ON THE PERMUTABILITY OF A FUNCTOR OF TENSOR COMPLETION WITH PRINCIPAL GROUP OPERATIONS

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Abstract

It is proved that the tensor completion is permutable with the operations of direct product and direct limit of exponential groups and, but in generally, is not permutable with the Cartesian product and the inverse limit of exponential groups.

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1 Introduction

The notion of an exponential group was introduced by R. Lyndon in [1]. In [2] A. G. Myasnikov refined the notion of a exponential group by introducing an additional axiom. In particular the new notion of a exponential group is a direct generalization of the notion of a module to the case of non-commutative groups. The systematic study of the category of exponential groups in the sense of A. G. Myasnikov was started in [2]–[5]. In paper [2] it is shown that in the investigation of exponential groups the decisive role is played by the notion of tensor completion. In the present paper we investigate the problem of the permutability of a functor of tensor completion with principal group operations.

2 Preliminary Information and Statement of the Problems

Let us give the main definitions from [2]. Fix an arbitrary associative ring A with unit, and also a group G. Let the action of A on G, i.e. the mapping $G \times A \to G$ be given. The result of the action of $\alpha \in A$ on $g \in G$ will be written in the form g^{α}.

We consider the axioms:

(1) $g^1 = g$, $g^0 = 1$, $1^\alpha = 1$;

(2) $g^{\alpha+\beta} = g^\alpha \cdot g^\beta$, $g^{\alpha\beta} = (g^\alpha)^\beta$;

(3) $(h^{-1}gh)^\alpha = h^{-1}g^\alpha h$;

(4) $[g, h] = 1 \Longrightarrow (gh)^\alpha = g^\alpha h^\alpha$.

Definition 1. A group G is called *an A-exponential group* or an A-group if the action of the ring A which satisfies axioms (1)–(4) is given on G.

Following [2] the class of all A-exponential groups will be denoted by $\mathfrak{M}_A$.

Definition 2. A homomorphism $\varphi : G \to H$ is called *an A-homomorphism* if

$$(g^\alpha)^\varphi = (g^\varphi)^\alpha, \ \ g \in G, \ \ \alpha \in A.$$

For the main definitions in the category $\mathfrak{M}_{\mathfrak{A}}$ and the results on these groups we refer the reader to [2]–[5].

For the completeness of our discussion we give here the definition of the notion of tensor completion.

Definition 3. Let G be an A-group, $\mu : A \to B$ be a ring homomorphism. Then a B-group G^B is called the *tensor B-completion* of an A-group G if G^B satisfies the following universal properties:

(1) there exists an A-homomorphism $i : G \to G^B$ such that $i(G)$ B-generates G^B, i.e. $\langle i(G)\rangle_B = G^B$;

(2) for any B-group H and any A-homomorphism $\varphi : G \to H$ there exists a B-homomorphism $\psi : G^B \to H$ that makes the following diagram commutative:

$$\begin{tikzcd} G \arrow[r, "i"] \arrow[d, "\varphi"'] & G^B \arrow[dl, "\psi"] \\ H & \end{tikzcd} \quad , \quad i\psi = \varphi.$$

Note that if G is an abelian A-group, then $G^B \cong G \underset{A}{\otimes} B$ is a tensor product of an A-module G by a ring B.

In [2] it is proved that for any A-group G and any homomorphism $\mu : A \to B$ the tensor completion G^B exists always and it is unique to within an isomorphism.

We are interested in the following two questions. Suppose we are given the family of A-groups $\{G_i, i \in I\}$.

(1) *Let* $G = \prod_i G_i$ *be a direct product of groups* G_i (see definition in [2]). *Is it true that*

$$G^B = \prod_i G_i^B\,?$$

In other words, is the functor of tensor completion permutable with direct product?

(2) *Let* $G = \varinjlim G_i$ *be the direct limit of groups* G_i (see definition in [2]). *Is it true that*

$$G^B = \varinjlim G_i^B\,?$$

Let us show that the answers to questions (1) and (2) are positive. The operation of tensor completion commutes with the operations of direct product and direct limit, but in general, does not commute with the operations of Cartesian product and inverse limit. The permutability of tensor completion with direct limits allows one to reduce many problems on completion to the case of a finitely generated group.

The basic results and notions which we will use below can be found in the books [6], [7].

In [7] it is proved that in the category of abelian groups the operations of a direct product of groups and of direct and inverse limits of groups possess the universal property. The corresponding operations in the category of exponential groups also possess analogous properties [2], since their proofs are parallel to the proofs from [7].

Theorem 1. The functor of tensor completion is permutable with a direct product. In other words, if

$$G = \prod_j G_j \text{ then } G^B = \prod_j G_j^B. \tag{2.1}$$

Before proving the theorem, let us formulate and prove the following lemma.

Lemma 1. Let for each j there exist an A-homomorphism $\varphi_j : G_j \to H$, where H is an A-group, and also

$$\big[\varphi_j(G_j), \varphi_k(G_k)\big] = 1 \tag{2.2}$$

for all pairs j, k, where $j \neq k$. Then there exists an A-homomorphism $\varphi : \prod_j G_j \to H$ that continues φ_j for all j.

Denote $H_0 = \langle(\varphi_j(G_j), j \in J\rangle_A$. Let us perform the linear ordering of the set of indices J and prove that any element $h \in H_0$ is representable in the form

$$g_{j_1}g_{j_2}g_{j_3}\cdots g_{j_s}, \;\; \textit{where} \;\; g_{j_k} \in \varphi_{j_k}(G_{j_k}), \;\; j_1 < j_2 < \cdots < j_k. \tag{2.3}$$

By virtue of condition (2.2) it is obvious that elements of form (2.2) make a subgroup. Since $\forall\,\alpha \in A$ by axiom (4) the element

$$(g_{j_1} g_{j_2} \cdots g_{j_s})^{\alpha} = g_{j_1}^{\alpha} g_{j_2}^{\alpha} \cdots g_{j_s}^{\alpha}$$

elements of form (2.3) make an A-subgroup. Now the required A-homomorphism ψ can be constructed as follows.

Let an element $g \in \prod\limits_j G_j$ be written in the form

$$g = (\dots, g_{j_1}, \dots, g_{j_s}, \dots),$$

where instead of the points there are units. Assume that

$$\psi(g) = g_{j_1} g_{j_2} \cdots g_{j_s}.$$

We can verify That φ is a homomorphism in a straightforward manner.

Let us return to the proof of the theorem. For this, between the groups G^B and $\prod\limits_j G_j^B$ we construct a pair of counter A-homomorphisms. Let

$$i_j : G_j \longrightarrow G_j^B$$

be a canonical A-homomorphism given by the definition of tensor completion. By Lemma 2 there exists an A-homomorphism

$$\varphi : G \longrightarrow \prod_j G_j^B.$$

Then it is obvious that

$$\big[i_j(G_j), i_k(G_k)\big] = 1, \ \text{ if } \ j \neq k.$$

By the definition of tensor completion there exists a B-homomorphism ψ_1 that makes the diagram

$$\begin{tikzcd} G = \prod\limits_j G_j \arrow[r, "i"] \arrow[d, "\varphi"'] & G^B \arrow[dl, "\psi_1"] \\ \prod G_j^B & \end{tikzcd}$$

commutative.

Let us verify that

$$\big[\langle i_j(G_j)\rangle_B, \langle i_k(G_k)\rangle_B\big] = 1.$$

This is so because the generatrices of the first subgroup commute with the generatrices of the second subgroup. By virtue of the definition of tensor completion there exists a B-homomorphism β_j that makes the diagram

$$\begin{tikzcd} G_j \arrow[r] \arrow[d, "i_j"'] & \langle i(G_j) \rangle_B \arrow[dl, "\beta_j"] \\ G_j^B & \end{tikzcd} \qquad (j \in J)$$

commutative.

Now by virtue of Lemma 2 there exists a B-homomorphism ψ_2 from $\prod_j G_j^B$ into G^B. That these are counter-homomorphisms can be verified in a straightforward manner. □

Theorem 2. The operation of tensor completion is permutable with direct limits.

Let us construct in a standard manner the counter-mappings between the group G_*^B and the group

$$H = \varinjlim G_j^B.$$

Denote

$$\varphi_j : G_j^B \longrightarrow H.$$

Then there exists a B-homomorphism

$$\psi_1 : H \longrightarrow G_*^B$$

that makes the diagram

$$\begin{tikzcd} & G_j^B \arrow[dl, "\varphi_j"'] \arrow[d, "\pi_j^B"] \\ H \arrow[r, "\psi_1"'] & G_*^B \end{tikzcd}$$

commutative. Here $\pi_j : G_j \to G_*$ is the projection, $\pi_j^B : G_j^B \longrightarrow G_*^B$ is the corresponding homomorphism of tensor completions.

Using the universal property of tensor completion, for every index j we construct the mapping $(\psi_2)_j$ that makes the diagram

$$\begin{tikzcd} & G_j^B \arrow[dl, "\varphi_j"'] \arrow[d, "\pi_j^B"] \\ H \arrow[r, "(\psi_2)_j"'] & G_*^B \end{tikzcd}$$

commutative.

Since the subgroups $\pi_j(G_j)$ cover the group G_* and since the homomorphisms $(\psi_2)_j$ and $(\psi_2)_k$ are consistent on the common elements, the A-homomorphism $\psi_2 : G_* \to H$ is well-defined. The B-homomorphism ψ_2^B will be the sought counter homomorphism for ψ_1.

Remark 1. The permutability of tensor completion with direct limits allows one to reduce many problems on completion to the case of a finitely generated group. In fact, let $\{G_i, i \in I; \pi_i^j\}$ be the direct limit of all finitely generated subgroups of the group G. Then $G = \varinjlim_{i\in I} G_i$ and $G^B \cong \varinjlim_{i\in I} G_i^B$.

Remark 2. Let us give an example showing that the Cartesin product operation is not permutable with the operation of tensor completion.

Denote

$$i : \overline{\prod_j} G_j \to \overline{\prod_j} G_j^B.$$

Then by virtue of the universal property of tensor completions we have the B-homomorphism

$$i^B : \Big(\overline{\prod_j} G_j\Big)^B \to \overline{\prod_j} G_j^B$$

which in the general case is not an isomorphism. An analogous example already exists in the theory of abelian groups.

Let us take as a ring A the field of rational numbers $\mathbb{Q}$, while as G_n, $n \in \mathbb{N}$, we take a cyclic group of order n. Let $G_n = \langle a_n \rangle$, $n \in \mathbb{N}$. Then

$$G_n^{\mathbb{Q}} = G_n \otimes \mathbb{Q} = 0.$$

Therefore

$$\overline{\prod_j} G_j^B = 0.$$

At the same time the group $\overline{\prod_n} G_n$ contains elements of infinite order and therefore the group

$$\Big(\overline{\prod_n} G_n\Big)^{\mathbb{Q}} = \overline{\prod_n} G_n \otimes \mathbb{Q}$$

is nonzero.

Let G^* be a limit group of the inverse spectrum

$$\mathbb{G} = \{G_j, j \in J, \pi_j^k\}.$$

We construct the B-homomorphism

$$\sigma : (G^*)^B \to \varprojlim G_i^B.$$

For this we denote by

$$\pi_j : G^* \to G_j$$

the projection of the limit group onto the j-th component. Then

$$\pi_j^B : (G^*)^B \to G_j^B$$

is the corresponding homomorphism of tensor completions. Let

$$\mu_j : \varprojlim G_i^B \to G_j^B$$

be the natural projection.

Then there exists a homomorphism

$$\sigma : (G^*)^B \to \varprojlim G_j^B,$$

that makes the diagram

$$\begin{tikzcd}
(G^*)^B \arrow[r, "\sigma"] \arrow[dr, "\pi_j^B"'] & \varprojlim G_j^B \arrow[d, "\mu_j"] \\
& G_j^B
\end{tikzcd}$$

commutative.

We will illustrate by an example that in the general case this homomorphism is not an isomorphism.

Let us consider G_k, $k \in \mathbb{N}$, $G_k = \langle a_k \rangle$, where a_k is an element of order p^k, p is a prime number. Then, as is known, $\varprojlim G_k \cong \mathbb{Z}_{|\infty}$ is an additive group of integer p-adic numbers,

$$\mathbb{Z}_{|\infty}^{\mathbb{Q}} = \mathbb{Z}_{|\infty} \otimes \mathbb{Q}$$

is the vector space over $\mathbb{Q}$ of continual cardinality. Simultaneously,

$$\varprojlim G_k^{\mathbb{Q}} = \varprojlim (G_k \otimes \mathbb{Q}) = \varprojlim 0 = 0.$$

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