ON RADON–NIKODYM DENSITY ESTIMATES OF THE SOLUTION OF DIFFERENTIAL EQUATIONS WITH A RANDOM RIGHT-HAND PART

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Abstract

The paper deals with some questions arising in the random analysis and statistical estimation theory. The approaches are outlined to the solution of the posed problems by methods of nonparametric statistical analysis.

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Statistical estimation for random processes is a rapidly developing area of research. It is understood that in the general case this problem is equivalent to the following problem: Let $X_1, X_2, \ldots, X_n$ be a sample of independent, identically distributed random elements in the separable Hilbert space H. Let B be the Borel σ-algebra in H, and μ be a probability measure on B which is the distribution of X_1. It is required to construct an estimate of μ by sample observation data.

In the general case this problem is not solvable to an extent to which it is solved in the finite-dimensional case (by the Glivenko–Cantelli theorem). Counter-examples of this kind were constructed by V. V. Sazonov [1] who indicated some situations in which the problem can be solved in particular cases. On the other hand, as recent studies have shown [2]–[4], we may succeed in estimating some important characteristics of the distribution of a measure in the Hilbert space, in particular, of a logarithmic derivative with respect to the direction of a subspace – the so-called Cameron–Martin subspace. Such estimates are based on the limit procedure by finite-dimensional approximations. Such a formulation makes it possible to apply the sufficiently well developed method of statistical estimation, in particular, estimation of the Rosenblatt–Parzen type. For practical reasons it would be desirable to find possibilities of applying these estimates in the problems traditionally considered for measure (distribution) transformations in the Hilbert space.

We give two examples of a possible application of these procedures.

Let us consider the differential equation

$$\begin{aligned} y''(t) + a(t)y(t) &= w'(t), \\ y'(0) = y'(1) &= 0, \quad 0 \le t \le 1, \end{aligned} \tag{1}$$

where $a(t)$ is a Gaussian random process related to the Wiener process

$$a(t) = b(t) + \int_0^1 A(t,s)\, dw(s).$$

Here $b(t)$ and $A(t,s)$ are nonrandom smooth functions. Problem (1) is understood as an equivalent form of the equation

$$y'(t) + \int_0^t \alpha(s) y(s)\, ds = w(t),$$

with the additional condition $\int_0^1 \alpha(s)y(s)\, ds = w(1) \pmod P$.

Our aim is to construct the solution of problem (1). For this we consider the direct and the inverse initial problem

$$\begin{aligned} y_1''(t) + \alpha(t)y_1(t) = 0, \\ y_1(0) = 1, \;\; y_1'(0) = 0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} y_2''(t) + \alpha(t)y_2(t) = 0, \\ y_2(1) = 1, \;\; y_2'(1) = 0. \end{aligned} \tag{3}$$

Using the method of successive approximations, for the solution we can write

$$y_1(t) = 1 + \sum_{k=1}^{\infty} (-1)^k \int_0^1 \int_0^{\lambda_1} \int_0^{\lambda_2} \cdots \int_0^{\lambda_{2k-1}} \alpha(\lambda_2)\alpha(\lambda_3)\cdots\alpha(\lambda_{2k})\, d\lambda_{2k} \cdots d\lambda_3\, d\lambda_2\, d\lambda_1$$

and

$$y_2(t) = 1 + \sum_{k=1}^{\infty} (-1)^k \int_t^1 \int_{\lambda_1}^1 \int_{\lambda_2}^1 \cdots \int_{\lambda_{2k-1}}^1 \alpha(\lambda_2)\alpha(\lambda_3)\cdots\alpha(\lambda_{2k})\, d\lambda_{2k} \cdots d\lambda_3\, d\lambda_2\, d\lambda_1.$$

The Wronskian of this system is

$$V(t) = y_1(1) \neq 0 \pmod P$$

and therefore the system y_1, y_2 is independent.

Let us now construct the Green function

$$G(t,s)=\begin{cases} y_1(t)y_2(s)V^{-1}(0) & \text{for } t\le s \\ y_1(t)y_2(s)V^{-1}(0) & \text{for } t>s \end{cases}. \tag{4}$$

As shown in [5], the solution of problem (1) can be written as an extended Skorokhod stochastic integral

$$y(t)=\int_0^1 \big\langle G(t,s),dw(s)\big\rangle. \tag{5}$$

If we now take into account the fact that an extended stochastic integral is in fact a logarithmic derivative in a concrete functional space (in our case in the space $L_2([0,1])$), then it becomes obvious that the stochastic estimation procedure can be used for estimating the solution of a problem of type (1). For this, keeping in mind that we construct estimates on the basis of finite-dimensional estimates, we must rewrite problem (1) as a system of equations in the finite-dimensional space R^n and write the estimate of finite-dimensional solutions using observations. Then we can be sure that the limit procedure converges.

More specifically, suppose we observe the solution of problem (1):

$$y_1(t),y_2(t),\dots,y_n(t).$$

We choose points $t_1,t_2,\dots,t_m$ so that $\lambda=\max(t_{i+1}-t_i)\to 0$. Let us rewrite (1) in the finite-difference form and consider the matrix of observations

$$\begin{array}{l} y_1(t_1),y_1(t_2),\dots,y_1(t_m) \\ y_2(t_1),y_2(t_2),\dots,y_2(t_m) \\ \dots\dots\dots\dots\dots\dots\dots \\ y_n(t_1),y_n(t_2),\dots,y_n(t_m). \end{array}$$

We construct the Rosenblatt–Parzen estimate by the following vector observations:

$$\widehat{l}_n^m(x_m)=\frac{\lambda_n\sum\limits_{i=1}^{n}\sum\limits_{s=1}^{m}a_n^sK'(\lambda_n(x_m^s-y_i(t_j)))\prod\limits_{\substack{j=1\\ j\neq s}}^{m}K(\lambda_n(x_m^j-y_i(t_j)))}{\sum\limits_{i=1}^{n}\prod\limits_{j=1}^{m}K(\lambda_n(x_m^j-y_i(t_j)))}.$$

The limit of this expression exists as $m \to \infty$ and give (5).

The other example concerns the problem

$$\begin{aligned} y' + f(t, y(t)) &= \xi'(t), \\ y(0) = 0 \ \text{ a.s. } \ 0 &\le t \le T, \end{aligned} \tag{6}$$

where $\xi(t)$ is some differentiable and smooth (for example, Gaussian) process.

As is known ([6]), under some natural conditions the measures μ_y and μ_ξ are equivalent and the Radon–Nikodym density can be written in the explicit form (the generalized Girsanov formula)

$$\frac{d\mu_y}{d\mu_\xi}(\xi) = \exp\left\{ -\int_0^T f(t, \xi(t))\, dw(t) - \frac{1}{2}\int_0^T f^2(t, \xi(t))\, dt \right\}.$$

The principal term in this expression is again the stochastic integral and it is not required to impose any restrictions like the dependence on "the past". Therefore this is an extended stochastic integral and it can be understood as a logarithmic derivative and we can construct as above finite-dimensional estimates of this summand and estimate by the limit procedure the Radon–Nikodym density. Of course, preliminarily, we should rewrite (7) as a difference equation in the same manner as we have done above for problem (1).

An analogous problem can be posed in the general case, too, using the results of [7]. The construction of statistical estimates of measure characteristics is also of interest in the case where these distributions are generated by random vector fields [8], [9].

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