ON ERROR ESTIMATION OF SYMMETRIC DECOMPOSITION SCHEME FOR MULTIDIMENSIONAL EVOLUTION PROBLEM

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Abstract

For an evolution problem in Hilbert space with self-adjoint positive definite operator, which in its turn represents a sum of self-adjoint positive definite operators (we call this case as a multi-dimensional), symmetric decomposition scheme of G. Baker and T. Oliphant is considered. On the basis of semigroup approximation it is shown that the norm of the error of approximate solution in the definition domain of the main operator is of order $O(\tau \ln(1/\tau))$, where τ is a time step.

Key words and phrases: Decomposition scheme, Operator splitting, Abstract parabolic equation.

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Let us consider the Cauchy abstract problem in H Hilbert space:

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = \varphi, \tag{1}$$

where A is a self-adjoint (generally unbounded), positive definite operator with the definition domain $D(A)$, which is everywhere dense in H, $\overline{D(A)} = H$, $\quad A = A^*$ and

$$(Au, u) \geq a \|u\|^2, \quad \forall u \in D(A), \quad a = const > 0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined correspondingly the norm and scalar product in H; φ is a given vector from H; $u(t)$ is a continuous and continuously differentiable, searched function with values in H.

Let $A = A_1 + ... + A_m$, where $A_1, ..., A_m$ are also self-adjoint positive definite operators. The solution of the problem (1) is given by the following formula (see for example [1]):

$$u(t) = U(t)\varphi. \tag{2}$$

where $U(t) = exp(-tA)$ is a strongly continuous semigroup.

We will consider the approximate solution of the problem (1) due to G. Baker and T.Oliphant symmetrical decomposition scheme. Our aim is to obtain explicit estimates for approximate solution error on $D(A)$. Under the explicit estimation we mean such a priori estimation for the solution error, where constants in the right-hand side do not depend on the solution of the initial continuous problem, i.e. are absolute constants.

Different types of decomposition schemes are examined in G. Marchuk's well-known book (see [2] and extensive bibliography added to it).

G. Baker and T. Oliphant symmetrical decomposition differential scheme for approximate solution of problem (1) has the form (see [3],[4]):

$$\begin{aligned}
\frac{du_k^{(1)}(t)}{dt}+\frac{1}{2}A_1u_k^{(1)}(t) &= 0, \quad u_k^{(1)}(t_{k-1})=u_{k-1}(t_{k-1}),\\
&\quad\; \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
\frac{du_k^{(m-1)}(t)}{dt}+\frac{1}{2}A_{m-1}u_k^{(m-1)}(t) &= 0, \quad u_k^{(m-1)}(t_{k-1})=u_k^{(m-2)}(t_k),\\
\frac{du_k^{(m)}(t)}{dt}+A_mu_k^{(m)}(t) &= 0, \quad u_k^{(m)}(t_{k-1})=u_k^{(m-1)}(t_k),\\
\frac{du_k^{(m+1)}(t)}{dt}+\frac{1}{2}A_{m-1}u_k^{(m-1)}(t) &= 0, \quad u_k^{(m+1)}(t_{k-1})=u_k^{(m)}(t_k), \qquad (3)\\
&\quad\; \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
\frac{du_k^{(2m-2)}(t)}{dt}+\frac{1}{2}A_2u_k^{(2m-2)}(t) &= 0, \quad u_k^{(2m-2)}(t_{k-1})=u_k^{(2m-3)}(t_k),\\
\frac{du_k(t)}{dt}+\frac{1}{2}A_1u_k(t) &= 0, \quad u_k(t_{k-1})=u_k^{(2m-2)}(t_k),
\end{aligned}$$

where $u_0(0)=\varphi$, $t\in[t_{k-1},t_k]$, $k=1,2,\ldots,t_k=k\tau,\tau>0$ is a time step.

Approximate value of exact solution of (3) problem at point $t=t_k$ is $u(t_k)$, $u(t_k)\approx u_k(t_k)$.

Solution of the scheme (3) is given by the following formula:

$$u_k(t_k)=V^k(\tau)\varphi, \tag{4}$$

where

$$\begin{aligned}
V(t) &= \exp\left(-\frac{1}{2}tA_1\right)...\exp\left(-\frac{1}{2}tA_{m-1}\right)\exp(-tA_m)\\
&\quad\times\exp\left(-\frac{1}{2}tA_{m-1}\right)...\exp\left(-\frac{1}{2}tA_1\right).
\end{aligned} \tag{5}$$

Let us note that, in the direction of estimate of error of semigroup decomposition formulas, important results are obtained by the authors (we mean estimate of error of semigroup decomposition formulas in the uniform topology): Rogava Dzh. L. [5], Ichinose T. and Tamura Hideo [6], [7] , Ichinose T. and Takanobu S. [8], Ichinose T., Tamura Hideo, Tamura

Hiroshi and Zagrebnov V. [9]. In our opinion, results obtained in [9] are most interesting. In this work it is proved that the following estimate holds:

$$\left\| V^n\left(\frac{t}{n}\right) - U(t) \right\| \le \frac{c}{n}, \tag{6}$$

where $c = const > 0$ and does not depend on t.

For estimate of error of the considered scheme, along with other results, we essentially base on estimate (6).

The following theorem takes place (below everywhere c denotes positive constant).

Theorem 1. *Let $A_1, ..., A_m$ $(m \ge 2)$ are self-adjoint positively defined operators in the Hilbert space H and operator $A = A_1 + ... + A_m$ is self-adjoint on $D(A) = D(A_1) \cap ... \cap D(A_m)$.*

Then the following estimation is true:

$$\|u(t_k) - u_k(t_k)\| \le c\tau \ln(t_k/\tau) \|A\varphi\|, \quad \varphi \in D(A).$$

Proof of Theorem 1 is based on the following lemmas.

Lemma 2. *Let A be a self-adjoint positive definite operator. Then the estimates are valid:*

$$\begin{aligned} \left\|(I - \exp(-tA)) A^{-1}\right\| &\le t, &(7)\\ \left\|A^{-1}((I - tA) - \exp(-tA)) A^{-1}\right\| &\le \frac{1}{2}t^2. &(8) \end{aligned}$$

Proof of estimates (7) and (8) is obvious. (7) is obtained from the formula (see [1], p. 603)

$$A\int\limits_0^t e^{-sA} ds = I - e^{-tA}, \tag{9}$$

and (8) - from the expansion

$$e^{-tA} = I - tA + A^2 \int\limits_0^t \int\limits_0^{s_1} e^{-sA} ds ds_1, \tag{10}$$

which obviously follows from (9).

Lemma 3. *Let the conditions of Theorem 1 be fulfilled. Then the following estimation is true:*

$$\begin{aligned} &\left\| \left(U_s\left(\frac{t}{2}\right)...U_{m-1}\left(\frac{t}{2}\right)U_m(t) \right.\right.\\ &\left.\left.\times\, U_{m-1}\left(\frac{t}{2}\right)...U_l\left(\frac{t}{2}\right)-I\right)A^{-1}\right\| \le ct, \end{aligned} \tag{11}$$

$$\begin{aligned} &\left\| A^{-1}\left(U_s\left(\frac{t}{2}\right)...U_{m-1}\left(\frac{t}{2}\right)U_m(t) \right.\right.\\ \times\quad &\left.\left. U_{m-1}\left(\frac{t}{2}\right)...U_l\left(\frac{t}{2}\right)-I\right)\right\| \le ct, \end{aligned} \tag{12}$$

where $1 \le s, l \le m-1$.

Proof. The representations are obvious:

$$\begin{aligned} & U_s\left(\frac{t}{2}\right)...U_{m-1}\left(\frac{t}{2}\right)U_m(t)\,U_{m-1}\left(\frac{t}{2}\right)...U_l\left(\frac{t}{2}\right)-I \\ = \; & U_s...U_{m-1}U_mU_{m-1}...U_{l+1}\left(U_l-I\right) \\ & +U_s...U_{m-1}U_mU_{m-1}...U_{l+2}\left(U_{l+1}-I\right)+...+\left(U_s-I\right), \end{aligned} \tag{13}$$

$$\begin{aligned} & U_s\left(\frac{t}{2}\right)...U_{m-1}\left(\frac{t}{2}\right)U_m(t)\,U_{m-1}\left(\frac{t}{2}\right)...U_l\left(\frac{t}{2}\right)-I \\ = \; & \left(U_s-I\right)U_{s+1}...U_{m-1}U_mU_{m-1}...U_l \\ & +\left(U_{s+1}-I\right)U_{s+2}...U_{m-1}U_mU_{m-1}...U_l+...+\left(U_l-I\right), \end{aligned} \tag{14}$$

where $U_j(t) = \exp(-tA_j)$, $j = 1, ..., m$.

It follows from the inclusion $D(A) \subset D(A_j)$, $j = 1, ..., m$, that the operator A_jA^{-1} is defined on the whole H-zė. In the other hand, the operator A_jA^{-1} is closed (as a product of closed and bounded operators). But closed operator defined on the whole space is bounded according to the theorem of closed graphic. In addition, the operator $A^{-1}A_j$ is also bounded, as we have $\left(A_jA^{-1}\right)^* \supset A^{-1}A_j$. From this fact and inequality (7) we have the estimates:

$$\begin{aligned} \left\|\left(I-U_j(t)\right)A^{-1}\right\| &= \left\|\left(I-U_j(t)\right)A_j^{-1}\left(A_jA^{-1}\right)\right\| \\ &\le \left\|A_jA^{-1}\right\|\left\|\left(I-U_j(t)\right)A_j^{-1}\right\| \le ct, \end{aligned} \tag{15}$$

$$\begin{aligned} \left\|A^{-1}\left(I-U_j(t)\right)\right\| &= \left\|\left(A^{-1}A_j\right)A_j^{-1}\left(I-U_j(t)\right)\right\| \\ &\le \left\|\left(A_jA^{-1}\right)^*\right\|\left\|\left(I-U_j(t)\right)A_j^{-1}\right\| \le ct. \end{aligned} \tag{16}$$

From the equalities (13) and (14), with account of equalities (15) and (16), we obtain the estimates (11) and (12).

Lemma 4. *Let the conditions of the Theorem 1 are fulfilled. Then the following estimate holds:*

$$\left\|A^{-\alpha}\left(U\left(t\right)-V\left(t\right)\right)A^{-\beta}\right\|\leq ct^{\alpha+\beta},\quad \alpha,\beta=0,1. \tag{17}$$

where $t\geq 0$.

Proof. Obviously the following representation is valid:

$$\begin{aligned} V\left(t\right) &= U_2\left(\frac{t}{2}\right)...U_{m-1}\left(\frac{t}{2}\right)U_m\left(t\right)U_{m-1}\left(\frac{t}{2}\right)...U_2\left(\frac{t}{2}\right) \\ &\quad -tA_1+R_1\left(t\right), \end{aligned} \tag{18}$$

where

$$\begin{aligned} R_1\left(t\right) &= \left(U_1...U_m...U_2-I\right)\left(U_1-I\right) \\ &\quad +\left(U_1-I\right)\left(U_2...U_m...U_2-I\right) \\ &\quad +2\left(U_1\left(\frac{t}{2}\right)-\left(I-\frac{t}{2}A_1\right)\right). \end{aligned} \tag{19}$$

Indeed we have

$$\begin{aligned} R_1\left(t\right) &= \left(U_1...U_m...U_2-I\right)\left(U_1-I\right)+\left(U_1-I\right) \\ &\quad +\left(U_1-I\right)+\left(U_1-I\right)\left(U_2...U_m...U_2-I\right)+tA_1 \\ &= U_1...U_m...U_2\left(U_1-I\right)+\left(U_1-I\right)U_2...U_m...U_2+tA_1 \\ &= U_1...U_m...U_1-U_2...U_m...U_2+tA_1. \end{aligned}$$

From the formula (18) it follows:

$$V\left(t\right)=I-tA+R\left(t\right), \tag{20}$$

where

$$R\left(t\right)=R_1\left(t\right)+R_2\left(t\right)+...+R_{m-1}\left(t\right),$$

where $R_2\left(t\right),...,R_{m-1}\left(t\right)$ are defined analogously to $R_1\left(t\right)$.

Clearly, from (8) the estimate follows:

$$\left\|A^{-1}\left(\left(I-tA_j\right)-\exp\left(-tA_j\right)\right)A^{-1}\right\|\leq ct^2. \tag{21}$$

Indeed we have

$$\begin{aligned} &\left\|A^{-1}\left(\left(I-tA_j\right)-\exp\left(-tA_j\right)\right)A^{-1}\right\| \\ = &\left\|A_j^{-1}\left(\left(I-tA_j\right)-\exp\left(-tA_j\right)\right)A_j^{-1}\left(A_jA^{-1}\right)\right\| \\ \leq &\left\|\left(A_jA^{-1}\right)^*\right\|\left\|\left(A_jA^{-1}\right)\right\|\left\|\left(\left(I-tA_j\right)-\exp\left(-tA_j\right)\right)A_j^{-1}\right\|\leq ct^2. \end{aligned}$$

Taking into account estimates (11),(12),(15),(16) and (21), we obtain:

$$\left\|A^{-1}R\left(t\right)A^{-1}\right\| \leq ct^2. \tag{22}$$

From equalities (10) and (20), with account of inequality (22), we obtain the proved estimate for $\alpha = \beta = 1$. Analogously are obtained estimates for other cases.

Proof of the theorem 1. According to formulas (2) and (4), we have:

$$\begin{aligned}
u\left(t_k\right) - u_k\left(t_k\right) &= \left[U\left(t_k\right) - V^k\left(\tau\right)\right]\varphi = \left[U^k\left(\tau\right) - V^k\left(\tau\right)\right]\varphi \\
&= \sum_{i=1}^{k} V^{k-i}\left(\tau\right)\left(U\left(\tau\right) - V\left(\tau\right)\right)U\left(\left(i-1\right)\tau\right)\varphi \\
&= \sum_{i=1}^{k-1}\left[V^{k-i}\left(\tau\right) - U^{k-i}\left(\tau\right)\right]\left(U\left(\tau\right) - V\left(\tau\right)\right)U\left(\left(i-1\right)\tau\right)\varphi \\
&\quad + \sum_{i=1}^{k-1} U^{k-i}\left(\tau\right)\left(U\left(\tau\right) - V\left(\tau\right)\right)U\left(\left(i-1\right)\tau\right)\varphi \\
&\quad + \left(U\left(\tau\right) - V\left(\tau\right)\right)U\left(\left(k-1\right)\tau\right)\varphi.
\end{aligned}$$

From here the inequality follows:

$$\begin{aligned}
\left\|u\left(t_k\right) - u_k\left(t_k\right)\right\| &\leq \sum_{i=1}^{k-1}\left\|V^{k-i}\left(\tau\right) - U^{k-i}\left(\tau\right)\right\|\left\|\left(U\left(\tau\right) - V\left(\tau\right)\right)A^{-1}\right\| \\
&\quad \times \left\|U\left(\left(i-1\right)\tau\right)\right\|\left\|A\varphi\right\| + \sum_{i=1}^{k-1}\left\|AU\left(t_{k-i}\right)\right\| \\
&\quad \times \left\|A^{-1}\left(U\left(\tau\right) - V\left(\tau\right)\right)A^{-1}\right\|\left\|U\left(\left(i-1\right)\tau\right)\right\|\left\|A\varphi\right\| \\
&\quad +. \left\|\left(U\left(\tau\right) - V\left(\tau\right)\right)A^{-1}\right\| \\
&\quad \times \left\|U\left(\left(k-1\right)\tau\right)\right\|\left\|A\varphi\right\|.
\end{aligned} \tag{23}$$

According to inequality (6), we have :

$$\left\|V^{k-i}\left(\tau\right) - U^{k-i}\left(\tau\right)\right\| = \left\|V^{k-i}\left(\frac{t_{k-i}}{k-i}\right) - U\left(t_{k-i}\right)\right\| \leq \frac{c}{k-i}. \tag{24}$$

The estimate holds (see. [1], Ch. IX):

$$\left\|AU\left(t\right)\right\| \leq \frac{c}{t}, \quad t > 0. \tag{25}$$

From (23), with account of estimates (24), (25) and (17), we obtain:

$$\begin{aligned}
\left\|u\left(t_k\right) - u_k\left(t_k\right)\right\| &\leq c\tau\left(\sum_{i=1}^{k-1}\frac{1}{k-i} + \sum_{i=1}^{k-1}\frac{\tau}{t_{k-i}} + 1\right)\left\|A\varphi\right\| \\
&\leq c\tau\ln\left(t_k/\tau\right)\left\|A\varphi\right\|. \quad \blacksquare
\end{aligned}$$

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