INTEGRAL SEMI-DISCRETE SCHEME FOR A KIRCHHOFF TYPE ABSTRACT EQUATION WITH THE GENERAL NONLINEARITY

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Abstract

Cauchy problem for a Kirchhoff-type abstract equation is considered with the general nonlinearity and self-adjoint positive definite operator, which is more than or equal to the square of the operator in the nonlinear term. Kirchhoff type equation for a beam represents a particular case of this equation. For the stated problem, the semi-discrete scheme is constructed, where for approximation of the term containing the gradient, the integral averaging is used. Stability of the scheme is proved and the error of the approximate solution is estimated.

Key words and phrases: Nonlinear abstract hyperbolic equation, Kirchhoff-type equation for a beam, Three-layer semi-discrete scheme

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Introduction

In the present work we consider a Kirchhoff type abstract equation for a beam in the Hilbert space with the self-adjoint positive definite operators A and B, which satisfy the condition $A^2 \leq c_0 B$ and with the general nonlinearity with respect to the gradient (here the role of the gradient is played $\left\|A^{1/2}u\right\|^2$, where u is a solution). This equation represents a generalization of the Kirchhoff type nonlinear equation for a beam (it was obtained by S. Woinowsky-Krieger [18]. Our aim is to find an approximate solution of the Cauchy problem stated for this equation. For this purpose we suggest the symmetric three-layer semi-discrete scheme, where for approximation of the term containing the gradient, the integral averaging is used.

Existence and uniqueness issues for local as well as global solutions of initial-boundary problem for the Kirchhoff string equation were first studied by Bernstein [2]. The issues of solvability of the classical and generalized Kirchhoff equations were later considered by many authors (see, for example, A. Arosio, S. Panizzi [1], L. Berselli, R. Manfrin [3], P. D'Ancona,

S. Spagnolo [5], [6], R. Manfrin [8], L.A. Medeiros [10], M. Matos [9], K. Nishihara [11], S. Panizzi [12], S.I. Pohozaev [15] and references therein).

The following works are devoted to approximate solution of nonlinear Kirchhoff type equation: A. I. Christie, J. Sanz-Serna [4], I. S. Liu, M. A. Rincon [7], J. Peradze [13], [14], J. Rogava, M. Tsiklauri [17]. In the work [14] the case is considered when the nonlinear term contains the sufficiently general function with respect to the gradient (the work [14] became known just before the publishing of the present work). Approximation of the term containing the gradient coincides with the approximation given in the work [14].

Investigation of the stability and convergence issues of the semi-discrete scheme given in this work are based upon the following two facts: (a) $(u_k - u_{k-1})/\tau$, $A^{1/2}u_k$ and $B^{1/2}u_k$ are uniformly bounded (u_k is an approximate solution, and τ is a step with respect to time variable); (b) For the corresponding linear discrete problem, the a priori estimate is valid, where in the left-hand side is the norm of $B^{1/2}u_k$, and in the right-hand side – the norm of f_k (f_k is a value of the right-hand side of the equation in the point $t = t_k = k\tau$). The mentioned fact allows to weaken the nonlinear term in the given nonlinear equation to such a degree that taking into account the fact (a), one can use Gronwell's lemma.

1 Statement of the problem and the integral semi-discrete scheme

Let us consider the Cauchy problem for an abstract hyperbolic equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + Bu\,(t) + \psi\left(\left\|A^{1/2}u\right\|^2\right) Au\,(t) = f\,(t)\,, \quad t \in [0,T]\,, \tag{1.1}$$

$$u\,(0) = \varphi_0, \quad \frac{du\,(0)}{dt} = \varphi_1. \tag{1.2}$$

where A and B are self-adjoint, positively defined (generally unbounded) operators with the definition domains $D\,(A)$ and $D(B)$ which are everywhere dense in H, besides, the following conditions are fulfilled

$$\|Au\|^2 \le c_0\,(Bu,u)\,, \quad \forall u \in D(B) \subset D\,(A)\,, \quad c_0 = const > 0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined correspondingly the norm and scalar product in H; scalar function $\psi(s)$, $s \in [0,+\infty)$ is continuous and twice continuously differentiable, in addition $\psi(s) \ge \lambda > 0$; φ_0 and φ_1 are given vectors from H; $u(t)$ is a continuous, twice continuously differentiable, sought-for function with values in H and $f(t)$ is a given continuous function with values in H (here continuity and differentiability is meant under the metric of H).

Existence and uniqueness of the solution of the problem (1.1), (1.2) (in case when $B = A^2$ and the scalar function $\psi(s) = \lambda + s$, $\lambda > 0$) is shown in [10]. Let us note that in this case equation (1.1) is an abstract analogue of Kirchhoff-type equation for a beam. Kirchhoff-type equation for a beam has the following form (see [18])

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} - \left(\lambda + \int_0^L u_\xi^2(\xi, t)\, d\xi\right) \frac{\partial^2 u}{\partial x^2} = f(x, t).$$

We are searching solution of the problem (1.1), (1.2) by the following semi-discrete scheme:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + B\frac{u_{k+1} + u_{k-1}}{2} + a_k \frac{Au_{k+1} + Au_{k-1}}{2} = f_k, \qquad (1.3)$$

where $k = 1, ..., n-1$, $\tau = T/n$ $(n > 1)$, $f_k = f(t_k)$, $t_k = k\tau$, $u_0 = \varphi_0$,

$$a_k = \widetilde{\psi}(\gamma_{k-1}, \gamma_{k+1}), \quad \gamma_k = \left\| A^{1/2} u_k \right\|^2,$$

and where the function $\widetilde{\psi}(a, b)$ is defined by the formula

$$\widetilde{\psi}(a, b) = \frac{1}{b-a} \int_a^b \psi(s) ds. \qquad (1.4)$$

It is obvious that if the length of the interval $b - a$ is small, then the formula (1.4) gives a good approximation of the value of the function $\psi(s)$ at the point $s = (a + b)/2$.

As an approximate solution $u(t)$ of problem (1.1), (1.2) at point $t_k = k\tau$ we state u_k, $u(t_k) \approx u_k$.

2 First step for the proof of the stability of the discrete problem (1.3)

In this section we will show in the standard way that $(u_k - u_{k-1})/\tau$, $A^{1/2}u_k$ and $B^{1/2}u_k$ are uniformly bounded for the discrete problem (1.3).

Our final aim is to obtain such a priori estimates for the scheme (1.3) from which follows the stability and convergence. Proof of the uniform boundedness of the solution to the discrete problem (1.3) and the difference analog of its corresponding first order derivative is a first step in this direction. Uniform boundedness means that for the concrete equation, during the numerical calculations, we should not expect the sharp increase of the solution value in case of the step's decrease.

The following theorem takes place (below everywhere c denotes positive constant).

Theorem 2.1 For the discrete problem (1.3), the vectors $\left(u_k - u_{k-1}\right)/\tau$, $A^{1/2}u_k$ and $B^{1/2}u_k$ are equally bounded, i.e there exist constants c_1, c_2 and c_3 (independent of n) such that

$$\left\|\frac{u_k - u_{k-1}}{\tau}\right\| \leq c_1, \quad \left\|B^{1/2}u_k\right\| \leq c_2, \quad \left\|A^{1/2}u_k\right\| \leq c_3, \quad k = 1, ..., n.$$

Proof. If we multiply scalarly both sides of equality (1.3) on vector $u_{k+1} - u_{k-1} = (u_{k+1} - u_k) + (u_k - u_{k-1})$, we obtain

$$\begin{aligned} & \left\|\frac{u_{k+1} - u_k}{\tau}\right\|^2 - \left\|\frac{u_k - u_{k-1}}{\tau}\right\|^2 + \frac{1}{2}\left(\left\|B^{1/2}u_{k+1}\right\|^2 - \left\|B^{1/2}u_{k-1}\right\|^2\right) \\ & + \frac{1}{2}a_k\left(\left\|A^{1/2}u_{k+1}\right\|^2 - \left\|A^{1/2}u_{k-1}\right\|^2\right) \\ = \; & \left(f_k, (u_{k+1} - u_k)\right) + \left(f_k, (u_k - u_{k-1})\right). \end{aligned} \tag{2.1}$$

Let us introduce denotations:

$$\alpha_k = \left\|\frac{u_k - u_{k-1}}{\tau}\right\|^2, \quad \beta_k = \left\|B^{1/2}u_k\right\|^2, \quad \gamma_k = \left\|A^{1/2}u_k\right\|^2.$$

If we take into account that according to the formula (1.4), we have

$$\begin{aligned} a_k\left(\gamma_{k+1} - \gamma_{k-1}\right) &= \widetilde{\psi}\left(\gamma_{k-1}, \gamma_{k+1}\right)\left(\gamma_{k+1} - \gamma_{k-1}\right) \\ &= \int_{\gamma_{k-1}}^{\gamma_{k+1}} \psi(s)ds = \int_0^{\gamma_{k+1}} \psi(s)ds - \int_0^{\gamma_{k-1}} \psi(s)ds, \end{aligned}$$

Then from (2.1) we obtain

$$\begin{aligned} & \alpha_{k+1} + \frac{1}{2}\left(\beta_{k+1} + \beta_k + \mu_{k+1} + \mu_k\right) \\ = \; & \alpha_k + \frac{1}{2}\left(\beta_k + \beta_{k-1} + \mu_k + \mu_{k-1}\right) \\ + \; & \left(f_k, (u_{k+1} - u_k)\right) + \left(f_k, (u_k - u_{k-1})\right), \end{aligned} \tag{2.2}$$

where

$$\mu_k = \int_0^{\gamma_k} \psi_1(s)ds.$$

According to Schwartz inequality, we have:

$$|(f_k, (u_{k+1} - u_k)) + (f_k, (u_k - u_{k-1}))| \leq \tau \left(\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}\right) \|f_k\| .$$

Taking into account this inequality, from (2.2) it follows:

$$\lambda_{k+1} \leq \lambda_k + \varepsilon_k, \tag{2.3}$$

where

$$\begin{aligned} \lambda_k &= \alpha_{k+1} + \frac{1}{2} \left(\beta_{k+1} + \beta_k + \mu_{k+1} + \mu_k\right), \\ \varepsilon_k &= \tau \left(\sqrt{\alpha_{k+1}} + \sqrt{\alpha_k}\right) \|f_k\| . \end{aligned}$$

Obviously from (2.3) we obtain

$$\begin{aligned} \lambda_{k+1} &\leq \lambda_1 + (\varepsilon_1 + \varepsilon_2 + ... + \varepsilon_k) \\ &= \lambda_1 + \tau \sum_{i=1}^{k} (\sqrt{\alpha_i} + \sqrt{\alpha_{i+1}}) \|f_i\| . \end{aligned}$$

Obviously from here we get:

$$\delta_{k+1}^2 \leq \delta_1^2 + \tau \sum_{i=1}^{k} (\delta_i + \delta_{i+1}) \|f_i\| , \quad \delta_k = \sqrt{\lambda_k}.$$

From here we obtain the following inequality

$$\delta_{k+1} \leq \delta_1 + 2\tau \sum_{i=1}^{k} \|f_i\|$$

From here it follows that α_k, β_k and γ_k are equally bounded. □

3 The a priori estimates for perturbation of the solution of the discrete problem

Our aim is to show the stability of the scheme (1.3). Since the additiveness does not take place for a nonlinear problem, we naturally try to obtain the a priori estimate directly for perturbation of the solution. Hence (analogously to the linear problem) there automatically follows the stability and convergence of the nonlinear scheme.

In this section, based on the results of the previous sections, we will obtain the a priori estimates for perturbations of the solution of the semi-discrete scheme (1.3) and the difference analog of its corresponding first order derivative.

The following theorem takes place (below everywhere c denotes positive constant).

Theorem 3.1 Let u_k and $\overline{u}_k$ be solutions of difference equation (1.3) corresponding to initial vectors (u_0, u_1, f_k) and $\left(\overline{u}_0, \overline{u}_1, \overline{f_k}\right)$, components of which are sufficiently smooth. Then for $z_k = u_k - \overline{u}_k$ the following estimates are true:

$$\begin{aligned} \left\|B^{1/2} z_{k+1}\right\| &\leq c\left(\left\|B^{1/2} z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\| + \tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\|\right. \\ &+ \left.\tau\sum_{i=1}^{k}\left\|f_i - \overline{f_i}\right\|\right), \qquad (3.1) \\ \left\|\frac{\Delta z_k}{\tau}\right\| &\leq c\left(\left\|B^{1/2} z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\| + \tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\|\right. \\ &+ \left.\tau\sum_{i=1}^{k}\left\|f_i - \overline{f_i}\right\|\right), \qquad (3.2) \end{aligned}$$

where $k = 1, ..., n-1$, $\Delta z_k = z_{k+1} - z_k$.

Proof of the theorem is base a upon lemma, which will be given below.

Let us consider in Hilbert space H the following difference linear equation:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + B\frac{u_{k+1} + u_{k-1}}{2} = f_k, \quad k = 1, ..., n-1, \qquad (3.3)$$

where u_0, u_1 and f_k are the given vectors from H.

The difference equation (3.3) represents a main part of the nonlinear difference equation (1.3).

The following lemma takes place.

Lemma 3.2 (see [16]). Let B be a self-adjoint positive definite operator. Then for the scheme (3.3), the following a priori estimates are valid:

$$
\begin{aligned}
\left\|B^{1/2}u_{k+1}\right\| &\leq \sqrt{2}\left(\left\|B^{1/2}u_0\right\| + \left\|\frac{\Delta u_0}{\tau}\right\|\right) + \tau\left\|B^{1/2}\frac{\Delta u_0}{\tau}\right\| \\
&+ \tau\sum_{i=1}^{k}\|f_i\|, \quad u_0, u_1 \in D(B^{1/2}), \qquad (3.4) \\
\left\|\frac{\Delta u_k}{\tau}\right\| &\leq \left\|B^{1/2}u_0\right\| + \sqrt{2}\left\|\frac{\Delta u_0}{\tau}\right\| \\
&+ \tau\sum_{i=1}^{k}\|f_i\|, \quad u_0 \in D(B^{1/2}), \qquad (3.5)
\end{aligned}
$$

where $k = 1, ..., n-1$, $\Delta u_k = u_{k+1} - u_k$.

Let us return to the proof of the Theorem 3.1.

Proof. According to (1.3) $z_k = u_k - \overline{u}_k$ will satisfy the following equation:

$$
\frac{z_{k+1} - 2z_k + z_{k-1}}{\tau^2} + B\frac{z_{k+1} + z_{k-1}}{2} = -\frac{1}{2}g_k, \quad k = 1, ..., n-1, \qquad (3.6)
$$

where

$$
g_k = a_k A\left(u_{k+1} + u_{k-1}\right) - \overline{a}_k A\left(\overline{u}_{k+1} + \overline{u}_{k-1}\right) - 2\left(f_k - \overline{f}_k\right).
$$

and where $\overline{a}_k = \widetilde{\psi}\left(\overline{\gamma}_{k+1}, \overline{\gamma}_{k-1}\right)$, $\overline{\gamma}_k = \left\|A^{1/2}\overline{u}_k\right\|^2$.

From (3.6) according to (3.4) it follows

$$
\begin{aligned}
\left\|B^{1/2}z_{k+1}\right\| &\leq \sqrt{2}\left(\left\|B^{1/2}z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\|\right) \\
&+ \tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\| + \frac{\tau}{2}\sum_{i=1}^{k}\|g_i\|. \qquad (3.7)
\end{aligned}
$$

Obviously for g_k we have

$$
g_k = \left(a_k - \overline{a}_k\right)\left(Au_{k+1} + Au_{k-1}\right) + \overline{a}_k\left(Az_k + Az_{k+1}\right) - 2\left(f_k - \overline{f}_k\right). \qquad (3.8)
$$

By simple transformations, for the difference $a_k - \overline{a}_k$, we obtain:

$$\begin{aligned}
a_k - \overline{a}_k &= \frac{1}{\gamma_{k+1} - \gamma_{k-1}} \int\limits_{\gamma_{k-1}}^{\gamma_{k+1}} \psi(s)\, ds - \frac{1}{\overline{\gamma}_{k+1} - \overline{\gamma}_{k-1}} \int\limits_{\overline{\gamma}_{k-1}}^{\overline{\gamma}_{k+1}} \psi(s)\, ds \\
&= \int\limits_0^1 \left[\psi\left(\gamma_{k-1} + \left(\gamma_{k+1} - \gamma_{k-1}\right)\xi\right) - \psi\left(\overline{\gamma}_{k-1} + \left(\overline{\gamma}_{k+1} - \overline{\gamma}_{k-1}\right)\xi\right)\right] d\xi \\
&= \int\limits_0^1 \int\limits_{\xi_k}^{\overline{\xi}_k} \psi'(\eta)\, d\eta d\xi, \qquad (3.9)
\end{aligned}$$

where

$$\begin{aligned}
\xi_k &= \gamma_{k-1} + \left(\gamma_{k+1} - \gamma_{k-1}\right)\xi, \\
\overline{\xi}_k &= \overline{\gamma}_{k-1} + \left(\overline{\gamma}_{k+1} - \overline{\gamma}_{k-1}\right)\xi.
\end{aligned}$$

From (3.9), according to Lemma 3.1, we have

$$\begin{aligned}
\left|a_k - \overline{a}_k\right| &\le \int\limits_0^1 \left| \int\limits_{\xi_k}^{\overline{\xi}_k} \psi'(\eta)\, d\eta \right| d\xi \le c \int\limits_0^1 \left|\overline{\xi}_k - \xi_k\right| d\xi \\
&= c \int\limits_0^1 \left|\left(\gamma_{k-1} - \overline{\gamma}_{k-1}\right) + \left(\left(\gamma_{k+1} - \overline{\gamma}_{k+1}\right) - \left(\gamma_{k-1} - \overline{\gamma}_{k-1}\right)\right)\xi\right| d\xi \\
&\le c\left(\left|\gamma_{k-1} - \overline{\gamma}_{k-1}\right| + \left|\gamma_{k+1} - \overline{\gamma}_{k+1}\right|\right).
\end{aligned}$$

Taking this into account, from (3.8) we obtain

$$\begin{aligned}
\|g_k\| \le\; & c\left(\left|\gamma_{k-1} - \overline{\gamma}_{k-1}\right| + \left|\gamma_{k+1} - \overline{\gamma}_{k+1}\right|\right)\left(\|Au_{k+1}\| + \|Au_{k-1}\|\right) \\
& + \left|\overline{a}_k\right|\left(\|Az_k\| + \|Az_{k+1}\|\right) + 2\left(\left\|f_k - \overline{f}_k\right\|\right). \qquad (3.10)
\end{aligned}$$

Obviously for $|\overline{a}_k|$ we have the estimate:

$$\begin{aligned}
\left|\overline{a}_k\right| &= \left| \frac{1}{\overline{\gamma}_{k+1} - \overline{\gamma}_{k-1}} \int\limits_{\overline{\gamma}_{k-1}}^{\overline{\gamma}_{k+1}} \psi(s)\, ds \right| \\
&= \left| \int\limits_0^1 \psi\left(\overline{\gamma}_{k-1} + \left(\overline{\gamma}_{k+1} - \overline{\gamma}_{k-1}\right)\xi\right) d\xi \right| \le c. \qquad (3.11)
\end{aligned}$$

According to Lemma 3.1, for $\gamma_k - \overline{\gamma}_k$, the following estimate is valid

$$\begin{aligned} |\gamma_k - \overline{\gamma}_k| &= \left(\sqrt{\gamma_k} + \sqrt{\overline{\gamma}_k}\right)\left|\sqrt{\gamma_k} - \sqrt{\overline{\gamma}_k}\right| \\ &\leq c\left|\left\|A^{1/2}u_k\right\| - \left\|A^{1/2}\overline{u}_k\right\|\right| \\ &\leq c\left\|A^{1/2}z_k\right\| \leq c\left\|Az_k\right\|. \qquad (3.12) \end{aligned}$$

If we insert these estimates in (3.10) and take into account that $\|Au_k\|$ is uniformly bounded, we obtain

$$\begin{aligned} \|g_k\| &\leq c\left(\|Az_{k+1}\| + \|Az_k\| + \left\|f_k - \overline{f}_k\right\|\right) \\ &\leq c\left(\left\|B^{1/2}z_{k+1}\right\| + \left\|B^{1/2}z_k\right\| + \left\|f_k - \overline{f}_k\right\|\right). \qquad (3.13) \end{aligned}$$

Taking this inequality into account, from (3.7) we obtain

$$\begin{aligned} \left\|B^{1/2}z_{k+1}\right\| &\leq \sqrt{2}\left(\left\|B^{1/2}z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\|\right) \\ &+\tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\| + c\tau\sum_{i=1}^{k+1}\left\|B^{1/2}z_i\right\| + c\tau\sum_{i=1}^{k}\left\|f_i - \overline{f}_i\right\|. \qquad (3.14) \end{aligned}$$

From here we have

$$\begin{aligned} \left\|B^{1/2}z_{k+1}\right\| \leq\ & c_0\left(\sqrt{2}\left(\left\|B^{1/2}z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\|\right. \\ & \left.+c\tau\sum_{i=1}^{k}\left\|B^{1/2}z_i\right\| + c\tau\sum_{i=1}^{k}\left\|f_i - \overline{f}_i\right\|\right), \end{aligned}$$

where $c_0 = \frac{1}{1-c\tau}$, $1 - c\tau > 0$.

Let us introduce the following notations:

$$\begin{aligned} \delta_k &= c\left(\left\|B^{1/2}z_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\| + \tau\left\|B^{1/2}\frac{\Delta z_0}{\tau}\right\| + \tau\sum_{i=1}^{k}\left\|f_i - \overline{f}_i\right\|\right), \\ \varepsilon_k &= \left\|B^{1/2}z_k\right\|. \end{aligned}$$

Then inequality (3.14) can be rewritten as

$$\varepsilon_{k+1} \leq c\tau\sum_{i=1}^{k}\varepsilon_i + \delta_k.$$

From here by the induction we can obtain (discrete analog of Gronwell's lemma):

$$\varepsilon_{k+1} \leq c\tau\left(1 + c\tau\right)^{k-1}\varepsilon_1 + \left(1 + c\tau\right)^{k-1}\delta_k = \left(1 + c\tau\right)^{k-1}\left(\delta_k + c\tau\varepsilon_1\right). \qquad (3.15)$$

If we take into consideration that $(1+c\tau)^k \le e^{ct_k}$, from (3.15) we obtain

$$\begin{aligned} \left\| B^{1/2} z_{k+1} \right\| \ \le\ & ce^{ct_k} \left(\left\| B^{1/2} z_0 \right\| + \left\| \frac{\Delta z_0}{\tau} \right\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\| . \right. \\ & \left. + \tau \sum_{i=1}^{k} \left\| f_i - \overline{f}_i \right\| + \tau \left\| B^{1/2} z_1 \right\| \right) . \end{aligned} \tag{3.16}$$

If we take into account that

$$\left\| B^{1/2} z_1 \right\| \le \left\| B^{1/2} z_0 \right\| + \tau \left\| B^{1/2} \frac{\Delta z_0}{\tau} \right\| ,$$

from (3.16) we obtain (3.1).

Let us show the estimate (3.2). From (3.6), according to (3.5), we obtain

$$\left\| \frac{\Delta z_k}{\tau} \right\| \le \left\| B^{1/2} z_0 \right\| + \sqrt{2} \left\| \frac{\Delta z_0}{\tau} \right\| + \frac{\tau}{2} \sum_{i=1}^{k} \left\| g_i \right\| .$$

Hence, taking into account (3.13), we obtain

$$\left\| \frac{\Delta z_k}{\tau} \right\| \le \left\| B^{1/2} z_0 \right\| + \sqrt{2} \left\| \frac{\Delta z_0}{\tau} \right\| + c\tau \sum_{i=1}^{k+1} \left\| B^{1/2} z_i \right\| + c\tau \sum_{i=1}^{k} \left\| f_i - \overline{f_i} \right\| .$$

From here an account of (3.1) we get (3.2).

4 Error estimate of the approximate solution on the smooth class of solutions

In this section, using the results of the previous sections, we prove the theorem regarding the error estimate of the approximate solution. It can be said that this theorem represents an almost trivial result of the theorem proved in the previous section. Before directly stating the theorem on the convergence of the scheme (1.3), we would like to make one remark regarding the well–posedness of the problem (1.1),(1.2). In fact we mean from the beginning that the initial continuous problem is well–posed and the solution is sufficiently smooth. Obviously we need the smoothness of the solution in order to find the convergence order. If we demand the minimal smoothness, which is necessary for well–posedness of the problem, the convergence will be guaranteed, but we will not be able to find the order. If we increase the smoothness order by one unit, the convergence order will be equal to one (in this case, as well as in the previous case it is sufficient to take $u_1 = \varphi_0 + \tau\varphi_1$). If we increase the smoothness order by two units, and define the starting vector u_1 by the following formula

$$u_1 = \varphi_0 + \tau\varphi_1 + \frac{\tau^2}{2}\varphi_2, \tag{4.1}$$

where

$$\varphi_2 = f_0 - \left(B\varphi_0 + \psi\left(\left\|A^{1/2}\varphi_0\right\|^2\right)A\varphi_0\right), \quad \varphi_0 \in D(B),$$

the convergence order will be equal to two. Further increase of the solution smoothness does not make sense, as the approximation order of the scheme (1.3) is not more than two (obviously the convergence order generally can not exceed the approximation order).

Let us state the theorem on the convergence of the semi-discrete scheme (1.3).

Theorem 4.1 Let the problem (1.1),(1.2) be well-posed. Besides, the following conditions are fulfilled: (a) $\varphi_0 \in D(B)$, $\varphi_1, \varphi_2 \in D\left(B^{1/2}\right)$; (b) Solution $u(t)$ of problem (1.1), (1.2) is continuously differentiable to third degree including and $u'''(t)$ satisfies the Lipschitz condition; (c) $u'(t) \in D(B)$ for every t from $[0,T]$ and function $Bu'(t)$ satisfy the Lipschitz condition.

Then for the scheme (1.3),(4.1) the following estimates are true:

$$\max_{1\le k\le n-1}\left\|B^{1/2}\widetilde{z}_k\right\| \le c\tau^2, \tag{4.2}$$

$$\max_{1\le k\le n-1}\left\|\frac{\Delta\widetilde{z}_k}{\tau}\right\| \le c\tau^2. \tag{4.3}$$

where: $\widetilde{z}_k = u(t_k) - u_k$ is an error of the approximate solution, $\Delta\widetilde{z}_k = \widetilde{z}_{k+1} - \widetilde{z}_k$.

Proof. Let us write down the equation (1.1) at point $t = t_k$ in the following form:

$$\begin{aligned} &\frac{\Delta^2 u(t_{k-1})}{\tau^2} + B\frac{u(t_{k+1}) + u(t_{k-1})}{2} \\ +\ &\psi\left(\left\|A^{1/2}u(t_k)\right\|^2\right)\frac{Au(t_{k+1}) + Au(t_{k-1})}{2} \\ =\ &f(t_k) + r_\tau(t_k), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} r_\tau(t_k) &= r_{0,\tau}(t_k) + r_{1,\tau}(t_k) + r_{2,\tau}(t_k), \qquad (4.5)\\ r_{0,\tau}(t_k) &= \frac{\Delta^2 u(t_{k-1})}{\tau^2} - u''(t_k),\\ r_{1,\tau}(t_k) &= \frac{1}{2} B\left(\Delta^2 u(t_{k-1})\right),\\ r_{2,\tau}(t_k) &= \frac{1}{2}\psi\left(\left\|A^{1/2}u(t_k)\right\|^2\right) A\left(\Delta^2 u(t_{k-1})\right). \end{aligned}$$

From (4.4) and (1.3) according to Theorem 3.1 we obtain

$$\begin{aligned} \left\|B^{1/2}\widetilde{z}_{k+1}\right\| &\le c\left(\left\|B^{1/2}\widetilde{z}_0\right\| + \left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\| + \tau\left\|B^{1/2}\frac{\Delta\widetilde{z}_0}{\tau}\right\|\right.\\ &+ \left.\tau\sum_{i=1}^{k}\left\|r_\tau(t_k)\right\|\right). \qquad (4.6) \end{aligned}$$

According to how smooth the function $u(t)$, is the following formulas are true:

$$\begin{aligned} \frac{\Delta^2 u(t_{k-1})}{\tau^2} - u''(t_k) = \frac{1}{\tau^2}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\int_{t_k}^{s}\left(u'''(\xi) - u'''(t_k)\right)d\xi ds dt\\ + \frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t}\int_{t_{k-1}}^{s}\left(u'''(t_k) - u'''(\xi)\right)d\xi ds dt, \qquad (4.7) \end{aligned}$$

$$\begin{aligned} \Delta^2 u(t_{k-1}) = \int_{t_k}^{t_{k+1}}\left(u'(t) - u'(t_k)\right)dt\\ + \int_{t_{k-1}}^{t_k}\left(u'(t_k) - u'(t)\right)dt, \qquad (4.8) \end{aligned}$$

$$u(t_1) = u_0 + \tau u'(0) + \int_0^\tau\left(u'(t) - u'(0)\right)dt, \qquad (4.9)$$

$$u(t_1) = u_0 + \tau u'(0) + \frac{\tau^2}{2}u''(0) + \int_0^\tau\int_0^t\int_0^s u'''(\xi)\,d\xi ds dt. \qquad (4.10)$$

According to conditions (a) and (c) of Theorem 4.1, from (4.9) there follows:

$$\left\|B^{1/2}\left(\Delta\widetilde{z}_0\right)\right\| = \left\|B^{1/2}\left(\widetilde{z}_1-\widetilde{z}_0\right)\right\| = \left\|B^{1/2}\widetilde{z}_1\right\| = \left\|B^{1/2}\left(u\left(t_1\right)-u_1\right)\right\|$$

$$= \left\|-\frac{\tau^2}{2}B^{1/2}\varphi_2+\int\limits_0^{\tau}B^{1/2}\left(u'\left(t\right)-u'\left(0\right)\right)dt\right\| \leq c\tau^2. \tag{4.11}$$

According to conditions (b) of Theorem 4.1, from (4.10) there follows

$$\left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\| = \frac{1}{\tau}\left\|u\left(t_1\right)-u_1\right\| = \frac{1}{\tau}\int\limits_0^{\tau}\int\limits_0^{t}\int\limits_0^{s}\left\|u'''\left(\xi\right)\right\|d\xi ds dt \leq c\tau^2. \tag{4.12}$$

According to condition (b) of Theorem 4.1, from (4.7) we have

$$\left\|r_{0,\tau}\left(t_k\right)\right\| = \left\|\frac{\Delta^2 u\left(t_{k-1}\right)}{\tau^2}-u''\left(t_k\right)\right\| \leq c\tau^2. \tag{4.13}$$

According to condition (c) of Theorem 4.1, from (4.8) we get

$$\left\|r_{j,\tau}\left(t_k\right)\right\| \leq c\tau^2, \quad j=1,2. \tag{4.14}$$

From (4.5), on account of inequalities (4.13) and (4.14), there follows

$$\left\|r_{\tau}\left(t_k\right)\right\| \leq c\tau^2. \tag{4.15}$$

From (4.6), on account of inequalities (4.11), (4.12) and (4.15), there follows (4.2).

According to (3.2), is obtained analogously (4.3). □

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