-wise interpolating) polynomials of higher order. Naturally, such formulas have somewhat more complicated structure, however, their numerical realization is not connected with some essential difficulties.

Above it was already mentioned about application of the Cauchy type integrals and their derivatives to the plane theory of elasticity. Thus, e.g. for calculation of components of vectors of stresses and displacements in problems of the plane theory of elasticity may be used formulas, offered in [2,3]. We give the corresponding formulas (referring, for clearness, to the first basic problem of the theory of elasticity):

$$X_x + Y_y = 2[\varphi_1'(z) + \overline{\varphi_1'(z)}],$$

$$Y_y - X_x + 2iX_y = 2[\overline{z}\varphi_1''(z) + \varphi_2'(z)],$$

where

$$\varphi_1(z) = \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)d\bar{t}}{t-z},$$

$$\varphi_2(z) = \frac{1}{2\pi i}\int\limits_L \frac{\overline{\omega}(t)dt}{t-z} + \frac{1}{2\pi i}\int\limits_L \frac{\omega(t)d\,\bar{t}}{t-z} - \frac{1}{2\pi i}\int\limits_L \frac{t\overline{\omega}(t)dt}{(t-z)^2},$$

with $\omega(t)$ being the solution to the integral equation, corresponding to the initial problem. From these integrals the two following

$$\int\limits_L \frac{\omega(t)d\,\bar{t}}{t-z}, \quad \int\limits_L \frac{\omega(t)d\,\bar{t}}{(t-z)^2}$$

are somehow different from the considered here $I(\varphi; z)$ and $I'(\varphi; z)$, Though, their calculation can be reduced to calculation of the considered above ones, using the following equality

$$\int\limits_L \frac{\omega(t)d\,\bar{t}}{t-z} = \int\limits_L \frac{\omega(t)\left(\frac{d\,\bar{t}}{dt}\right)}{t-z}dt$$

and, analogously, for the second integral. Nevertheless, in dependence of the way of giving the contour L, calculation of the values of derivative $\dfrac{d\,\bar{t}}{dt}$ at the knots is not desirable always (especially, when the boundary L of the considered domain is not given by its exact equation, which generally occurs in practical problems). We can bypass the situation using approximation of the function $\omega(t)$ again, applied here. In this case in the role of coefficients of the corresponding quadrature formulas we will have integrals of type

$$\int\limits_{\tau_j\tau_{j+1}} \frac{(t-t_0)l_{jk}(t)}{t-z}d\,\bar{t}, \quad \int\limits_{\tau_j\tau_{j+1}} \frac{(t-t_0)^2 l_{jk}(t)}{(t-z)^2}d\,\bar{t}, \tag{17}$$

at the values j and k known already. The first of the integrals in (17) will be represented as a sum

$$\int\limits_{\tau_j\tau_{j+1}} l_{jk}(t)d\,\bar{t} + (z-t_0)\int\limits_{\tau_j\tau_{j+1}} \frac{l_{jk}(t)}{t-z}d\,\bar{t}. \tag{18}$$

We apply formula of integration by parts to the first integral in the present sum

$$\int\limits_{\tau_j\tau_{j+1}} l_{jk}(t)d\,\bar{t} = \overline{\tau}_{j+1}l_{jk}(\tau_{j+1}) - \overline{\tau}_j l_{jk}(\tau_j) - \int\limits_{\tau_j\tau_{j+1}} \bar{t}\,l'_{jk}(t)dt \quad (k=0,1).$$

At this we remember that $l_{j0}(\tau_{j+1}) = 0, l_{j1}(\tau_{j+1}) = 1,\ \ l_{j0}(\tau_j) = 1,\ \ l_{j1}(\tau_j) = 0$. And the integrals of the form $\int\limits_{\tau_j\tau_{j+1}} \bar{t}\, l'_{jk}(t)$ can be calculated approximately via approximation (namely, linear interpolation) of the function $\bar{t}$. The second term in (18) we represent so:

$$(z - t_0)\left[l_{jk}(z) \int\limits_{\tau_j\tau_{j+1}} \frac{d\,\bar{t}}{t - z} + \int\limits_{\tau_j\tau_{j+1}} \frac{l_{jk}(t) - l_{jk}(z)}{t - z} d\,\bar{t}\right]. \tag{19}$$

The first integral in (19) equals

$$\frac{\bar{t}_{j+1}}{\bar{t}_{j+1} - z} - \frac{\bar{t}_j}{\bar{t}_j - z} + \int\limits_{\tau_j\tau_{j+1}} \frac{\bar{t}dt}{(t - z)^2},$$

at that

$$\int\limits_{\tau_j\tau_{j+1}} \frac{\bar{t}dt}{(t - z)^2} = \bar{z} \int\limits_{\tau_j\tau_{j+1}} \frac{dt}{(t - z)^2} + \int\limits_{\tau_j\tau_{j+1}} \frac{(\bar{t} - \bar{z})dt}{(t - z)^2}.$$

Now the question reduces to approximate calculation of the second integral in the previous equality. Representing the last as

$$\int\limits_{\tau_j\tau_{j+1}} \frac{(\bar{t} - \bar{z})^2 dt}{|t - z|^2 (t - z)} = \int\limits_{\tau_j\tau_{j+1}} \frac{e^{(i \arg{(\bar{t} - \bar{z})}^2}}{t - z} dt,$$

we can apply the approximation scheme (used in the beginning) to its density $e^{(i \arg(\bar{t} - \bar{z})}$. The second integral in (19) can be transformed as well, using integration by parts, which, similar to the case of integral $\int\limits_{\tau_j\tau_{j+1}} \bar{t}\, l'_{jk}(t)$ with appropriate approximation of $\bar{t}$ brings us to calculation of certain integrals from polynomials. As a result, the expression (19), thanks to presence of factor $z - t_0$, under agreed selection of t_0, turns out to be bounded while arbitrary approaching z to the points of contour L (the same factor will play similar role at estimation of the remainder of approximate expressions used here).

While computing approximately the following integrals

$$\int\limits_{\tau_j\tau_{j+1}} \bar{t}\, l'_{jk}(t), \int\limits_{\tau_j\tau_{j+1}} \frac{e^{i \arg{(\bar{t} - \bar{z})}^2}}{t - z} dt,$$

from the viewpoint of obtaining final higher accuracy it may turn out to be more reasonable for better approximation of the functions $\bar{t}$ $e^{i \arg{(\bar{t} - \bar{z})}^2}$ to introduce additional points (with appropriate division of arcs $\tau_j\tau_{j+1}$).

Similar approaches may be used for calculation of the second integral in (17).

Integral $\varphi_1'(z)$ can be computed with the help of above indicated approximate scheme for integral $I'(\varphi; z)$. As for calculation of integral $\varphi_1''(z)$ and the third integral in expression $\varphi_2'(z)$, for their approximate calculation (naturally, for z close to boundary points) we should use more precise quadrature formulas (mentioned earlier). By that, the corresponding question requires more detailed consideration.

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