ON THE SOLUTION OF BOUNDARY VALUE PROBLEM FOR DIFFERENTIAL EQUATIONS GIVEN ON GRAPHS

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Abstract

Mathematical modeling of various processes in the nets of gas pipeline, system of submission and distribution of water, drainpipe, also long current lines and different types of engineering constructions quite naturally leads to the consideration of partial differential equations on graphs with the boundary value data on the tops of graphs, with conditions of conjunctions in the nodes and given initial conditions ([1]-[4]). Not so it is a lot of papers, devoted to the theoretical investigation of boundary value problems, considered on graphs (see, for example, [5]-[6] and the references therein). In the present work boundary value problems for ordinary differential equations on graphs are investigated; correctness of the stated problem is proved; let's notice, that the special attention is given to the construction and research of difference analogues, which is a little concern in papers of other authors; estimation of precision is given; formulas of double-sweep method type are suggested for finding the solution of difference scheme ([7],[8]).

Key words and phrases: Tops of graphs, conjunctions in the nodes, difference scheme, differential equations on graphs, double-sweep method.

AMS subject classification: 65L07, 65L10, 65L20.

1. Ordinary differential equations of the second order on graphs

Let us consider a graph $G = (V, E)$, where $V = (a_0, a_1, \ldots, a_n)$ is a set of tops of this graph, a_0 is a node of graph and E is a set of ribs of graph $\{\overline{a_0a_1},\ \overline{a_0a_2}, \cdots,\ \overline{a_0a_n}\}$. Denote the rib $\overline{a_0A_i}$ by Γ_i. On each rib introduce a local coordinate system with the origin in the node a_0 and coordinate $x_\alpha \in (0,\ l_\alpha)$, where l_α is length of curve $\Gamma_\alpha(\alpha = 1,\ 2, \ldots, n)$.

Let us state the following problem: find the functions $Vu_\alpha(x_\alpha)(\alpha = 1,\ 2, \ldots, n)$, which satisfy the differential equations

$$\frac{d}{dx_\alpha}\left(K_\alpha\left(x_\alpha\right)\frac{du_\alpha\left(x_\alpha\right)}{dx_\alpha}\right) - q_\alpha\left(x_\alpha\right)u_\alpha\left(x_\alpha\right) = f_\alpha\left(x_\alpha\right) \qquad (1.1)$$

$$\alpha = \overline{1,\ n}, \quad x_\alpha \in (0, l_\alpha)$$

boundary conditions

$$u_\alpha(l_\alpha) = u^{(\alpha)}, \quad \alpha = \overline{1,\ n}, \tag{1.2}$$

and conditions of conjunctions

$$u_\alpha(a_0) = u_\beta(a_0), \quad \alpha,\ \beta = \overline{1,\ n}, \tag{1.3}$$

$$\sum_{\alpha=1}^{n} K_\alpha(x_\alpha) \left. \frac{du_\alpha(x_\alpha)}{dx_\alpha} \right|_{x_\alpha=0} = b. \tag{1.4}$$

where $(x_\alpha) \in C^1[0, l_\alpha]$, $K_\alpha(x_\alpha) > C_0 = const > 0$, $q_\alpha(x_\alpha) \in C^1[0, l_\alpha]$, $q_\alpha(x_\alpha) \geq 0$, $f_\alpha(x_\alpha) \in C^0[0, l_\alpha]$ are the given functions and $b,\ u^{(\alpha)}\left(\alpha = \overline{1, n}\right)$ are the given numbers.

Theorem 1.1. *There exists a unique regular solution of problem* $(1.1) - (1.4)$, *i.e. exists unique functions* $u_\alpha(x_\alpha) \in C^2]0, l_\alpha[\cap; C^1[0, l_\alpha]$, $\left(\alpha = \overline{1,\ n}\right)$, *which satisfy equations* (1.1), *boundary conditions* (1.2) *and conditions of conjunctions* (1.3), (1.4).

Proof. First we prove the uniqueness of solution of the problem (1.1)-(1.4). Let the problem (1.1)-(1.4) have two solutions: $\left(u_\alpha^{(1)}(x_\alpha)\right)^n$ and $\left(u_\alpha^{(2)}(x_\alpha)\right)^n$. Then the difference , $w_\alpha(x_\alpha) = u_\alpha^{(1)}(x_\alpha) - u_\alpha^{(2)}(x_\alpha)$, $\alpha = 1,\ 2, \ldots,\ n$, is the solution of the following homogeneous equation:

$$\frac{d}{dx_\alpha}\left(K_\alpha(x_\alpha)\frac{dw_\alpha(x_\alpha)}{dx_\alpha}\right) - q_\alpha(x_\alpha)\, w_\alpha(x_\alpha) = 0 \tag{1.5}$$

$$\alpha = \overline{1,\ n}, \quad x_\alpha \in (0, l_\alpha),$$

$$w_\alpha(l_\alpha) = 0, \quad \alpha = \overline{1,\ n}, \tag{1.6}$$

$$w_\alpha(0) = w_\beta(0), \quad \alpha,\ \beta = \overline{1,\ n}, \tag{1.7}$$

$$\sum_{\alpha=1}^{n} K_\alpha(0) \left. \frac{du_\alpha(x_\alpha)}{dx_\alpha} \right|_{x_\alpha=0} = 0. \tag{1.8}$$

Multiply equalities (1.5) on $w_\alpha(x_\alpha)$, $\alpha = 1,\ 2, \ldots,\ n$, and integrate the obtained equalities accordingly in the interval $(0, l_\alpha)$):

$$\int_0^{l_\alpha} \frac{d}{dx_\alpha}\left(K_\alpha(x_\alpha)\frac{dw_\alpha(x_\alpha)}{dx_\alpha}\right) w_\alpha(x_\alpha)\, dx_\alpha - \int_0^{l_\alpha} q_\alpha(x_\alpha)\, w_\alpha^2(x_\alpha) dx_\alpha = 0.$$

Using the formula of integration by parts we obtain the following equality $(\alpha = \overline{1,\ n})$:

$$K_\alpha(x_\alpha)\, w_\alpha(x_\alpha) \frac{dw_\alpha(x_\alpha)}{dx_\alpha}\bigg|_0^{l_\alpha} - \int_0^{l_\alpha} K_\alpha(x_\alpha)\left(\frac{dw_\alpha(x_\alpha)}{dx_\alpha}\right)^2 dx_\alpha - \\ - \int_0^{l_\alpha} q_\alpha(x_\alpha)\, w_\alpha^2(x_\alpha) dx_\alpha = 0. \tag{1.9}$$

Summing up the equalities (1.9) and taking into account relations (1.6)-(1.8), we obtain:

$$\sum_{\alpha=1}^{n} \int_0^{l_\alpha} \left[K_\alpha(x_\alpha)\left(\frac{dw_\alpha(x_\alpha)}{dx_\alpha}\right)^2 + q_\alpha(x_\alpha)\, w_\alpha^2(x_\alpha)\right] dx_\alpha = 0. \tag{1.10}$$

As the functions $K_\alpha(x_x) > 0, \quad q_\alpha(x_x) \geq 0 \quad \left(\alpha = \overline{1,\ n}\right)$, from the equality (1.10) it follows, that $\dfrac{dw_\alpha(x_\alpha)}{dx_\alpha} = 0$, i.e. $w_\alpha(x_\alpha) = const$.

Taking into account, that $w_\alpha(l_x) = 0$, finally we obtain $w_\alpha(x_x) \equiv 0,\ \ \alpha = \overline{1,\ n}$. Thereby, the uniqueness of solution (1.1)-(1.4) is proved.

To prove the existence of solution of the problem (1.1)-(1.4) it is sufficient to prove the existence of solution of the corresponding problem for homogeneous equation:

$$\frac{d}{dx_\alpha}\left(K_\alpha(x_\alpha)\frac{dw_\alpha(x_\alpha)}{dx_\alpha}\right) - q_\alpha(x_\alpha)\, w_\alpha(x_\alpha) = 0. \tag{1.11}$$

$$\alpha = \overline{1,\ n}, \quad x_\alpha \in (0, l_\alpha),$$

$$w_\alpha(l_\alpha) = u^{(\alpha)}, \quad \alpha = \overline{1,\ n}, \tag{1.12}$$

$$w_\alpha(0) = w_\beta(0), \quad \alpha,\ \beta = \overline{1,\ n}, \tag{1.13}$$

$$\sum_{\alpha=1}^{n} K_\alpha(0)\, \frac{du_\alpha(x_\alpha)}{dx_\alpha}\bigg|_{x_\alpha=0} = 0. \tag{1.14}$$

Denote by $w_{\alpha 1}(x_\alpha)$ the solution of equation (1.11), which satisfies the boundary conditions

$$w_{\alpha 1}(0) = 1, \quad w_{\alpha 1}(l_\alpha) = 0, \quad \alpha = \overline{1,\ n}, \tag{1.15}$$

and by $w_{\alpha 2}(x_\alpha)$ the solution of equation (1.11), which satisfies the boundary conditions

$$w_{\alpha 2}(0) = 0, \quad w_{\alpha 2}(l_\alpha) = 1, \quad \alpha = \overline{1,\ n}. \tag{1.16}$$

It is known, that solutions of these problems exist and belong to the class $C^2]0, l_\alpha[\cap C^1[0, l_\alpha]$. Represent the solution of problem (1.11)-(1.14) in the following form

$$w_\alpha(x_\alpha) = a_{\alpha 1} w_{\alpha 1}(x_\alpha) + a_{\alpha 2} w_{\alpha 2}(x_\alpha), \quad \alpha = \overline{1,\ n}, \tag{1.17}$$

where $a_{\alpha 1}$ and $a_{\alpha 2}$ are constants yet unknown. It is obvious, that these functions are solutions of the equation (1.11). Choose constants $a_{\alpha 1}, a_{\alpha 2} \quad \left(\alpha = \overline{1,\ n}\right)$, such, that the conditions (1.12)-(1.14) be fulfilled.

From the conditions (1.12),(1.13) it follows, that

$$a_{\alpha 1} = a = const, \quad a_{\alpha 2} = u^{(\alpha)} \quad \left(\alpha = \overline{1,\ n}\right).$$

Inserting these values in the equality (1.17), we obtain

$$w_\alpha(x_\alpha) = a w_{\alpha 1}(x_\alpha) + u^{(\alpha)} w_{\alpha 2}(x_\alpha), \quad \alpha = \overline{1,\ n}, \quad x_\alpha \in [0,\ l_\alpha], \tag{1.18}$$

where a is an arbitrary constant. To define this constant we use the condition (1.14) Differentiating equality (1.18) and inserting it in (1.14), we obtain

$$\sum_{\alpha=1}^{n} K_\alpha(0) \left(a \frac{dw_{\alpha 1}(0)}{dx_\alpha} + u^{(\alpha)} \frac{dw_{\alpha 2}(0)}{dx_\alpha} \right) = 0.$$

From this equality it follows, that

$$a = \frac{-\sum\limits_{\alpha=1}^{n} K_\alpha(0)\, u^{(\alpha)} \frac{dw_{\alpha 2}(0)}{dx_\alpha}}{\sum\limits_{\alpha=1}^{n} K_\alpha(0) \frac{dw_{\alpha 1}(0)}{dx_\alpha}}, \tag{1.19}$$

if $\sum\limits_{\alpha=1}^{n} K_\alpha(0) \dfrac{dw_{\alpha 1}(0)}{dx_\alpha} \neq 0$. Therefore, to prove the existence of the solution (1.11)-(1.14), it is sufficient to prove that the denominator of expression (1.19) is not equal to zero.

It is easy to show, that the solution of problem (1.11),(1.15) is monotonoūsly decreasing function. Then

$$\frac{dw_{\alpha 1}(0)}{dx_\alpha} \leq 0. \tag{1.20}$$

Let us prove, that at $x_a = 0$ the strict inequality holds

$$\frac{dw_{\alpha 1}(0)}{dx_\alpha} < 0. \tag{1.21}$$

As $K_\alpha(x_\alpha) > 0$, from the inequality (1.20) we obtain that

$$K_\alpha(x_\alpha)\frac{dw_{\alpha 1}(x_\alpha)}{dx_\alpha} \le 0, \quad x_\alpha \in [0,\ l_\alpha], \quad \alpha = \overline{1,\ n}. \tag{1.22}$$

Taking into account, that and from the equation(1.11) it follows

$$\frac{d}{dx_\alpha}\left(K_\alpha(x_\alpha)\frac{dw_{\alpha 1}(x_\alpha)}{dx_\alpha}\right) \ge 0.$$

This means, that $K_\alpha(x_\alpha)\dfrac{dw_{\alpha 1}(x_\alpha)}{dx_\alpha}$ is monotonously increasing function.

Further assume that the inequality (1.21) does not hold, i.e. the equality takes place

$$\frac{dw_{\alpha 1}(0)}{dx_\alpha} = 0.$$

Then we will have $K_\alpha(0)\dfrac{dw_{\alpha 1}(0)}{dx_\alpha} = 0$. In this case there exists $\varepsilon_\alpha \in (0, l_\alpha)$, such that $K_\alpha(\varepsilon_\alpha)\dfrac{dw_{\alpha 1}(\varepsilon_\alpha)}{dx_\alpha} > 0$, that contradicts to the condition (1.22). Thus, we obtain, that $\dfrac{dw_{\alpha 1}(0)}{dx_\alpha} < 0, \quad \alpha = \overline{1,\ n}$. This means, that the denominator of expression (1.19) is not equal to zero. Thereby, the existence of the solution of problem (1.11)-(1.14) is proved. Theorem 1.1 is completely proved.

2. Difference scheme for numerical solution of problem (1.1)-(1.4)

On $\Gamma_\alpha\,(\alpha = 1,\ 2, \ldots,\ n)$ we introduce an uniform mesh with step h_α:

$$\bar{\omega}_h^{(\alpha)} = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h_\alpha,\ i_\alpha = 0,\ 1,\ 2, \ldots, N_\alpha;\ x_\alpha^{(0)} = 0\,;\ h_\alpha N_\alpha = l_\alpha\right\}\ \alpha = \overline{1,\ n}\,.$$

If on mesh we substitute differential operator by the difference operator, we obtain the following difference scheme:

$$\left(K_\alpha y_{\alpha,\bar{x}_\alpha}\right)_{x_\alpha}^{(i_\alpha)} - q_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha)} = f_\alpha^{(i_\alpha)}, \quad i_\alpha = \overline{1,\ N_\alpha - 1},\ a = \overline{1, n}, \tag{2.1}$$

$$y_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1,\ n}, \tag{2.2}$$

$$y_\alpha^{(0)} = y_\beta^{(0)}, \quad \alpha,\ \beta = \overline{1,\ n}, \tag{2.3}$$

$$\sum_{\alpha=1}^{n} K_\alpha^{(1)} \frac{y_\alpha^{(1)} - y_\alpha^{(0)}}{h_\alpha} = 0, \tag{2.4}$$

where $y_\alpha^{(i_\alpha)} = y_\alpha(i_\alpha h_\alpha)$, $\quad y_{\alpha,\bar{x}_\alpha} = \dfrac{y_\alpha^{(i_\alpha)} - y_\alpha^{(i_\alpha - 1)}}{h_\alpha}, \quad y_{\alpha,x_\alpha} = \dfrac{y_\alpha^{(i_\alpha+1)} - y_\alpha^{(i_\alpha)}}{h_\alpha}.$

Theorem 2.1. *Difference scheme* (2.1) – (2.4) *has no more then one solution.*

Proof. Let us assume, that problem (2.1) – (2.4) has two solutions: $\bar{y}_\alpha\left(x_\alpha^{(i_\alpha)}\right)$ and $\bar{\bar{y}}_\alpha\left(x_\alpha^{(i_\alpha)}\right)$, $\alpha = 1,\ 2, \ldots, n$. From the relations (2.1)–(2.4) it follows, that the difference $w_\alpha\left(x_\alpha^{(i_\alpha)}\right) = \bar{y}_\alpha\left(x_\alpha^{(i_\alpha)}\right) - \bar{\bar{y}}_\alpha\left(x_\alpha^{(i_\alpha)}\right)$ is a solution of the following difference scheme:

$$(K_\alpha W_{\alpha,\bar{x}_\alpha})_{x_\alpha}^{(i_\alpha)} - q_\alpha^{(i_\alpha)} W_\alpha^{(i_\alpha)} = 0,\ \imath_\alpha = \overline{1,\ N_\alpha - 1},\ i_\alpha = \overline{1,\ N_\alpha - 1}, \tag{2.5}$$

$$W_\alpha^{(N_\alpha)} = 0, \quad \alpha = \overline{1,\ n}, \tag{2.6}$$

$$W_\alpha^{(0)} = W_\beta^{(0)}, \quad \alpha,\ \beta = \overline{1,\ n}, \tag{2.7}$$

$$\sum_{\alpha=1}^{n} K_\alpha^{(1)} W_{\alpha,x_\alpha}^{(0)} = 0. \tag{2.8}$$

Introduce the scalar products

$$(v_\alpha, z_\alpha) = \sum_{i_\alpha=1}^{N_\alpha-1} v_\alpha^{(i_\alpha)} z_\alpha^{(i_\alpha)} h_\alpha, \qquad (v_\alpha, z_\alpha] = \sum_{i_\alpha=1}^{N_\alpha} v_\alpha^{(i_\alpha)} z_\alpha^{(i_\alpha)} h_\alpha$$

and the following norms introduced by these scalar products:

$$\|y_\alpha\| = (y_\alpha,\ y_\alpha)^{1/2}, \qquad \|y_\alpha\| = (y_\alpha, y_\alpha]\,.$$

Multiply the equalities (2.5) on $W_\alpha^{(i_\alpha)} h_\alpha \quad \left(\alpha = \overline{1,\ n}\right)$ and sum up the obtained equalities by I_α accordingly from 1 to $N_\alpha - 1$. Using the Green first difference formula [4] and taking into account boundary conditions (2.6), we obtain:

$$-\left(K_\alpha W_{\alpha,\bar{x}_\alpha},\ W_{\alpha,\bar{x}_\alpha}\right] - K_\alpha^{(1)} W_{\alpha,\bar{x}_\alpha}^{(1)} W_\alpha^{(0)} - \sum_{i_\alpha=1}^{N_\alpha-1} q_\alpha^{(i_\alpha)} \left(W_\alpha^{(i_\alpha)}\right)^2 h_\alpha = 0. \tag{2.9}$$

Further, sum up these equations by from 1 to n and take into account relations (2.7), (2.8). Then we obtain:

$$\sum_{\alpha=1}^{n} \left\{ \left(K_\alpha, (\ W_{\alpha,\bar{x}_\alpha})^2\right] + \sum_{i_\alpha=1}^{N_\alpha-1} q_\alpha^{(i_\alpha)} \left(W_\alpha^{(i_\alpha)}\right)^2 h_\alpha \right\} = 0 \tag{2.10}$$

As $K_\alpha(x_\alpha) > 0$ and $q_\alpha(x_\alpha) \geq 0$ from the equality (2.10) immediately follows that

$W_\alpha(x_\alpha) \equiv 0, \quad x_\alpha \in \omega_h^\alpha, \quad \left(\alpha = \overline{1,\ n}\right).$

The theorem is proved.

Theorem 2.2. *There exists a solution of difference scheme* (2.1)-(2.4).

Proof. To prove the existence of the solution of difference scheme it is sufficient to show, that the corresponding homogeneous difference scheme has only trivial solution. The homogeneous difference scheme corresponding to difference scheme (2.1)-(2.4) has the form (2.5)-(2.8). The fairness of the statement of the theorem directly follows from the equality (2.10). The theorem is proved.

Theorem 2.3. *Let* $u_\alpha \in C^3[0,l_\alpha]$, $\alpha=\overline{1,\ n}$. *Then the solution of the difference scheme* $(2.1)-(2.4)$ *uniformly converges to the solution of problem* $(1.1)-(1.4)$ *at the rate of* $O(h)$ *when* $h\to 0$, *where* $h=\max\limits_{1\le\alpha\le n} h_\alpha$.

Proof. Introduce the mesh function of error

$$z_\alpha(x_\alpha^{(i_\alpha)})=y_\alpha(x_\alpha^{(i_\alpha)})-u_\alpha(x_\alpha^{(i_\alpha)}),\quad i_\alpha=\overline{1,\ N_\alpha},\quad \alpha=\overline{1,\ n},\tag{2.11}$$

where $y_\alpha(x_\alpha^{(i_\alpha)})$ is a solution of the difference scheme (2.1)-(2.4) and function $u_\alpha(x_\alpha^{(i_\alpha)})$ is a solution of differential equation (1.1)-(1.4). Define from the equality (2.11) $y_\alpha(x_\alpha^{(i_\alpha)})=u_\alpha(x_\alpha^{(i_\alpha)})+z_\alpha(x_\alpha^{(i_\alpha)})$ and substitute it in the difference scheme (2.1)-(2.4). Then for the error function we obtain the following problem:

$$(K_\alpha z_{\alpha,\bar{x}_\alpha})_{x_\alpha}^{(i_\alpha)}-q_\alpha^{(i_\alpha)}z_\alpha^{(i_\alpha)}=-\Psi_\alpha^{(i_\alpha)},\quad i_\alpha=\overline{1,\ N_\alpha-1},\quad \alpha=\overline{1,\ n},\tag{2.12}$$

where

$$\Psi_\alpha^{(i_\alpha)}=(K_\alpha u_{\alpha,\bar{x}_\alpha})_{x_\alpha}^{(i_\alpha)}-q_\alpha^{(i_\alpha)}u_\alpha^{(i_\alpha)},\tag{2.13}$$

$$z_\alpha^{(i_\alpha)}=0,\quad \alpha=\overline{1,\ n}.\tag{2.14}$$

$$z_\alpha^{(0)}=z_\beta^{(0)},\quad \alpha,\beta=\overline{1,\ n}.\tag{2.15}$$

$$\sum_{\alpha=1}^{n}\frac{K_\alpha^{(1)}}{h_\alpha}\left(z_\alpha^{(1)}-z_\alpha^{(0)}\right)=-\Theta_0,\tag{2.16}$$

where

$$\Theta_0=\sum_{\alpha=1}^{n}\frac{K_\alpha^{(1)}}{h_\alpha}\left(u_\alpha^{(1)}-u_\alpha^{(0)}\right).\tag{2.17}$$

It can be easily shown, that if $u_\alpha\left(x_\alpha\right)\in C^3\left(\Gamma_\alpha\right),\quad \alpha=\overline{1,\ n}$, then

$$\left\|\Psi_\alpha^{(i_\alpha)}\right\|=O\left(h\right),\quad \left|\Theta_0\right|=O\left(h\right),\quad i_\alpha=\overline{1,\ N_\alpha-1},\quad \alpha=\overline{1,\ n}\,.\tag{2.18}$$

Multiply the equalities (2.13) on $z_\alpha^{(i_\alpha)}h_\alpha,\quad \alpha=\overline{1,\ n}$, and sum up the obtained equalities w.r.t i_α from 1 to $N_\alpha-1$. Taking into account conditions (2.14) and using the first difference formula of Green we obtain:

$$-\left(K_\alpha,\left(z_{\alpha,\bar{x}_\alpha}\right)^2\right]-K_\alpha^{(1)}z_{\alpha\bar{x}_\alpha}^{(1)}z_1^{(0)}-\left(q_\alpha,\left(z_\alpha\right)^2\right)=\left(\Psi_\alpha,z_\alpha\right).$$

Further sum up these equalities w.r.t α and take into account conditions (2.15, (2.16). We will have

$$\sum_{\alpha=1}^{n}\left\{\ \left(K_\alpha, (z_{\alpha,\bar{x}_\alpha})^2\right] + \left(q_\alpha, (z_\alpha)^2\right)\ \right\} = \sum_{\alpha=1}^{n}(\Psi_\alpha, z_\alpha) + z_1^{(0)}\Theta_0. \qquad (2.19)$$

Introduce the following notation

$$q_{\alpha,\min} = \min_{1\le i_\alpha \le N_\alpha - 1} q_\alpha^{(i_\alpha)} \ge 0, \quad K_{\alpha,\min} = \min_{1\le i_\alpha \le N_\alpha - 1} K_\alpha^{(i_\alpha)} > 0,$$

then we will have

$$\sum_{\alpha=1}^{n}\left\{\ \left(K_\alpha, (z_{\alpha,\bar{x}_\alpha})^2\right] + \left(q_\alpha, (z_\alpha)^2\right)\ \right\} \ge \sum_{\alpha=1}^{n}\left\{K_{\alpha,\min}\left\|z_{\alpha,\bar{x}_\alpha}\right\|^2 + q_{\alpha,\min}\left\|z_\alpha\right\|^2\right\}.$$

Further, using the inequality [7], from the equality (2.19) we obtain

$$\sum_{\alpha=1}^{n}\left\{K_{\alpha,\min}\left\|z_{\alpha,\bar{x}_\alpha}\right\|^2 + q_{\alpha,\min}\left\|z_\alpha\right\|^2\right\} \le$$

$$\le \sum_{\alpha=1}^{n}\left(\varepsilon_\alpha\left\|z_{\alpha,\bar{x}_\alpha}\right\|^2 + \frac{1}{4\varepsilon_\alpha}\left\|\Psi_\alpha\right\|^2\right) + \varepsilon_0\left\|z_1\right\|^2 + \frac{1}{4\varepsilon_0}\left|\Theta_0\right|^2. \qquad (2.20)$$

We choose constants ε_i $(i = 0, 1, \ldots, n)$ in the following way:
a) $q_{\alpha,\min} > 0, \quad \alpha = 1, 2, \ldots, n$, then

$$\varepsilon_0 + \varepsilon_1 = q_{1,\min}, \quad \varepsilon_\alpha = q_{\alpha,\min}, \quad \alpha = 2, 3, \ldots, n.$$

In this case from the inequality (2.20) we obtain

$$\sum_{\alpha=1}^{n} K_{\alpha,\min}\left\|z_{\alpha,\bar{x}_\alpha}\right\|^2 \le M_1\left|\Theta_0\right|^2 + M_2\sum_{\alpha=1}^{n}\left\|\Psi_\alpha\right\|^2, \qquad (2.21)$$

where $M_1 > 0$, $M_2 > 0$ are some constants. Further we use the following embedding theorem [9]. For any mesh function $v(x)$, $x \in [0, l]$, given on the arbitrary nonuniform mesh $\bar{\omega}$ and being equal to zero only at $x = 0$ or at $x = l$, the inequality is fair

$$\left\|v\right\|_C^2 \le l\left\|v_{\bar{x}}\right\|^2.$$

Using this theorem from the inequality (2.21) we obtain the following estimate:

$$\sum_{\alpha=1}^{n}\frac{K_{\alpha,\min}}{l_\alpha}\left\|z_\alpha\right\|_C^2 \le M_1\left|\Theta_0\right|^2 + M_2\sum_{\alpha=1}^{n}\left\|\Psi_\alpha\right\|^2, \qquad (2.22)$$

Taking into account, that $\|\Psi_\alpha\| = O(h)$, $|\Theta_0| = O(h)$, $\alpha = \overline{1,\ n}$, finally we obtain:

$$|z_\alpha| = O(h), \quad \text{where} \quad h = \max_{1\le\alpha\le n} h_\alpha,$$

that implies the uniform convergence of the difference scheme (2.1)-(2.4), when $h \to 0$.

b) Some of $q_{a,\min}$ are equal to zero. If $q_{1,\min} = 0$, then constants ε_0 and ε_1 can be chosen in the following way:

$$\varepsilon_{\alpha_0} < \frac{K_{\alpha_0,\min}}{l_\alpha}.$$

If $q_{\alpha_0,\min} = 0$, $2 \le \alpha_0 \le n$, then the corresponding value ε_{a_0} can be chosen in the following way

$$\varepsilon_{\alpha_0} < \frac{K_{\alpha_0,\min}}{l_\alpha}.$$

The theorem is proved.

Remark. Let $u_\alpha \in C^3[0, l_\alpha]$ $\left(\alpha = \overline{1,\ n}\right)$. Then $\left\|\Psi_\alpha^{(i_\alpha)}\right\| = O\left(h^2\right)$, $i_\alpha = \overline{1,\ N_\alpha - 1}$, $\alpha = \overline{1,\ n}$. Instead of condition (2.4) we consider the following approximation of the conjunction conditions:

$$\sum_{\alpha=1}^{n} K_\alpha(0.5h_\alpha)\frac{y_\alpha^{(1)} - y_\alpha^{(0)}}{h_\alpha} - 0.5\sum_{\alpha=1}^{n} h_\alpha\left[q_\alpha^{(0)}y_\alpha^{(0)} + f_\alpha^{(0)}\right] = 0 \qquad (2.23)$$

Then the error approximation

$$\Theta_0 = \sum_{\alpha=1}^{n} K_\alpha(0.5h_\alpha)\frac{u_\alpha^{(1)} - u_\alpha^{(0)}}{h_\alpha} - 0.5\sum_{\alpha=1}^{n} h_\alpha\left[q_\alpha^{(0)}y_\alpha^{(0)} + f_\alpha^{(0)}\right]$$

will have the order $O(h^2)$.

Indeed,

$$\begin{aligned}\Theta_0 &= \sum_{\alpha=1}^{n}\Bigg\{K_\alpha(0)\left.\frac{du_\alpha}{dx_\alpha}\right|_{x_\alpha=0} - 0.5h_\alpha\left(K_\alpha'(0)\,u_\alpha'(0) + K_\alpha(0)\,u_\alpha''(0)\right) - \\ &\quad -0.5h_\alpha\left(q_\alpha^{(0)}u_\alpha^{(0)} + f_\alpha^{(0)}\right)\Bigg\} + O\left(h^2\right) = O\left(h^2\right),\end{aligned}$$

as $\displaystyle\sum_{\alpha=1}^{n} K_\alpha(0)\left.\frac{du_\alpha}{dx_\alpha}\right|_{x_\alpha=0} = 0$ (conjunction condition) and

$$K_\alpha'(0)\,u_\alpha'(0) + K_\alpha(0)\,u_\alpha''(0) - q_\alpha^{(0)}u_\alpha^{(0)} - f_\alpha^{(0)} = \frac{\partial}{\partial x}\left.\left(K_\alpha\frac{du_\alpha}{dx_\alpha}\right)\right|_{x_\alpha=0} -$$

$$-q_\alpha u_\alpha^{(0)} - f_\alpha = 0.$$

Therefore, $\|\Psi_\alpha\| = O\left(h^2\right), \quad \alpha = \overline{1,\ n}\,, \quad |\Theta_0| = O\left(h^2\right)$, if $u_\alpha \in\ C^3[0, l_\alpha]$.

The a priori estimate for difference scheme (2.1)-(2.3), (2.23) may be obtained using the techniques by means of which the a priori estimate (2.22) was obtained. But in this case we will obtain the uniform convergence of the difference scheme (2.1)-(2.3), (2.23) at rate $O(h^2)$ when $h \to 0$.

3. Variant of double-sweep method for difference equations (2.1)-(2.4)

Let us write the difference scheme (2.1)-(2.4) as a system of linear algebraic equations:

$$\frac{K_\alpha^{(i_\alpha-1)}}{h_\alpha^2} y_\alpha^{(i_\alpha-1)} - \left(\frac{K_\alpha^{(i_\alpha-1)} + K_\alpha^{(i_\alpha)}}{h_\alpha^2} + q_\alpha^{(i_\alpha)}\right) y_\alpha^{(i_\alpha)} + \frac{K_\alpha^{(i_\alpha)}}{h_\alpha^2} y_\alpha^{(i_\alpha+1)} = f_\alpha^{(i_\alpha)},$$

$$i_\alpha = \overline{1,\ N_\alpha - 1}, \quad \alpha = \overline{1, n}.$$

$$y_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1,\ n},$$

$$\sum_{\alpha=1}^{n} \frac{K_\alpha^{(1)}}{h_\alpha}\left(y_\alpha^{(1)} - y_\alpha^{(0)}\right) = 0. \tag{3.1}$$

Introduce the following denotations:

$$a_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha-1)}}{h_\alpha^2}, \quad b_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha)}}{h_\alpha^2}, \quad c_\alpha^{(i_\alpha)} = \frac{K_\alpha^{(i_\alpha-1)} + K_\alpha^{(i_\alpha)}}{h_\alpha^2}, \quad m_\alpha = \frac{K_\alpha^{(0)}}{h_\alpha}.$$

$$i_\alpha = \overline{1,\ N_\alpha - 1}, \quad \alpha = \overline{1, n}.$$

Then the system of equations (3.1) can be rewritten in the following form:

$$a_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} - c_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha)} + b_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha+1)} = f_\alpha^{(i_\alpha)} \tag{3.2}$$

$$y_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1,\ n}, \tag{3.3}$$

$$y_\alpha^{(0)} = y_\beta^{(0)}, \quad \alpha,\ \beta = \overline{1,\ n}, y_\alpha^{(0)} = y_\beta^{(0)}, \quad \alpha,\ \beta = \overline{1,\ n}, \tag{3.4}$$

$$\sum_{\alpha=1}^{n} m_\alpha \left(y_\alpha^{(1)} - y_\alpha^{(0)}\right) = 0. \tag{3.5}$$

Suppose, that for the solution of difference equation (3.2) the relation holds:

$$y_\alpha^{(i_\alpha+1)} = \xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)}, \quad i_\alpha = \overline{0,\ N_\alpha - 1}, \quad \alpha = \overline{1,\ n}. \tag{3.6}$$

then

$$y_\alpha^{(i_\alpha)} = \xi_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} + \eta_\alpha^{(i_\alpha)}, \tag{3.7}$$

Substituting expression (3.6) in the equation (3.2) we obtain

$$a_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha-1)} - c_\alpha^{(i_\alpha)} y_\alpha^{(i_\alpha)} + b_\alpha^{(i_\alpha)} \left(\xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)} \right) = f_\alpha^{(i_\alpha)}.$$

From this equation we define $y_\alpha^{(i_\alpha)}$:

$$y_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}} y_\alpha^{(i_\alpha-1)} + \frac{b_\alpha^{(i_\alpha)} \eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}}$$

$$\left(c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)} \neq 0 \right).$$

Comparing this equality with the equality (3.7) we obtain:

$$\xi_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}}, \quad \eta_\alpha^{(i_\alpha)} = \frac{b_\alpha^{(i_\alpha)} \eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)}}. \tag{3.8}$$

Using the boundary conditions (3.3) to define $\xi_\alpha^{(N_\alpha)}$ and $\eta_\alpha^{(N_\alpha)}$, we obtain:

$$\xi_\alpha^{(N_\alpha)} = 0, \quad \eta_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1,\ n}. \tag{3.9}$$

Recurrent relations (3.8), (3.9) allow to define coefficients $\xi_\alpha^{(i_\alpha)}$ and $\eta_\alpha^{(i_\alpha)}$ ($i_\alpha = N_\alpha - 1,\ N_\alpha - 2,\ \ldots,\ 1 \quad \alpha = \overline{1,\ n}.$), if $c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)} \xi_\alpha^{(i_\alpha+1)} \neq 0$.

As $\left| c_\alpha^{(i_\alpha)} \right| \geq \left| a_\alpha^{(i_\alpha)} \right| + \left| b_\alpha^{(i_\alpha)} \right|$, $i_\alpha = \overline{1,\ N_\alpha - 1}$, $\alpha = \overline{1, n}$, therefore repeating the reasoning from [4], it can be proved, that

$$\left| \xi_\alpha^{(i_\alpha)} \right| < 1 \quad \text{and} \quad \left| \mathrm{c}_\alpha^{(\mathrm{i}_\alpha)} - \mathrm{b}_\alpha^{(\mathrm{i}_\alpha)} \xi_\alpha^{(\mathrm{i}_\alpha+1)} \right| \geq \left| \mathrm{a}_\alpha^{(\mathrm{i}_\alpha)} \right|.$$

Thus, we have proved that by means of recurrent formulas (3.8), (3.9) uniquely can be defined values of the coefficient $\xi_\alpha^{(i_\alpha)}$, $\eta_\alpha^{(i_\alpha)}$ ($i_\alpha = N_\alpha - 1,\ N_\alpha - 2,\ \ldots,\ 1, \quad \alpha = \overline{1, n}$).

Write out formulas (3.6) in case of $i_\alpha = 0$:

$$y_\alpha^{(1)} = \xi_\alpha^{(1)} y_\alpha^{(0)} + \eta_\alpha^{(1)}, \quad \alpha = \overline{1, n}.$$

Insert these equalities in (3.5) and take into account relations (3.4), then we obtain:

$$\sum_{\alpha=1}^{n} m_\alpha \left(\xi_\alpha^{(1)} y_\alpha^{(0)} - \eta_\alpha^{(1)} - y_\alpha^{(0)} \right) = 0.$$

As $\left| \xi_\alpha^{(1)} \right| < 1$, from the last equality we obtain:

$$y_1^{(0)} = y_\alpha^{(0)} = \frac{\sum\limits_{\alpha=1}^{n} m_\alpha \eta_\alpha^{(1)}}{\sum\limits_{\alpha=1}^{n} m_\alpha - \sum\limits_{\alpha=1}^{n} m_\alpha \xi_\alpha^{(1)}}, \quad \alpha = 2, 3, \ldots, n. \tag{3.10}$$

Collect all formulas of double-sweep method and write them down in order of application:

$$\xi_\alpha^{(i_\alpha)} = \frac{a_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)}}, \quad \eta_\alpha^{(i_\alpha)} = \frac{b_\alpha^{(i_\alpha)}\eta_\alpha^{(i_\alpha+1)} - f_\alpha^{(i_\alpha)}}{c_\alpha^{(i_\alpha)} - b_\alpha^{(i_\alpha)}\xi_\alpha^{(i_\alpha+1)}}$$

$$i_\alpha = N_\alpha - 1,\ N_\alpha - 2, \ldots, 0, \quad \alpha = \overline{1,n}.$$

$$\xi_\alpha^{(N_\alpha)} = 0, \quad \eta_\alpha^{(N_\alpha)} = u^{(\alpha)}, \quad \alpha = \overline{1,n}.$$

$$y_\alpha^{(i_\alpha+1)} = \xi_\alpha^{(i_\alpha+1)} y_\alpha^{(i_\alpha)} + \eta_\alpha^{(i_\alpha+1)}, \quad i_\alpha = \overline{0,\ N_\alpha - 1}, \quad \alpha = \overline{1,n}.$$

$$y_\alpha^{(0)} = \frac{\sum\limits_{\alpha=1}^{n} m_\alpha \eta_\alpha^{(1)}}{\sum\limits_{\alpha=1}^{n} m_\alpha - \sum\limits_{\alpha=1}^{n} m_\alpha \xi_\alpha^{(1)}}, \quad \alpha = 1, 2, 3, \ldots, n.$$

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