AN ANALOGUE OF THE BITSADZE–SAMARSKI PROBLEM FOR A MIXED-TYPE EQUATION

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Abstract

The paper considers an analogue of the Bitsadze-Samarskii problem for a mixed type equation when a skew derivative is given on the boundary of an elliptic domain. It is shown that this problem is Noetherian.

Key words and phrases: Mixed type equation, Cauchy solution, Singular integral equation, Index of problem.

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Let us consider the equation

$$0 = \begin{cases} u_{xx} + u_{yy} + au_x + bu_y + cu, & y > 0, \\ u_{xx} - u_{yy}, & y < 0, \end{cases} \tag{1}$$

where a, b, c are the given analytic functions of their arguments that have real values for real (x, y), and $u(x, y)$ is the real function we want to define.

Let Ω be a simply connected domain on the plane of a complex variable $z = x + iy$, bounded by a curve σ from the class C^2 having ends at the points $C_1(0, 0)$ and $C_2(1, 0)$, lying in an upper half-plane $y > 0$ and having the characteristics $CC_1 : y = -x$, $CC_2 : y = x-1$, $C = \left(\frac{1}{2}, -\frac{1}{2}\right)$ of equation (1).

We use the following notation: Ω^+, Ω^- are respectively an elliptic and a hyperbolic part of the mixed domain Ω; $\mathcal{J} = \{x : 0 < x < 1\}$ is the unit interval of the straight line $y = 0$; $\theta(x) = \frac{x}{2} - i\,\frac{x}{2}$ is an affix of the intersection point of equation (1) coming out from a point $x \in \mathcal{J}$ with characteristic CC_1.

Under a regular solution of equation (1) in the domain Ω we will understand a function $u(x, y) \in C(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^+ \cup \Omega^-)$ that satisfies the equation in $\Omega^+ \cup \Omega^-$ and is such that at the ends of the interval $\mathcal{J}$ $u_y(x, 0)$ may tend to infinity of order less than one.

Problem A. Find a regular solution $u(x,y)$ of equation (1) in the domain Ω that satisfies the conditions

$$(pu_x + qu_y + \lambda u)\big|_{\sigma} = \varphi \quad \forall (x,y) \in \sigma, \tag{2}$$

$$\frac{d}{dx}\, u[\theta(x)] = \alpha^*(x) u_y(x,0) + \beta^*(x) \quad \forall x \in \mathcal{J}, \tag{3}$$

where φ, p, q, $\lambda \in C^{1,h}$, $0 \le h = const < 1$, $a \in C(\overline{\mathcal{J}}) \cap C^1(\mathcal{J})$, $b \in C(\mathcal{J})$ are the given functions, and $p^2 + q^2 \neq 0$, $\alpha^*(x) \neq -\frac{1}{2}$ $\forall x \in \mathcal{J}$.

Let us assume that there exists a Cauchy solution for a string oscillation equation in the domain Ω^- under the initial conditions

$$u(x,0) = \tau(x), \quad u_y(x,0) = \nu(x),$$

$$u(x,y) = \frac{\tau(x+y) + \tau(x-y)}{2} + \frac{1}{2}\int_{x-y}^{x+y} \nu(t)\, dt.$$

If we take condition (3) into account, it is not difficult to obtain the following functional relation between the functions $\tau(x)$ and $\nu(x)$ transferred from Ω^- onto $\mathcal{J}$

$$(1 + 2\alpha^*(x))\, \nu(x) = \tau'(x) - 2\beta^*(x). \tag{4}$$

Instead of the real variables x and y, $(x,y) \in \Omega^+$, let us introduce the complex variables $z = x+iy$, $\overline{z} = x-iy$. Then equation (1) can be rewritten as follows:

$$\frac{\partial^2}{\partial z \partial \overline{z}} + A(z,\overline{z})\,\frac{\partial u}{\partial z} + \overline{A(z,\overline{z})}\,\frac{\partial u}{\partial \overline{z}} + C(z,\overline{z})u = 0, \tag{5}$$

where

$$4A(z,\overline{z}) = a\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + ib\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right),$$

$$4C(z,\overline{z}) = c\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right).$$

If we use the formulas

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \overline{z}}, \quad \frac{\partial}{\partial y} = i\,\frac{\partial}{\partial z} - i\,\frac{\partial}{\partial \overline{z}},$$

then the boundary condition (2) takes the form

$$H(s)\,\frac{\partial u}{\partial t} + \overline{H(s)}\,\frac{\partial}{\partial \overline{t}} + \lambda(s)u = \varphi(s), \qquad H(s) = p(s) + iq(s). \tag{6}$$

Let us make use of the general representation of regular solutions of equation (1) in Ω^+ by analytic functions $\omega(z)$ [1]

$$u(x,y) = \operatorname{Re}\left\{\alpha(z,\overline{z})\omega(z) + \int_{p_0}^{z} \beta(z,\overline{z},t)\omega(t)\,dt\right\}, \tag{7}$$

where $\omega(z)$ is an arbitrary analytic function in the domain Ω^+ that satisfies the condition $\operatorname{Im}\omega(p_0) = 0$, $p_0 \in \Omega^+$, and $\alpha(z,\overline{z})$, $\beta(z,\overline{z},t)$ are the entire functions of their arguments

$$\alpha(z,\overline{z}) = \exp\left(-\int_0^{\overline{z}} A(z,\overline{t})\,d\overline{t}\right),$$

$$\beta(z,\overline{z},t) = \int_0^{\overline{z}} V(z,\overline{z};t,\overline{t})\,d\overline{t},$$

where $V(z,\overline{z};t,\overline{t})$ is uniquely defined by the following conditions: 1) $V(z,\overline{z};t,\overline{t})$ is a solution of the differential equation (5), and 2)

$$V(z,\overline{t};t,\overline{t}) = \gamma(t,\overline{t})\exp\left(-\int_t^{z} \overline{A}(t_1,t)\,dt_1\right),$$

$$V(t,\overline{z};t,\overline{t}) = \gamma(t,\overline{t})\exp\left(-\int_{\overline{t}}^{\overline{z}} A(t,\overline{t}_1)\,d\overline{t}_1\right),$$

$$-\gamma(z,\overline{z}) = \frac{\partial\alpha(z,\overline{z})}{\partial z\partial\overline{z}} + A(z,\overline{z})\frac{\partial\alpha(z,\overline{z})}{\partial z} + \overline{A(z,\overline{z})}\frac{\partial\alpha(z,\overline{z})}{\partial z} + C(z,\overline{z})\alpha(z,\overline{z}).$$

To define the function V, we must solve the Goursat problem in a complex domain. The solution of the problem can always be found by the method of successive approximations [1].

I. N. Vekua proved [1] that if $\omega(z) \in C^{1,h}(\Omega^+)$ is an analytic function in the simply connected domain Ω^+ that satisfies the condition $\operatorname{Im}\omega(z)(p_0) = 0$, then there exists a unique real function $\mu(t) \in C^{0,h}$ such that the formula

$$\omega(z) = \int_{\partial\Omega^+} \mu(t)\log e\left(1 - \frac{z}{t}\right)\,dS_t \tag{8}$$

holds, where dS_t is an element of an arc of the boundary $\partial\Omega^+$, while $\log e\left(1 - \frac{\overline{z}}{t}\right)$, $z \in \Omega^+$, $t \in \partial\Omega^+$, is understood as the branch of this function which is equal to zero for $z = 0$.

By means of (7) and (8), the boundary condition (6) can be rewritten as

$$\alpha_1(t)\mu(t) + \beta_1(t)\int_{\partial\Omega^+} \frac{\mu(t_1)\,dt_1}{t_1 - t} + \int_{\partial D^+} K(t,t_1)\mu(t_1)\,dt_1 = \varphi(t), \quad t \in \partial D^+ \setminus C_1C_2, \tag{9}$$

where

$$\alpha_1(t) = \mathrm{Re}\left(\pi i \bar{t}\,'\alpha(t,\bar{t})H(t)\right),$$
$$\beta_1(t) = \mathrm{Im}\left(-i\bar{t}\,'\alpha(t,\bar{t})H(t)\right)$$

and $\int\limits_{\partial\Omega^+} K(t,t_1)\mu(t_1)\,dt_1$ is a completely defined general operator.

Using the general representation of regular solutions in Ω^+, we find

$$\tau'(t) = \widetilde{\alpha}_1(t)\mu(t) + \widetilde{\beta}_1(t)\int_{\partial\Omega^+}\frac{\mu(t_1)\,dt_1}{t_1 - t} + \mathcal{K}_\infty(\mu), \tag{10}$$

$$\nu(t) = \widetilde{\alpha}_2(t)\mu(t) + \widetilde{\beta}_2(t)\int_{\partial\Omega^+}\frac{\mu(t_1)\,dt_1}{t_1 - t} + \mathcal{K}_\in(\mu), \tag{11}$$

where

$$\widetilde{\alpha}_1(t) = \mathrm{Re}\left(-\pi i\alpha(t,\bar{t})\bar{t}\,'\right), \qquad \widetilde{\beta}_1(t) = \mathrm{Im}\left(-i\alpha(t,\bar{t})\bar{t}\,'\right),$$
$$\widetilde{\alpha}_2(t) = \mathrm{Re}\left(\pi\alpha(t,\bar{t})\bar{t}\,'\right), \qquad \widetilde{\beta}_1(t) = \mathrm{Im}\left(i\alpha(t,\bar{t})\bar{t}\,'\right),$$

and $\mathcal{K}_\infty(\mu)$, $\mathcal{K}_\in(\mu)$ are completely defined integral operators.

The substitution of (10) and (11) into (4) gives

$$\alpha_3(t)\mu(t) + \beta_3(t)\int_{\partial D^+}\frac{\mu(t_1)\,dt_1}{t_1 - t} + \mathcal{K}_\ni(\mu) = \widetilde{\psi}(\sqcup), \quad \sqcup \in (/,\infty), \tag{12}$$

where

$$\alpha_3(t) = \widetilde{\alpha}_1(t) - (1 + 2\alpha^*(t))\widetilde{\alpha}_2(t) = \mathrm{Re}\left(-\pi(i + 1 + 2\alpha^*(t))\alpha(t,\bar{t})\bar{t}\,'\right),$$
$$\beta_3(t) = \widetilde{\beta}_1(t) - (1 + 2\alpha^*(t))\widetilde{\beta}_2(t) = \mathrm{Im}\left(-(i + 1 + 2\alpha^*(t))\alpha(t,\bar{t})\bar{t}\,'\right).$$

Let us write equations (9) and (12) in the form of one singular integral equation on the entire boundary ∂D^+:

$$\alpha_4(t)\mu(t) + \beta_4(t)\int_{\partial D^+}\frac{\mu(t_1)\,dt_1}{t_1 - t} + \mathcal{K}_\triangle(\mu) = \{(\sqcup), \tag{13}$$

where

$$\alpha_4(t) = \begin{cases}\alpha_1(t), & t \in \partial D^+ \setminus C_1C_2,\\ \alpha_3(t), & t \in C_1C_2,\end{cases} \qquad \beta_4(t) = \begin{cases}\beta_1(t), & t \in \partial D^+ \setminus C_1C_2,\\ \beta_3(t), & t \in C_1C_2,\end{cases}$$

$$f(t) = \begin{cases}\varphi(t), & t \in \partial D^+ \setminus C_1C_2,\\ 2\beta^*(t), & t \in C_1C_2,\end{cases}$$

$\mathcal{K}_\triangle(\mu)$ is a completely defined compact linear integral operator.

Remark. In the sequel it will be assumed that $\alpha_4(t)$ and $\beta_4(t)$ are continuous at the point $t = 0$, i.e. $p(0)+q(0) = 0$ and $(1+i)p(0) = i+1+2\alpha^*(0)$.

A solution $\mu(t)$ of the singular integral equation (13) is sought in the space $H^*(\partial D^+)$; the node of ∂D^+ is assumed to lie at the point $C_2(1, 0)$ [3].

Assuming that

$$H(t) = (p + iq)(t) \neq 0, \quad t \in \sigma,$$

we write

$$\omega(t) = \frac{\alpha_4(t) - i\pi\beta_4(t)}{\alpha_4(t) + i\pi\beta_4(t)}\,.$$

The index $\varkappa$ of the singular integral equation (13) in the class $H^*(\partial D^+)$ is defined in the following manner [3].

We denote by $\arg\omega_-(t)$ and $\arg\omega_+(t)$ the continuous branches of the argument of functions on $\partial D^+ \setminus \sigma$ and σ, respectively.

Let

$$d = \frac{1}{2\pi}\,\arg\omega_-(C_2) - \frac{1}{2\pi}\,\arg\omega_+(C_2).$$

The index $\varkappa$ is defined by the formula

$$\varkappa = \begin{cases} -[d] - 1 & \text{if } d \notin Z, \\ -d & \text{if } d \in Z, \end{cases} \tag{14}$$

where Z is the set of integer numbers, $[d]$ is an integer part of the number d.

Theorem 1. *Let the following conditions be fulfilled:*

$$H(t) = (p + iq)(t) \neq 0, \quad t \in \sigma,$$
$$(1 + i)p(0) = 1 + 2\alpha^*(0) + i,$$
$$p(0) + q(0) = 0, \quad \varphi(0) = \beta^*(0) = 0.$$

Then Problem A is Noetherian and its index is given by formula (14).

References

1. I. N. Vekua, New methods for solving elliptic equations. Translated from the Russian. North-Holland Series in Applied Mathematics and Mechanics, Vol. 1 North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York (1967), 358 pp.

2. B. V. Khvedelidze, The Poincaré problem for the linear second order differential equation of elliptic type. *Trudy Tbiliss. Mat. Inst. Razmadze* (*Proc. A. Razmadze Math. Inst.*) **12** (1943), 47-77 (in Georgian).

3. N. I. Muskhelishvili, Singular integral equations. Boundary problems of function theory and their application to mathematical physics. Translated from the Russian. Corrected reprint of the 1953 English translation. Dover Publications, Inc., New York (1992), 447 pp.