THE DIRICHLET BOUNDARY VALUE PROBLEM OF THE THEORY OF CONSOLIDATION WITH DOUBLE POROSITY

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Abstract

The purpose of this paper is to consider three-dimensional version of quasistatic Aifantis' equation of the theory of consolidation with double porosity and to study the uniqueness and existence of solution of the Dirichlet boundary value problem (BVP). Using the fundamental matrix we will construct the simple and double layer potentials and study their properties. Using the potential method, for the Dirichlet BVP we construct Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

Key words and phrases: Steady-state quasistatic equations, porous media, double porosity, fundamental solution.

AMS subject classification: 74G25,74G30.

Introduction

A theory of consolidation with double porosity has been proposed by Aifantis. This theory unifies a model proposed by Biot for the consolidation of deformable single porosity media with a model proposed by Barenblatt for seepage in undeformable media with two degrees of porosity. In a material with two degrees of porosity, there are two pore systems, the primary and the secondary. For example in a fissured rock (i.e., a mass of porous blocks separated from each other by an interconnected and continuously distributed system of fissures) most of the porosity is provided by the pores of the blocks or primary porosity, while most of permeability is provided by the fissures or the secondary porosity. When fluid flows and deformations processes occur simultaneously , three coupled partial differential equations can be derived [1],[2] to describe the relationships governing pressure in the primary and secondary pores (and therefore the mass exchange between them) and the displacement of the solid. Inertia effects are neglected as they are in Biot's theory.

The physical and mathematical foundations of the theory of double porosity were considered in the papers [1]-[3]. In part I of a series of paper on the subject, R. K. Wilson and E. C. Aifantis [1] gave detailed physical interpretations of the phenomenological coefficients appearing in the double porosity theory.They also solved several representative boundary value problems. In part II of this series, uniqueness and variational principles were established by D. E. Beskos and E. C. Aifantis [2] for the equations of double porosity,while in part III Khaled, Beskos and Aifantis [3] provided a related finite element to consider the numerical solution of Aifantis' equations of double porosity (see [1],[2],[3] and references cited therein). The basic results and the historical information on the theory of porous media were summarized by the Boer [4].

In this paper for the solution of the Dirichlet BVP we construct a Fredholm type integral equation of the second kind and prove the existence theorem of solution for the finite and infinite domains.

1 Basic Equations, Boundary Value Problem and Uniqueness Theorems

The basic steady-state quasistatic Aifantis' equations of the theory of consolidation with double porosity are given by the partial differential equations in the form [1], [2]

$$\begin{gathered}\mu\Delta u+(\lambda+\mu) graddivu - grad(\beta_1 p_1+\beta_2 p_2)=0,\\ i\omega\beta_1 divu+(m_1\Delta+\alpha_3)p_1+kp_2=0, \\ i\omega\beta_2 divu+kp_1+(m_2\Delta+\alpha_4)p_2=0,\end{gathered}\tag{1.1}$$

where $u=(u_1,u_2,u_3)$ is the displacement vector, p_1 is the fluid pressure within the primary pores and p_2 is the fluid pressure within the secondary pores. $\alpha_3=i\omega\alpha_1-k,\ \ \alpha_4=i\omega\alpha_2-k,\ \ m_j=\dfrac{k_j}{\mu^*}, j=1,2$. The constant λ is the Lame modulus, μ is the shear modulus and the constants β_1 and β_2 measure the change of porosities due to an applied volumetric strain. The constants α_1 and α_2 measure the compressibilities of primary and secondary pores filled with pore fluid.The constants k_1 and k_2 are the permeabilities of the primary and secondary systems of pores, the constant μ^* denotes the viscosity of the pore fluid and the constant k measures the transfer of fluid from the secondary pores to the primary pores. The quantities

$\lambda,\ \mu,\ \alpha_j,\ \beta_j,\ k_j\ (j=1,2)$ and μ^* are all positive constants. $\triangle$ is the Laplace operator, ω is the oscilation frequency $(\omega > 0)$.

We also rewrite the equation (1.1) in the matrix form

$$B(\partial x)U = 0, \tag{1.2}$$

where

$$B(\partial x) = \parallel B_{pq}(\partial x) \parallel_{5x5}, \ \ p,q = 1,2,3,4,5,$$

$$B_{jj}(\partial x) = \mu\Delta + (\lambda+\mu)\frac{\partial^2}{\partial x_j^2}, \ \ j = 1,2,3,$$

$$B_{1j}(\partial x) = B_{j1}(\partial x) = (\lambda+\mu)\frac{\partial^2}{\partial x_1 \partial x_j}, j = 2,3,$$

$$B_{23}(\partial x) = B_{32}(\partial x) = (\lambda+\mu)\frac{\partial^2}{\partial x_3 \partial x_2},$$

$$B_{j4}(\partial x) = -\beta_1\frac{\partial}{\partial x_j}, \quad B_{j5}(\partial x) = -\beta_2\frac{\partial}{\partial x_j}, \quad j = 1,2,3,$$

$$B_{4j}(\partial x) = i\omega\beta_1\frac{\partial}{\partial x_j}, \quad B_{5j}(\partial x) = i\omega\beta_2\frac{\partial}{\partial x_j}, \quad j = 1,2,3,$$

$$B_{44}(\partial x) = m_1\Delta + \alpha_3, \ \ B_{45} = k\ \ , B_{54} = k,$$

$$B_{55}(\partial x) = m_2\Delta + \alpha_4, \ \ U(u_1,u_2,u_3,p_1,p_2).$$

The conjugate system of the equation (1.2) is

$$\widetilde{B}(\partial x)U = B^T(-\partial x)U = 0.$$

Throughout this paper the superscript "T" denotes transposition.

Write now the expressions for the components of the generalizes stress vector. Denoting the generalized stress tensor by $\overset{\kappa}{\mathrm{R}}(\partial x, n)$ where κ is an arbitrary constant, we have

$$\overset{\kappa}{\mathrm{R}}(\partial x, n) = \parallel \overset{\kappa}{\mathrm{R}}_{kj}(\partial x, n) \parallel_{5\times 5}, \tag{1.3}$$

where [6]

$$\begin{aligned} \overset{\kappa}{\mathrm{R}}_{kj}(\partial x, n) &= \mu\delta_{kj}\frac{\partial}{\partial n} + (\lambda+\mu)n_k\frac{\partial}{\partial x_j} \\ &+\kappa\left(n_j\frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial x_j}\right), \quad k,j = 1,2,3, \end{aligned} \tag{1.4}$$

$$\overset{\kappa}{\mathrm{R}}_{j4}(\partial x,n)=-\beta_1 n_j,\quad \overset{\kappa}{\mathrm{R}}_{j5}(\partial x,n)=-\beta_2 n_j,$$
$$\overset{\kappa}{\mathrm{R}}_{4j}(\partial x,n)=\overset{\kappa}{\mathrm{R}}_{5j}(\partial x,n)=\overset{\kappa}{\mathrm{R}}_{45}(\partial x,n)=\overset{\kappa}{\mathrm{R}}_{54}(\partial x,n)=0,$$
$$\overset{\kappa}{\mathrm{R}}_{44}(\partial x,n)=m_1\frac{\partial}{\partial n},\quad \overset{\kappa}{\mathrm{R}}_{55}(\partial x,n)=m_2\frac{\partial}{\partial n},\quad j=1,2.3.$$

$n=(n_1,n_2,n_3)$ is the unit normal vector.

If $\kappa=\mu$, we have the stress vector $P(\partial x,n)u$. The operator, which will be obtained from $\overset{\kappa}{\mathrm{P}}(\partial x,n)$, for $\kappa=\kappa_n=\dfrac{\mu(\lambda+\mu)}{\lambda+3\mu}$, will be called the operator $N(\partial x,n)$, and vector $N(\partial x,n)u$ will be called the pseudo-stress vector. The pseudo-stress operator succeeded in obtaining the Fredholm integral equation of the second kind for the Dirichlet boundary value problem.

Let $D^+(D^-)$ be a finite (an infinite) three-dimensional domain bounded by the surface S. Suppose that $S\in C^{1,\beta},\quad 0<\beta\le 1$.

We introduce the following definition:

Definition 1. A vector-function $U(x)=(u_1,u_2,u_3,p_1,p_2)$ defined in the domain $D^+(D^-)$ is called regular if it has integrable continuous second order derivatives in $D^+(D^+)$, and U and its first order derivatives are continuously extendable at every point of the boundary of $D^+(D^-)$, i.e., $U\in C^2(D^+)\bigcap C^1(\overline{D^+}),\quad (U\in C^2(D^+)\bigcap C^1(\overline{D^+}))$. Note that for the infinite domain D^- the vector $U(x)$ additionally satisfies the following conditions at infinite:

$$U(x)=O(|x|^{-1}),\quad \frac{\partial U_k}{\partial x_j}=O(|x|^{-2}),\quad |x|^2=x_1^2+x_2^2+x_3^2,\ \ j=1,2,3. \tag{1.5}$$

For the equation (1.1) we pose the following boundary value problem: find in the domain $D^+(D^-)$ a regular solution U of equation (1.1), satisfying on the boundary S the following Dirichlet boundary conditions:

Problem 1. The displacement vector and the fluid pressures are given in the form

$$u^{\pm}(u_1,u_2,u_3)=f(z)^{\pm},\quad p_1^{\pm}=f_4^{\pm},\ \ p_2^{\pm}=f_5^{\pm},\quad z\in S.$$

Generalized Green's Formulas: Let U and $\overline{u}$ be two regular solutions of equation (1.1) in D^+. Multiply the first equation of (1.1) by $\overline{u}=(\overline{u_1},\overline{u_2},\overline{u_3})$, the second one by $\overline{p_1}$ and the third by $\overline{p_2}$, where $\overline{u}$, $\overline{p_1}$ and $\overline{p_2}$ are the complex conjugate of u, p_1 and p_2 respectively. Integration of the results over D^+ and then their addition, after some simplification, gives

$$\int_{D^+}\left[E(u,\bar{u})+\alpha_1|p_1|^2+\alpha_2|p_2|^2+\frac{k}{i\omega}|p_1-p_2|^2+\frac{m_1}{i\omega}|gradp_1|^2\right.$$

$$\left.+\frac{m_2}{i\omega}|gradp_2|^2\right]dV=\int_S\left[\overline{u}\overset{\kappa}{\mathrm{P}}(\partial x,n)u+\frac{m_1}{i\omega}p_1\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}p_2\frac{\partial\overline{p_2}}{\partial n}\right]dS,$$

$$\overset{\kappa}{\mathrm{P}}(\partial x,n)u=\overset{\kappa}{\mathrm{T}}(\partial x,n)u-n(\beta_1p_1+\beta_2p_2),\quad u=(u_1,u_2,u_3),$$

$$\overset{\kappa}{\mathrm{T}}(\partial x)_{kj}=\mu\delta_{kj}\frac{\partial}{\partial n}+(\lambda+\mu)n_k\frac{\partial}{\partial x_j}$$

$$+\kappa\left(n_j\frac{\partial}{\partial x_k}-n_k\frac{\partial}{\partial x_j}\right),\quad k,j=1,2,3, \tag{1.6}$$

where [6]

$$E(u,u)=(\lambda+\mu-\kappa)\left(\sum_k\frac{\partial u_k}{\partial x_k}\right)^2+\frac{\mu+\kappa}{4}\sum_{k,q}\left(\frac{\partial u_k}{\partial x_q}+\frac{\partial u_q}{\partial x_k}\right)^2$$

$$+\frac{\mu-\kappa}{4}\sum_{k,q}\left(\frac{\partial u_k}{\partial x_q}-\frac{\partial u_q}{\partial x_k}\right)^2.$$

For the positive definiteness of the potential energy the inequalities $\lambda+\mu-\kappa>0,\quad \mu+\kappa>0,\quad \mu-\kappa>0,$ are necessary and sufficient.

One can generalize the formula (1.6) for an infinite domain D^- , provided the conditions

$$\lim_{R\to\infty}\int_{S(0,R)}\left[\overline{u}\overset{\kappa}{\mathrm{P}}(\partial x,n)u+\frac{m_1}{i\omega}p_1\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}p_2\frac{\partial\overline{p_2}}{\partial n}\right]dS=0, \tag{1.7}$$

are fulfilled, where $S(0,R)$ is a sphere of radius R with center at the point O, lying inside D^+ . The radius R is taken so large that the region D^+ lies entirely inside the sphere $S(0,R)$.

Obviously, the conditions (1.7) are fulfilled if the vector u and $\overline{u}$ satisfy the conditions (1.5).

If (1.7) are fulfilled the Green's formula for the domain D^- takes the form

$$\int\limits_{D^-}\left[E(u,u)+\alpha_1|p_1|^2+\alpha_2|p_2|^2+\frac{k}{i\omega}|p_1-p_2|^2+\frac{m_1}{i\omega}|gradp_1|^2\right.$$

$$\left.+\frac{m_2}{i\omega}|gradp_2|^2\right]dV=-\int\limits_{S}\left[\overline{u}\overset{\kappa}{\mathrm{P}}(\partial x,n)u+\frac{m_1}{i\omega}p_1\frac{\partial\overline{p_1}}{\partial n}+\frac{m_2}{i\omega}p_2\frac{\partial\overline{p_2}}{\partial n}\right]dS, \tag{1.8}$$

The Uniqueness Theorems. In this subsection we investigate the question of the uniqueness of solution of the above-mentioned problem.

Now Let us prove the following theorems:

Theorem 1. The first boundary value problem has at most one regular solution in the finite domain D^+.

Proof: Let the first BVP have in the domain D^+ two regular solutions $U^{(1)}$ and $U^{(2)}$. We write $v=U^{(1)}-U^{(2)}$. Evidently the vector v satisfies (1.1) and the boundary condition $v^+=0$ on S . Note that if v is a regular solution of the equation (1.1), we have the Green's formula (1.6). Using (1.6) and taking into account the fact that the pseudopotential energy is positive definite, we conclude that $v=C$, $x\in D^+$, where $C=const$. Since $v^+=0$, we have $C=0$ and $v(x)=0$, $x\in D^+$.

Theorem 2. The first boundary value problem has at most one regular solution in the infinite domain D^-.

Proof: The vectors $U^{(1)}$ and $U^{(2)}$ in the domain D^- must satisfy the condition at infinite (1.5). In this case the formula (1.8) is valid and $v(x)=C$, $x\in D^-$, where C is again the constant vector. But v on the boundary satisfies the condition $v^-=0$, which implies that $C=0$ and $v(x)=0$, $x\in D^-$.

2 Matrix of Fundamental Solutions

The matrix of fundamental solutions for the system (1.1) is given in the work of M. Svanadze [5]. We rewrite it in the form

$$\Gamma(x-y)=\parallel\Gamma_{pq}(\partial x)\parallel_{5\times5},\quad p,q=1,2,3,4,5, \tag{2.1}$$

where

$$\Gamma_{kj} = \Gamma_{jk} = \frac{\delta_{kj}}{\mu r} - \frac{\partial^2 \Psi_{11}}{\partial x_j x_k}, \quad \Gamma_{j4} = \frac{1}{am_1 m_2}\frac{\partial \Psi_{12}}{\partial x_j},$$

$$\Gamma_{j5} = \frac{1}{am_1 m_2}\frac{\partial \Psi_{13}}{\partial x_j}, \quad \Gamma_{4j} = \frac{-i\omega}{am_1 m_2}\frac{\partial \Psi_{12}}{\partial x_j},$$

$$\Gamma_{5j} = \frac{-i\omega}{am_1 m_2}\frac{\partial \Psi_{13}}{\partial x_j}, \quad \Gamma_{44} = \frac{1}{m_1 m_2}\Psi_{44}, \quad \Gamma_{45} = \Gamma_{54} = \frac{1}{m_1 m_2}\Psi_{45},$$

$$\Gamma_{55} = \frac{1}{m_1 m_2}\Psi_{55}, \quad k, j = 1, 2, 3, \quad \Psi_{11} = \frac{\alpha_{11} r}{2} + \frac{\alpha_{12}}{r}$$

$$+\frac{\beta_{11} e^{i\lambda_1 r} + \beta_{12} e^{i\lambda_2 r}}{r}, \quad \Psi_{12} = \frac{\gamma_{11}}{r} + \frac{\gamma_{12} e^{i\lambda_1 r} + \gamma_{13} e^{i\lambda_2 r}}{r},$$

$$\Psi_{13} = \frac{\delta_{11}}{r} + \frac{\delta_{12} e^{i\lambda_1 r} + \delta_{13} e^{i\lambda_2 r}}{r}, \quad \Psi_{44} = \frac{\eta_{11} e^{i\lambda_1 r} + \eta_{12} e^{i\lambda_2 r}}{r},$$

$$\Psi_{55} = \frac{\gamma_{21} e^{i\lambda_1 r} + \gamma_{22} e^{i\lambda_2 r}}{r}, \quad \Psi_{45} = \gamma_{45}\frac{e^{i\lambda_2 r} - e^{i\lambda_1 r}}{r},$$

$$\alpha_{11} = \frac{\lambda + \mu}{a\mu} + \frac{i\omega(\alpha_4 \beta_1^2 + \alpha_3 \beta_2^2 - 2k\beta_1\beta_2)}{a^2 m_1 m_2 \lambda_1^2 \lambda_2^2},$$

$$\alpha_{12} = \frac{i\omega}{a^2 m_1 m_2 \lambda_1^2 \lambda_2^2}\left[m_2\beta_1^2 + m_1\beta_2^2 - \frac{(\lambda_1^2 + \lambda_2^2)(\alpha_4\beta_1^2 + \alpha_3\beta_2^2 - 2k\beta_1\beta_2)}{\lambda_1^2\lambda_2^2}\right],$$

$$\beta_{1k} = \frac{(-1)^k i\omega}{a^2 m_1 m_2 (\lambda_1^2 - \lambda_2^2)\lambda_2^2}\left[-m_2\beta_1^2 - m_1\beta_2^2 + \frac{\alpha_4\beta_1^2 + \alpha_3\beta_2^2 - 2k\beta_1\beta_2}{\lambda_k^2}\right],$$
$k = 1, 2,$

$$\gamma_{11} = \frac{\alpha_4\beta_1 - k\beta_2}{\lambda_1^2\lambda_2^2}, \quad \gamma_{1k} = \frac{(-1)^k}{(\lambda_1^2 - \lambda_2^2)}\left[-m_2\beta_1 + \frac{\alpha_4\beta_1 - k\beta_2}{m_2\lambda_{k-1}^2}\right], \quad k = 2, 3,$$

$$\delta_{11} = \frac{\alpha_3\beta_2 - k\beta_1}{\lambda_1^2\lambda_2^2}, \quad \delta_{1k} = \frac{(-1)^k}{\lambda_1^2 - \lambda_2^2}\left[-m_1\beta_2 + \frac{\alpha_3\beta_2 - k\beta_1}{m_1\lambda_{k-1}^2}\right], \quad k = 2, 3$$

$$\eta_{1k} = \frac{(-1)^k}{\lambda_1^2 - \lambda_2^2}\left[-m_2\lambda_k^2 + \alpha_4 + \frac{i\omega\beta_2^2}{a}\right], \quad k = 1, 2, \quad \gamma_{45} = -\frac{ka + i\omega\beta_1\beta_2}{a(\lambda_1^2 - \lambda_2^2)},$$

$$\gamma_{2k} = \frac{(-1)^k}{\lambda_1^2 - \lambda_2^2}\left[-m_1\lambda_k^2 + \alpha_3 + \frac{i\omega\beta_1^2}{a}\right], \quad k = 1, 2$$

$$\alpha_{12} + \beta_{11} + \beta_{12} = 0, \quad \delta_{11} + \delta_{12} + \delta_{13} = 0, \quad \eta_{11} + \eta_{12} = m_2, \tag{2.2}$$

$$\gamma_{11}+\gamma_{12}+\gamma_{13}=0, \qquad \gamma_{21}+\gamma_{22}=m_1,$$

$$r^2=(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2.$$

Here λ_j^2, $j=1,2$ are roots of the characteristic equation

$$m_1m_2x^2-\left[\alpha_4m_1+\alpha_3m_2+\frac{i\omega}{a}\left(\beta_2^2m_1+\beta_1^2m_2\right)\right]x+\alpha_3\alpha_4-k^2$$

$$+\frac{i\omega}{a}\left(\alpha_4\beta_1^2+\alpha_3\beta_2^2-2k\beta_1\beta_2\right)=0, \;\; a=\lambda+2\mu.$$

Without loss of generality we assume that $Im\lambda_j>0$, $j=1,2$.

Analogously we construct the matrix $\widetilde{\Gamma}(x)=\Gamma^T(-x)$.

3 Singular Matrix of Solutions

In solving boundary value problems of the theory of consolidation with double porosity by the method of potential theory, the fundamental matrix and some other matrices of singular solutions to equation (1.1) are of great importance. These matrices will be constructed explicitly in the present section with the help of elementary functions. Using the basic fundamental matrix, we will construct the so-called singular matrices of solutions. For simplicity, we will introduce the special generalized stress vector.

We introduce the following notation $\overset{\kappa}{\widetilde{\mathrm{R}}}(\partial x,n)$

$$\overset{\kappa}{\widetilde{\mathrm{R}}}(\partial x,n)=\|\overset{\kappa}{\widetilde{\mathrm{R}}}_{kj}(\partial x,n)\|_{5\times5}, \quad p,q=1,2,3,4,5,$$

$$\overset{\kappa}{\widetilde{\mathrm{R}}}_{kj}(\partial x,n)=\mu\delta_{kj}\frac{\partial}{\partial n}+(\lambda+\mu)n_k\frac{\partial}{\partial x_j}$$

$$+\kappa\left(n_j\frac{\partial}{\partial x_k}-n_k\frac{\partial}{\partial x_j}\right), \quad k,j=1,2,3,$$

$$\overset{\kappa}{\widetilde{\mathrm{R}}}_{j4}(\partial x,n)=-i\omega\beta_1n_j, \quad \overset{\kappa}{\widetilde{\mathrm{R}}}_{j5}(\partial x,n)=-i\omega\beta_2n_j, j=1,2,3,$$

$$\overset{\kappa}{\widetilde{\mathrm{R}}}_{4j}(\partial x,n)=\overset{\kappa}{\widetilde{\mathrm{R}}}_{5j}(\partial x,n)=\overset{\kappa}{\widetilde{\mathrm{R}}}_{45}(\partial x,n)=\overset{\kappa}{\widetilde{\mathrm{R}}}_{54}(\partial x)=0,$$

$$\overset{\kappa}{\widetilde{\mathrm{R}}}_{44}(\partial x,n)=m_1\frac{\partial}{\partial n}, \quad \overset{\kappa}{\widetilde{\mathrm{R}}}_{55}(\partial x,n)=m_2\frac{\partial}{\partial n}.$$

Applying the operator $\overset{\kappa}{\mathrm{R}}(\partial x, n)$ to the matrix $\Gamma(x)$ we construct the so-called singular matrix of solutions. Let us consider the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y-x)]^*$, which is obtained from $\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma(x-y) = (\overset{\kappa}{\mathrm{P}}_{pq})_{5\times 5}$ by transposition of the columns and rows and the variables x and y. It is not difficult to prove that every column of the matrix $[\overset{\kappa}{\mathrm{R}}(\partial y, n)\Gamma(y-x)]^*$ is a solution of the system $\widetilde{B}(\partial x)U = 0$ with respect to the point x, if $x \neq y$ and all elements of $\overset{\kappa}{\mathrm{P}}_{pq}$ have a singularity of type $O(|x|^{-2})$.

The elements $\overset{\kappa}{\mathrm{P}}_{pq}$ are given as follows:

$$\begin{aligned}
\overset{\kappa}{\mathrm{P}}_{11} &= \frac{\partial}{\partial n}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2} - \frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_3}\right],\\
\overset{\kappa}{\mathrm{P}}_{12} &= -\frac{\kappa}{\mu}\frac{\partial}{\partial s_3}\frac{1}{r} - (\kappa+\mu)\left[\frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_2\partial x_3} - \frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_2^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{13} &= \frac{\kappa}{\mu}\frac{\partial}{\partial s_2}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_2\partial x_3} - \frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_3^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{1j} &= \frac{\kappa+\mu}{am_1m_2}\left[\frac{\partial}{\partial s_2}\frac{\partial\Psi_{1,j-2}}{\partial x_3} - \frac{\partial}{\partial s_3}\frac{\partial\Psi_{1,j-2}}{\partial x_2}\right], \quad j=4,5\\
\overset{\kappa}{\mathrm{P}}_{21} &= \frac{\kappa}{\mu}\frac{\partial}{\partial s_3}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_3} - \frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_1^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{22} &= \frac{\partial}{\partial n}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_2\partial x_3} - \frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2}\right],\\
\overset{\kappa}{\mathrm{P}}_{23} &= -\frac{\kappa}{\mu}\frac{\partial}{\partial s_1}\frac{1}{r} + (\kappa+\mu)\left[-\frac{\partial}{\partial s_3}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_3} + \frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_3^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{2j} &= \frac{\kappa+\mu}{am_1m_2}\left[\frac{\partial}{\partial s_3}\frac{\partial\Psi_{1,j-2}}{\partial x_1} - \frac{\partial}{\partial s_1}\frac{\partial\Psi_{1,j-2}}{\partial x_3}\right], \quad j=4,5,\\
\overset{\kappa}{\mathrm{P}}_{31} &= -\frac{\kappa}{\mu}\frac{\partial}{\partial s_2}\frac{1}{r} + (\kappa+\mu)\left[-\frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_2\partial x_1} + \frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_1^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{32} &= \frac{\kappa}{\mu}\frac{\partial}{\partial s_1}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_2} - \frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_2^2}\right],\\
\overset{\kappa}{\mathrm{P}}_{33} &= \frac{\partial}{\partial n}\frac{1}{r} + (\kappa+\mu)\left[\frac{\partial}{\partial s_2}\frac{\partial^2\Psi_{11}}{\partial x_1\partial x_3} - \frac{\partial}{\partial s_1}\frac{\partial^2\Psi_{11}}{\partial x_2\partial x_3}\right],
\end{aligned} \tag{3.1}$$

$$\overset{\kappa}{\mathrm{P}}_{3j} = \frac{\kappa+\mu}{am_1m_2}\left[-\frac{\partial}{\partial s_2}\frac{\partial\Psi_{1,j-2}}{\partial x_1} + \frac{\partial}{\partial s_1}\frac{\partial\Psi_{1,j-2}}{\partial x_2}\right], \quad j=4,5,$$

$$\overset{\kappa}{\mathrm{P}}_{4j} = -\frac{i\omega}{am_2}\frac{\partial}{\partial x_j}\frac{\partial}{\partial n}\Psi_{12}, \quad \overset{\kappa}{\mathrm{P}}_{5j} = -\frac{i\omega}{am_1}\frac{\partial}{\partial x_j}\frac{\partial}{\partial n}\Psi_{13}, \quad j=1,2,3,$$

$$\overset{\kappa}{\mathrm{P}}_{44} = \frac{1}{m_2}\frac{\partial}{\partial n}\Psi_{44}, \quad \overset{\kappa}{\mathrm{P}}_{45} = \frac{1}{m_2}\frac{\partial}{\partial n}\Psi_{45}, \quad \overset{\kappa}{\mathrm{P}}_{54} = \frac{1}{m_1}\frac{\partial}{\partial n}\Psi_{45},$$

$$\overset{\kappa}{\mathrm{P}}_{55} = \frac{1}{m_1}\frac{\partial}{\partial n}\Psi_{55}, \quad \frac{\partial}{\partial s_3} = n_1\frac{\partial}{\partial x_2} - n_2\frac{\partial}{\partial x_1},$$
$$\frac{\partial}{\partial s_2} = n_3\frac{\partial}{\partial x_1} - n_1\frac{\partial}{\partial x_3}, \quad \frac{\partial}{\partial s_1} = n_2\frac{\partial}{\partial x_3} - n_3\frac{\partial}{\partial x_2}.$$

Analogously we obtain the matrix

$$\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x) = ([\widetilde{\mathrm{R}}^{\kappa}\widetilde{\Gamma}]_{pq})_{5\times 5}.$$

The matrix $[\widetilde{\mathrm{R}}^{\kappa}(\partial y, n)\widetilde{\Gamma}(y-x)]^*$ is a solution of the system (1.1). It shows, that the matrixes $[\widetilde{\mathrm{R}}^{\kappa}(\partial x, n)\widetilde{\Gamma}]^*$ and $[\overset{\kappa}{\mathrm{R}}(\partial x, n)\Gamma]^*$ contains a singular part, which is integrable in sense the principal Cauchy value sense.

4 Potentials and Their Properties

We introduce the following definitions:

Definition 1. The vector-functions defined by the equalities

$$V^{(1)}(x) = \frac{1}{2\pi}\int_S\!\!\int \Gamma(y-x)h(y)dS_y, \tag{4.1}$$

$$V^{(2)}(x) = \frac{1}{2\pi}\int_S\!\!\int \widetilde{\Gamma}(x-y)h(y)dS_y$$

where $\widetilde{\Gamma}(x,y)$ is the fundamental matrix, $\widetilde{\Gamma}(x) = \Gamma^T(-x)$, h is a continuous (or Hölder continuous) vector and S is a closed Lyapunov surface, will be called simple layer potentials.

Definition 2. The vector-functions defined by the equalities

$$\begin{aligned} U^{(1)}(x) &= \frac{1}{2\pi}\int_S\!\!\int \left[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-x)\right]^* h(y)dS_y, \\ U^{(2)}(x) &= \frac{1}{2\pi}\int_S\!\!\int [N(\partial y, n)\Gamma(y-x)]^* h(y)dS_y, \end{aligned} \tag{4.2}$$

are called double layer potentials, where $h\ (y)$ is a real 5-dimensional vector.

The potentials $V^{(1)}$, $U^{(1)}$ are solutions of the system (1.1) and the potentials $V^{(2)}$, $U^{(2)}$ are solutions of the system $\widetilde{B}(\partial x)U = 0$ in the domains D^+ and D^-.

The following theorems are valid.

Theorem 3. If S is a closed surface of the class $S \in L_1(\alpha)$ and $h \in S^{1,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then the potentials (4.2) are regular in $D^\pm$ there exist boundary values of the vectors $U^{(1)}(x)$, $U^{(2)}(x)$ from inside and outside of the surface S, and there take place the equalities

$$\begin{aligned} U^{(1)\pm} &= \pm h(z) + \frac{1}{2\pi}\int\limits_S\int \Big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-z)\Big]^* h(y)dS_y, \\ U^{(2)\pm} &= \pm h(z) + \frac{1}{2\pi}\int\limits_S\int [N(\partial y, n)\Gamma(y-z)]^* h(y)dS_y. \end{aligned} \tag{4.3}$$

Theorem 4. If S is a closed surface of the class $S \in L_1(\alpha)$ and $h \in S^{0,\beta}(S)$, $0 < \beta < \alpha \leq 1$, then
(a) the potentials (4.1) are regular in $D^\pm$;
(b) there exist boundary values of the vectors $N(\partial x, n)V^{(1)}(x)$, $N(\partial x, n)V^{(2)}(x)$ from inside and outside of the surface S, and there take place the equalities

$$\begin{aligned} \big[N(\partial y, n)V^{(1)}(z)\big]^{\mp} &= \mp h(z) + \frac{1}{2\pi}\int\limits_S\int N(\partial y, n)\Gamma(z-y)h(y)dS_y, \\ [\widetilde{N}(\partial y, n)V^{(2)}(z)]^{\mp} &= \mp h(z) + \frac{1}{2\pi}\int\limits_S\int \widetilde{N}(\partial y, n)\widehat{\Gamma}(z-y)h(y)dS_y, \end{aligned} \tag{4.4}$$

Let us note that $[N(\partial x, n)\Gamma(x-y)]^*$ is a weakly singular kernel .

5 Solution of the Boundary Value Problems

Problem $(I)^+$. Let us first prove the existence of solution of the first boundary value problem in the domain D^+. A solution is sought in the form of the double layer potential

$$U(x) = \frac{1}{2\pi}\int\limits_S\int \Big[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-x)\Big]^* h(y)dS_y. \tag{5.1}$$

Then for determining the unknown real vector function h we obtain the following Fredholm integral equation of the second kind

$$-h(z)+\frac{1}{2\pi}\int_S\int\Big[\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y-z)\Big]^* h(y)dS_y=f^+. \tag{5.2}$$

Let us prove that the equation (5.2) is solvable for any continuous right-hand side. Consider the associated to (5.2) homogeneous equation

$$-h(z)+\frac{1}{2\pi}\int_S\int N(\partial y,n)\Gamma(y-z)h(y)dS_y=0. \tag{5.3}$$

and prove that it has only the trivial solution. Assume the contrary and denote by $\varphi(z)$ a nontrivial solution of (5.3). Compose the simple layer potential

$$V(x)=\frac{1}{2\pi}\int_S\int\Gamma(y-x)\varphi(y)dS_y. \tag{5.4}$$

It is obvious from (5.3), that

$$[N(\partial z,n)V(z)]^-=0,\quad \int_S\int\varphi(y)ds=0.$$

Using the formula (1.8) for $\kappa=\kappa_n$ in D^- , we obtain $V(z)=0,\ \ z\in D^-$.

Now taking into account the continuity of the simple layer potential and using the uniqueness theorem for the solution of the first boundary value problem, we will have $V(x)=0,\ \ x\in D^+$.

Note that $[NV]^+-[NV]^-=2\varphi(x)=0$ and hence the equation (5.3) has only the trivial solution. This implies that the associated to (5.3) homogeneous equation also has only the trivial solution, and the equation(5.2) is solvable for any continuous right-hand side (according to the Fredholm theorem).

For the regularity of the double layer potential in the domain D^+ it is sufficient to assume that $S\in C^{2,\beta}$, $(0<\beta<1)$ and $\dfrac{\partial f}{\partial s}$ is Holder continuous, $f\in C^{1,\ \alpha}(S), (\ 0<\alpha<\beta)$.

Problem $(I)^-$. Consider the first boundary value problem in the domain D^-. Its solution is sought in the form

$$U(x)=\frac{1}{2\pi}\int_S\int\Big[[\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y-x)]^*-[\widetilde{N}(\partial y,n)\widetilde{\Gamma}(y)]^*\Big]\psi(y)dS_y. \tag{5.5}$$

To define the unknown real vector function ψ we obtain the following Fredholm integral equation of the second kind

$$\psi(z) + \frac{1}{2\pi}\int\limits_S\!\!\int\Big[[\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y-z)]^* - [\widetilde{N}(\partial y, n)\widetilde{\Gamma}(y)]^*\Big]\,\psi(y)dS_y = f^-. \tag{5.6}$$

Prove that the equation (5.6) is solvable for any continuous right-hand side. To this end, we consider the associated to (5.6) homogeneous equation

$$h(z) + \frac{1}{2\pi}\int\limits_S\!\!\int[N(\partial z, n)\Gamma(z-y) + N(\partial z, n)\Gamma(z)]\,h(y)dS_y = 0. \tag{5.7}$$

Let us prove that (5.7) has only the trivial solution. Suppose that it has a nontrivial solution $h(z)$. From (5.7) by integration we obtain

$$\int\limits_S hdS = 0.$$

In this case the equation (5.7) corresponds to the boundary condition $[N(\partial x, n)V]^+ = 0$, where

$$V(x) = \frac{1}{2\pi}\int\limits_S\!\!\int\Gamma(y-x)h(y)dS_y, \tag{5.8}$$

and find that $V = C,\ x \in D^+$, where C is a constant vector.

Taking into account the equation $\int\limits_S hdS = 0$ and the fact that the single layer potential is continuous while passing through the boundary and using the Green's formula for $\kappa = \kappa_n$, we obtain $V = 0,\ x \in D^-$. Since $[NV]^+ - [NV]^- = 2h(x) = 0$, and $[NV]^+ = 0$, $[NV]^- = 0$, then $h(x) = 0$.

Thus we conclude that the associated to (5.7) homogeneous equation has only the trivial solution, and the equation (5.6) is solvable for any continuous right-hand side.

To prove the regularity of the potential (5.5) in the domain D^-, it is sufficient to assume that $S \in C^{2,\beta}(0 < \beta < 1)$ and $f \in C^{1,\alpha}(S), (0 < \alpha < \beta)$.

Finally, on the basis of the general theory, the following theorems are valid.

Theorem 5. If $S \in L_2(\alpha)$ and $f \in C^{1,\beta(S)}$, then the BVP $(I)^+$ has a unique solution. Moreover, this solution is given in the form of the double layer potential (5.1), where h is a solution of equation (5.2).

Theorem 6. The problem $(I)^-$ is solvable for an arbitrary vector $f \in C^{1,\alpha}(S)$, for $S \in L_2(\alpha)$, and its solution is representable by formula (5.5).

For the proof of Theorems 1,2,...,6, we used the method given in [6] which is applied to the proof of analogous theorems for isotropic media.

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