THE PROBLEM OF VIBRATION OF AN ELASTIC SEMI-PLANE WITH THE CRACK PERPENDICULAR TO ITS BOUNDARY

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Abstract

In the paper an elastic isotropic semi-plane with the crack perpendicular to its boundary is considered. The crack is affected by transversal effort. The displacement takes place in the direction perpendicular to the possible propagation of the crack. For solving of the problem integral transformation of Kontorovich-Lebedev is used.The problem is reduced to the functional equation of Wiener-Hopf and is solved by the method of factorization.

Key words and phrases: Integral transformation of Kontorovich-Lebedev, vibration, elastic semi-plane, boundary condition, Wiener-Hopf method, factorization.

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In the present article an elastic isotropic semi-plane with the crack perpendicular to its boundary is considered. The crack is affected by transversal effort. The displacement takes place in the direction perpendicular to the possible propagation of the crack. According to the known classification [1] the considered case belongs to the type III called a pure displacement or an anti-plane deformation.

Components of displacement are $u_1 = 0$, $v_1 = 0$, $w_1 = w_1(x, y, t)$. The components of strain with value different from zero are defined by the relations $\tau_{xz} = \mu\frac{\partial w_1}{\partial x}$, $\tau_{yz} = \mu\frac{\partial w_1}{\partial y}$.

Components of displacement $w_1(x, y, t)$ satisfy differential equation

$$\frac{\partial^2 w_1}{\partial x^2} + \frac{\partial^2 w_1}{\partial y^2} = \frac{1}{c_2^2}\,\frac{\partial^2 w_1}{\partial t^2}\,, \tag{1}$$

where $c_2 = (\frac{\mu}{\rho})^{1/2}$ is the velocity of propagation of waves of displacement, ρ is density of material and μ is modulus of displacement.

Let us consider steady periodic vibrations with circle frequency ω and assume that $w_1(x, y, t) = w(x, y)e^{i\omega t}$. Then the equation (1) will have a form

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k^2 w = 0, \tag{2}$$

where $k = \frac{\omega}{c_2}$.

We assume that the concentrated transversal efforts $\tau\,\delta(x_0)$ are applied to the point x_0 of the boundary of the crack. The boundary conditions are as follows

$$\begin{aligned} &\mu\,\frac{\partial w}{\partial y} = \tau\,\delta(x_0), \quad && x < l, \quad y = 0, \\ &w(x,0) = 0, && x > l, \quad y = 0, \\ &\frac{\partial w}{\partial x} = 0, && x = 0, \quad 0 < y < \infty. \end{aligned} \tag{3}$$

Because of the symmetry we consider the upper quarter plane. We write the equation (2) and (3) by means of polar coordinates

$$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^2}\,\frac{\partial^2 w}{\partial \alpha^2} + \frac{1}{r}\,\frac{\partial w}{\partial r} + k^2 w = 0, \tag{4}$$

$$\begin{aligned} &\frac{\partial w}{\partial \alpha} = \frac{\tau\,\delta(x_0)}{\mu}\,r, \quad && 0 < r < l, \quad \alpha = 0, \\ &w = 0, && r > l, \quad \alpha = 0, \\ &\frac{\partial w}{\partial \alpha} = 0, && 0 < r < \infty, \quad \alpha = \frac{\pi}{2}\,. \end{aligned} \tag{5}$$

For the differential equation (4) and the boundary condition (5) we use integral transformation of Kontorovich-Lebedev [2].

$$\begin{aligned} \Phi(\lambda) &= \int_0^\infty W(r)\,\frac{H_\lambda^{(2)}(kr)}{r}\,dr, \\ W(r) &= -\frac{1}{2}\int_{-i\infty}^{i\infty} \lambda\,\Phi(\lambda)\,J_\lambda(kr)\,d\lambda. \end{aligned} \tag{6}$$

The first integral (6) has a meaning only in case if $W(r)$ also tends to zero when $r = 0$. In order to avoid this difficulty let us introduce function v instead of the function W. The function v is connected to the function W by the relation [2]

$$v = W - W(0)\,e^{-ikr},$$

where $W(0)$ is the value of the function W when $r = 0$. Then the differential equation (4) will have a form

$$\frac{\partial^2 v}{\partial \alpha^2} + r^2\,\frac{\partial^2 v}{\partial r^2} + r\,\frac{\partial v}{\partial r} + k^2 r^2 v = ikr\,W(0)\,e^{-ikr}. \tag{7}$$

Let us multiply both sides of the equation by $\frac{H_\lambda^2(kr)}{r}$ are integrate them within the limit from $r = 0$ to $r = \infty$. Taking into consideration the equation which is satisfied by the function $H_\lambda^{(2)}(kr)$ and having used integration by parts, after carrying out some transformations we will have

$$\frac{d^2\,\overline{v}}{d\alpha^2} + \lambda^2\,\overline{v} = ikW(0)\int_0^\infty e^{-ikr}\,H_\lambda^{(2)}(kr)\,dr, \tag{8}$$

where $\overline{v}$ is an integral transformation of the function v.

We calculate integral

$$I = \int_0^\infty e^{-ikr}\,H_\lambda^{(2)}(kr)\,dr.$$

We substitute integral representation of the Henkel function [2] for the expression of the integral I

$$H_\lambda^{(2)}(kr) = -\frac{e^{i\frac{\pi}{2}\lambda}}{\pi i}\int_{-\infty}^{\infty} e^{-ikrch\eta-\lambda\eta}\,d\eta.$$

Using the theory of residues for the integral I we obtain

$$I = -\frac{2\lambda e^{i\frac{\pi}{2}\lambda}}{k\sin\lambda\pi}\,.$$

Then the differential equation (8) reduces to

$$\frac{d^2\,\overline{v}}{d\alpha^2} + \lambda^2\,\overline{v} = \frac{2\lambda W(0)\,e^{i\frac{\pi}{2}\lambda}}{k\sin\lambda\pi}\,. \tag{9}$$

The equation (9) coincides with equation (25) from [2].

The general solution of the equation (9) is

$$\overline{v} = A\cos\lambda\alpha + B\sin\lambda\alpha + \frac{2W(0)\,e^{i\frac{\pi}{2}\lambda}}{i\lambda\sin\lambda\pi}\,. \tag{10}$$

The transformed boundary conditions are

$$\begin{aligned} \frac{d\,\overline{v}}{d\alpha} &= \frac{\tau}{\mu}\,H_\lambda^{(2)}(kr_0) + \Phi_1, \qquad && \alpha = 0,\\ \overline{v} &= \frac{4ikW(0)}{\lambda}\,e^{-i\frac{\pi}{2}\lambda} + \Phi_2, && \alpha = 0,\\ \frac{d\,\overline{v}}{d\alpha} &= 0, && \alpha = \frac{\pi}{2}\,, \end{aligned} \tag{11}$$

where Φ_1, ϕ_2 are unknown functions.

Substituting the general solution (10) for the boundary solutions (11) we will receive the following functional equation

$$\Phi_+ - \lambda\,\mathrm{tg}\,\frac{\lambda\pi}{2}\,\Phi_- = \frac{iW(0)e^{i\frac{\pi}{2}\lambda}}{\cos^2\frac{\lambda\pi}{2}} + 4ikW(0)e^{-i\frac{\pi}{2}\lambda}\,\mathrm{tg}\,\frac{\lambda\pi}{2}\,. \tag{12}$$

Φ_+ is the function which is analytical in the upper semi-plane and Φ_- is the function which is analytical in the lower semi-plane. We represent the function $\operatorname{tg}\frac{\lambda\pi}{2}$ in the form of the infinite product

$$\operatorname{tg}\frac{\lambda\pi}{2} = \frac{\lambda(\lambda+2)\prod\limits_1^\infty\left(1-\frac{\lambda}{2n}\right)\prod\limits_1^\infty\left(1+\frac{1}{n}+\frac{\lambda}{2n}\right)}{(1-\lambda^2)\prod\limits_1^\infty\left(1+\frac{1}{2n}-\frac{\lambda}{2n}\right)\prod\limits_1^\infty\left(1+\frac{1}{n}+\frac{\lambda}{2n}\right)}\,. \tag{13}$$

We substitute the expression (13) for the equation (12) and make a factorization [3]

$$\frac{(1+\lambda)\prod\limits_1^\infty\left(1+\frac{1}{2n}+\frac{\lambda}{2n}\right)}{(\lambda+2)\prod\limits_1^\infty\left(1+\frac{1}{n}+\frac{\lambda}{2n}\right)}\Phi_+ - \frac{\lambda^2\prod\limits_1^\infty\left(1-\frac{\lambda}{2n}\right)}{(1-\lambda)\prod\limits_1^\infty\left(1+\frac{1}{n}-\frac{\lambda}{2n}\right)}\Phi_- =$$

$$= \frac{iW(0)\,e^{i\frac{\pi}{2}\lambda}}{\cos^2\frac{\lambda\pi}{2}}\,\frac{(1+\lambda)\prod\limits_1^\infty\left(1+\frac{1}{2n}+\frac{\lambda}{2n}\right)}{(\lambda+2)\prod\limits_1^\infty\left(1+\frac{1}{n}+\frac{\lambda}{2n}\right)}+$$

$$+\,4ik\,W(0)e^{-i\frac{\pi}{2}\lambda}\,\frac{\lambda\prod\limits_1^\infty\left(1-\frac{\lambda}{2n}\right)}{(1-\lambda)\prod\limits_1^\infty\left(1+\frac{1}{n}-\frac{\lambda}{2n}\right)}\,.$$

According to the generalized theorem of Liouville

$$\frac{\lambda^2\prod\limits_1^\infty\left(1-\frac{\lambda}{2n}\right)}{(1-\lambda)\prod\limits_1^\infty\left(1+\frac{1}{n}-\frac{\lambda}{2n}\right)}\Phi_- =$$

$$= 4ikW(0)e^{-i\frac{\pi}{2}\lambda}\,\frac{\lambda\prod\limits_1^\infty\left(1-\frac{\lambda}{2n}\right)}{(1-\lambda)\prod\limits_1^\infty\left(1+\frac{1}{n}-\frac{\lambda}{2n}\right)}+C\lambda,$$

where

$$C = \frac{4ik\,W(0)}{\prod\limits_1^\infty\left(1+\frac{1}{2n}\right)}\,.$$

After carrying out some elementary transformations we will have

$$\Phi_- = -4ik\,W(0)\,e^{-i\frac{\pi}{2}\lambda}\,\frac{1}{\lambda}+\frac{4ikW(0)}{\prod\limits_1^\infty\left(1+\frac{1}{2n}\right)}\,\frac{1}{\lambda}\,\operatorname{ctg}\frac{\lambda\pi}{2}\,. \tag{14}$$

We use the reverse transformation of Kontorovich-Lebedev [2] for the equation (14).

We will have

$$v = 8kW(0)\left(\sum_{1}^{\infty}(-1)^n Y_{2n}(kr) + \frac{1}{\prod\limits_{1}^{\infty}\left(1+\frac{1}{2n}\right)} \sum_{1}^{\infty} J_{2n}(kr)\right). \quad (15)$$

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