LARGE TIME BEHAVIOR OF SOLUTIONS AND NUMERICAL APPROXIMATION OF NONLINEAR INTEGRO-DIFFERENTIAL EQUATION ASSOCIATED WITH THE PENETRATION OF A MAGNETIC FIELD INTO A SUBSTANCE

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Abstract

Large time behavior of solutions and numerical approximation of a nonlinear integro-differential equation associated with the penetration of a magnetic field into a substance is studied. The initial-boundary value problem with Dirichlet boundary conditions is investigated. Exponential stabilization of solution is established.

Key words and phrases: Nonlinear integro-differential equation, large time behavior, finite difference scheme.

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1 Introduction

Process of penetration of the magnetic field into a substance is modelled by Maxwell's system of partial differential equations [1]. If the coefficient of thermal heat capacity and electroconductivity of the substance depend on temperature, then Maxwell's system can be rewritten in the integro-differential form [2]. For the one-component magnetic field the one-dimensional case of this model is given by following integro-differential equation:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x}\left[a\left(\int_0^t \left(\frac{\partial W}{\partial x}\right)^2 d\tau \right) \frac{\partial W}{\partial x} \right], \tag{1.1}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

The existence of solutions of the initial-boundary value problem for the case $a(S) = 1 + S$ and the uniqueness for more general cases are studied in [2]. In [3] the existence and uniqueness properties are studied for the case $a(S) = (1 + S)^p$, $0 < p \leq 1$.

In the work [4] some generalization of equations of type (1) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e., depending on time, but independent of the space coordinates, the same process of penetration of the magnetic field into the material is modeled by the integro-differential equation, one-dimensional analogue of which has the form [4]:

$$\frac{\partial W}{\partial t} = a\left(\int_0^t \int_0^1 \left(\frac{\partial W}{\partial x}\right)^2 dx d\tau\right)\frac{\partial^2 W}{\partial x^2}. \tag{1.2}$$

The existence and uniqueness properties of the solutions of the initial-boundary value problems for the equations of (1.2) type were first studied in the work [5]. Investigation of (1.1) and (1.2) type models were continued in a number of other works (see, for example, [6], [7] and references there in). The existence theorems proved in [2],[3],[5] are based on a priori estimates, Galerkin's method and compactness arguments as it is done in [8],[9] for nonlinear problems.

Note that in [10] and [11] difference schemes for these and such type models were investigated.

The large time behavior of the solutions of the initial-boundary value problems for (1.1) type model for the case $a(S) = (1+S)^p$, $0 < p \leq 1$ is studied in [6], [7]. Note that, in these works exponential stabilization of solution of problem with homogeneous boundary condition are given, while stabilization results of the solutions of problem with nonhomogeneous boundary data on one side of lateral boundary has the power-like form [7]. Analogous result for solutions of (1.2) type model is proven in [5].

The purpose of this note is to continue the study of large time behavior of solutions of the first boundary value problem as well as investigation of difference scheme for the equation (1.2). Here attention is paid to the case $a(S) = 1 + S$. It is shown that for the solution of initial-boundary value problem with nonhomogeneous data on part of lateral boundary the exponential stabilization takes place as in homogeneous case.

2 Large time behavior of solution

In the domain $Q = (0,1) \times (0,\infty)$ let us consider the following initial-boundary value problem:

$$\frac{\partial W}{\partial t} = (1+S)\frac{\partial^2 W}{\partial x^2}, \quad (x,t) \in Q,$$

$$W(0,t) = 0, \quad W(1,t) = \psi, \quad t \geq 0, \tag{2.1}$$

$$W(x,0) = W_0(x), \quad x \in [0,1],$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial W}{\partial x}\right)^2 dx d\tau,$$

$W_0(x)$ is a given function of its argument and $\psi = Const \geq 0$.

Let us introduce the notation

$$U(x,t) = W(x,t) - \psi x. \tag{2.2}$$

So, instead of (2.1) we have following problem:

$$\frac{\partial U}{\partial t} = (1+S)\frac{\partial^2 U}{\partial x^2}, \quad (x,t) \in Q, \tag{2.3}$$

$$U(0,t) = U(1,t) = 0, \quad t \geq 0, \tag{2.4}$$

$$U(x,0) = W_0(x) - \psi x, \quad x \in [0,1], \tag{2.5}$$

where

$$S(t) = \int_0^t \int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx d\tau.$$

Recall that the L_2 norm of a function v is given by:

$$\|u\| = \left[\int_0^1 u^2(x) dx\right]^{1/2}.$$

Theorem 2.1. *If $W_0 \in H_0^1(0,1)$, then the solution of the problem (2.1) satisfies the following estimate*

$$\|W - \psi x\| + \left\|\frac{\partial W}{\partial x} - \psi\right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Remark. Note that here and below in this section C denote positive constants independent of t and $H^k(0,1)$ and $H_0^k(0,1)$ denote usual Sobolev spaces.

Proof. Let us multiply (2.3) by U and integrate over $(0,1)$. After integrating by parts and using the boundary conditions (2.4) we get

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \int_0^1 (1+S)\left(\frac{\partial U}{\partial x}\right)^2 dx = 0.$$

Since $1 + S \geq 1$ we have

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2 \leq 0. \tag{2.6}$$

Using Poincare-Friedrichs inequality we obtain

$$\frac{1}{2}\frac{d}{dt}\|U\|^2 + \|U\|^2 \leq 0. \tag{2.7}$$

Now multiply (2.3) by $\dfrac{\partial^2 U}{\partial x^2}$ and integrate over $(0,1)$. Using again integration by parts and the boundary conditions (2.4) we get

$$\frac{\partial U}{\partial t}\frac{\partial U}{\partial x}\bigg|_0^1 - \int_0^1 \frac{\partial^2 U}{\partial x \partial t}\frac{\partial U}{\partial x}dx = \int_0^1 (1+S)\left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx,$$

$$\frac{1}{2}\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2 + (1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 = 0, \tag{2.8}$$

or

$$\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2 \leq 0. \tag{2.9}$$

From (2.6),(2.7) and (2.9) we find

$$\frac{d}{dt}\left[\exp(t)\left(\|U\|^2 + \left\|\frac{\partial U}{\partial x}\right\|^2\right)\right] \leq 0.$$

This inequality immediately proves Theorem 2.1.

Note that Theorem 2.1 gives exponential stabilization of the solution of the problem (2.1) in the norm of the space $H^1(0,1)$. Let us show that the stabilization is also achieved in the norm of the space $C^1(0,1)$. In particular, let us show that the following statement takes place.

Theorem 2.2. *If $W_0 \in H^4(0,1) \cap H_0^1(0,1)$, then the solution of the problem (2.1) satisfies the following estimates:*

$$\left|\frac{\partial W(x,t)}{\partial x} - \psi\right| \leq C\exp\left(-\frac{\alpha t}{2}\right), \quad \left|\frac{\partial W(x,t)}{\partial t}\right| \leq C\exp\left(-\frac{\beta t}{2}\right),$$

where $\alpha = Const$, $\beta = Const$, $0 < \beta < \alpha < 1$.

To this end we need the following auxiliary result.

Lemma 2.1. *The following estimate holds*

$$\left\|\frac{\partial U}{\partial t}\right\| \leq C\exp\left(-\frac{\alpha t}{2}\right).$$

Proof. Differentiating (2.3) with respect to t we get

$$\frac{\partial^2 U}{\partial t^2} = (1+S)\frac{\partial^3 U}{\partial x^2 \partial t} + \left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right] \frac{\partial^2 U}{\partial x^2}. \tag{2.10}$$

Multiply (2.10) by $\dfrac{\partial U}{\partial t}$ and integrate over $(0,1)$. Using the boundary conditions (2.4) we deduce

$$\begin{gathered}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + 2(1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx = \\ = -2\left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right] \int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t} dx.\end{gathered} \tag{2.11}$$

Let us estimate the right hand side of the equality (2.11).

$$\begin{gathered}-2\left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right] \int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t} dx = \\ = -2\int_0^1 \left\{(1+S)^{-1/2}\left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right] \frac{\partial U}{\partial x}\right\} \times \\ \times \left\{(1+S)^{1/2}\frac{\partial^2 U}{\partial x \partial t}\right\} dx.\end{gathered} \tag{2.12}$$

From this, using the Schwarz's inequality we get

$$-2\left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right] \int_0^1 \frac{\partial U}{\partial x}\frac{\partial^2 U}{\partial x \partial t} dx \le (2-\alpha)(1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx +$$

$$+\frac{1}{2-\alpha}(1+S)^{-1}\left[\int_0^1 \left(\frac{\partial U}{\partial x} + \psi\right)^2 dx\right]^2 \int_0^1 \left(\frac{\partial U}{\partial x}\right)^2 dx \le$$

$$\le (2-\alpha)(1+S)\int_0^1 \left(\frac{\partial^2 U}{\partial x \partial t}\right)^2 dx + \tag{2.13}$$

$$+\frac{8}{2-\alpha}(1+S)^{-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2dx\right]^3+$$

$$+\frac{8\psi^4}{2-\alpha}(1+S)^{-1}\int_0^1\left(\frac{\partial U}{\partial x}\right)^2dx.$$

Combining (2.11)-(2.13) we have

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2dx+\alpha(1+S)\int_0^1\left(\frac{\partial^2 U}{\partial x\partial t}\right)^2dx\leq$$

$$\leq\frac{8}{2-\alpha}(1+S)^{-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2dx\right]^3+\frac{8\psi^4}{2-\alpha}(1+S)^{-1}\int_0^1\left(\frac{\partial U}{\partial x}\right)^2dx.$$

Using Poincare-Friedrichs inequality, notation $U(x,t)=W(x,t)-\psi x$, Theorem 2.1 and nonnegativity of $S(t)$ we arrive at

$$\frac{d}{dt}\int_0^1\left(\frac{\partial U}{\partial t}\right)^2dx+\alpha\int_0^1\left(\frac{\partial U}{\partial t}\right)^2dx\leq C\exp(-t).$$

After multiplying by $\exp(\alpha t)$, the last inequality gives

$$\frac{d}{dt}\left(\exp(\alpha t)\left\|\frac{\partial U}{\partial t}\right\|^2\right)\leq C\exp(-(1-\alpha)t).$$

Therefore,

$$\exp(\alpha t)\left\|\frac{\partial U}{\partial t}\right\|^2\leq C\int_0^t\exp(-(1-\alpha)\tau)d\tau\leq\frac{C}{1-\alpha},$$

i.e.

$$\left\|\frac{\partial U}{\partial t}\right\|\leq C\exp\left(-\frac{\alpha t}{2}\right).$$

So, Lemma 2.1 is proven.

Now, let us estimate $\frac{\partial^2 U}{\partial x^2}$ in the norm of the space $L_1(0,1)$. From (2.3) we have

$$\frac{\partial^2 U}{\partial x^2}=(1+S)^{-1}\frac{\partial U}{\partial t}. \tag{2.14}$$

Integrating on $(0,1)$ and using Schwarz's inequality we get

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx = \int_0^1 \left|(1+S)^{-1}\frac{\partial U}{\partial t}\right| dx \leq \left[\int_0^1 (1+S)^{-2}dx\right]^{1/2} \left[\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx\right]^{1/2}.$$

Applying Lemma 2.1 and taking into account the nonnegativity of $S(t)$ we derive

$$\int_0^1 \left|\frac{\partial^2 U}{\partial x^2}\right| dx \leq C\exp\left(-\frac{\alpha t}{2}\right).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial x} = \int_0^1 \frac{\partial U(y,t)}{\partial y}dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy$$

and the boundary conditions (2.4) it follows that

$$\left|\frac{\partial U(x,t)}{\partial x}\right| = \left|\int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial \xi^2}d\xi dy\right| \leq \int_0^1 \left|\frac{\partial^2 U(y,t)}{\partial y^2}\right| dy \leq C\exp\left(-\frac{\alpha t}{2}\right).$$

So, for the solution of the initial-boundary value problem (2.1) we have

$$\left|\frac{\partial W(x,t)}{\partial x} - \psi\right| \leq C\exp\left(-\frac{\alpha t}{2}\right).$$

Now let us estimate $\frac{\partial U}{\partial t}$ in the norm of the space $C^1(0,1)$. Let us multiply (2.3) by $\frac{\partial^3 U}{\partial x^2 \partial t}$ and integrate over $(0,1)$. Using integration by parts we get

$$\frac{\partial U}{\partial t}\frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = (1+S)\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}dx. \tag{2.15}$$

Taking into account the equality

$$\int_0^1 \frac{\partial^3 U}{\partial x^2 \partial t}\frac{\partial^2 U}{\partial x^2}dx = \frac{1}{2}\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2$$

and the boundary conditions (2.4) we arrive at

$$\frac{1+S}{2}\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 = 0,$$

i.e.

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 \leq 0. \tag{2.16}$$

Note that from (2.15) we have

$$\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 \leq \frac{(1+S)}{2}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2 + \frac{(1+S)}{2}\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2. \tag{2.17}$$

Now multiply (2.10) by $\dfrac{\partial^3 U}{\partial x^2 \partial t}$ scalarly and integrate the left hand side by parts

$$\frac{\partial^2 U}{\partial t^2}\frac{\partial^2 U}{\partial x \partial t}\bigg|_0^1 - \int_0^1 \frac{\partial^3 U}{\partial x \partial t^2}\frac{\partial^2 U}{\partial x \partial t}dx = (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 +$$
$$+\left[\int_0^1\left(\frac{\partial U}{\partial x}+\psi\right)^2 dx\right]\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}dx.$$

Taking into account the boundary conditions (2.4) we have

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + 2(1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 = -2\left[\int_0^1\left(\frac{\partial U}{\partial x}+\psi\right)^2 dx\right]\int_0^1 \frac{\partial^2 U}{\partial x^2}\frac{\partial^3 U}{\partial x^2 \partial t}dx.$$

We estimate the right hand side in a similar fashion to (2.12),(2.13). It is easy to see that

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \leq$$
$$\leq (1+S)^{-1}\left[\int_0^1\left(\frac{\partial U}{\partial x}+\psi\right)^2 dx\right]^2 \int_0^1\left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx \leq$$
$$\leq 8(1+S)^{-1}\left\{\left[\int_0^1\left(\frac{\partial U}{\partial x}\right)^2 dx\right]^2 + \psi^4\right\}\int_0^1\left(\frac{\partial^2 U}{\partial x^2}\right)^2 dx.$$

Using Theorem 2.1, (2.14) and Lemma 2.1 we have

$$\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x \partial t}\right\|^2 + (1+S)\left\|\frac{\partial^3 U}{\partial x^2 \partial t}\right\|^2 \leq C\exp(-\alpha t). \tag{2.18}$$

Combining (2.6)-(2.8), (2.16), (2.17) and (2.18) we get

$$\|U\|^2+\frac{d}{dt}\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2+2(1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+$$
$$+\beta\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+(1+S)\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2\leq$$
$$\leq\frac{\beta}{2}(1+S)\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{\beta}{2}(1+S)\left\|\frac{\partial^3 U}{\partial x^2\partial t}\right\|^2+C\exp(-\alpha t).$$

From this, keeping in mind the nonnegativity of $S(t)$ and inequalities $0<\beta<\alpha<1$, we deduce

$$\beta\|U\|^2+\frac{d}{dt}\|U\|^2+\beta\left\|\frac{\partial U}{\partial x}\right\|^2+\frac{d}{dt}\left\|\frac{\partial U}{\partial x}\right\|^2+\beta\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+$$
$$\beta\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2+\frac{d}{dt}\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\leq C\exp(-\alpha t).$$

After multiplying by the function $\exp(\beta t)$ we get

$$\frac{d}{dt}\left[\exp(\beta t)\left(\|U\|^2+\left\|\frac{\partial U}{\partial x}\right\|^2+\left\|\frac{\partial^2 U}{\partial x^2}\right\|^2+\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\right)\right]\leq C\exp(-(\alpha-\beta)t).$$

Since $\beta<\alpha$ we get

$$\left\|\frac{\partial^2 U}{\partial x\partial t}\right\|^2\leq C\exp(-\beta t).$$

From this, taking into account the relation

$$\frac{\partial U(x,t)}{\partial t}=\int_0^1\frac{\partial U(y,t)}{\partial t}dy+\int_0^1\int_y^x\frac{\partial^2 U(\xi,t)}{\partial t\partial\xi}d\xi dy$$

and Lemma 2.1, we obtain

$$\left|\frac{\partial W(x,t)}{\partial t}\right|=\left|\frac{\partial U(x,t)}{\partial t}\right|\leq C_1\exp\left(-\frac{\alpha t}{2}\right)+C_2\exp\left(-\frac{\beta t}{2}\right)\leq C\exp\left(-\frac{\beta t}{2}\right).$$

Thus Theorem 2.2 has been proven.

3 Numerical implementation remark

In this section we consider numerical approximation of problem (2.1). Let us investigate problem (2.3)-(2.5) which is the equivalent to problem (2.1). In order to describe the finite difference method for this problem, we introduce a net whose mesh points are denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M;\ j=0,1,...,N$, with $h=\frac{1}{M}$, $\tau=\frac{T}{N}$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j and the exact solution to the problem (2.3)-(2.5) by U_i^j. We will use the following known notations:

$$u_{x,i}^{j+1} = \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h}, \quad u_{\bar{x},i}^{j+1} = \frac{u_i^{j+1} - u_{i-1}^{j+1}}{h}.$$
$$u_{t,i}^{j} = \frac{u_i^{j+1} - u_i^{j}}{\tau}, \quad u_{\bar{t},i}^{j} = u_{t,i}^{j-1} = \frac{u_i^{j} - u_i^{j-1}}{\tau}.$$

Let us correspond to the problem (2.3)-(2.5) the following difference scheme:

$$\frac{u_i^{j+1} - u_i^j}{\tau} - \left[1 + \tau h \sum_{l=1}^{M} \sum_{k=1}^{j+1} (u_{\bar{x},l}^k)^2\right] u_{\bar{x}x,i}^{j+1} = f_i^j, \qquad (3.1)$$
$$i = 1, 2, ..., M-1; \;\; j = 0, 1, ..., N-1,$$

$$u_0^j = u_M^j = 0, \quad j = 0, 1, ..., N, \qquad (3.2)$$

$$u_i^0 = U_{0,i}, \quad i = 0, 1, ..., M, \qquad (3.3)$$

where $f_i^j = f(x_i, t_j)$ is a known function. For the problem (2.3)-(2.5) $f_i^j \equiv 0$.

The following statement of convergence takes place [10].

Theorem 3.1. *If problem (2.3)-(2.5) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u^j = (u_1^j, u_2^j, \dots, u_{M-1}^j)$, $j = 1, 2, \dots, N$, of the finite difference scheme (3.1) tends to the $U^j = (U_1^j, U_2^j, \dots, U_{M-1}^j)$ for $j = 1, 2, \dots, N$, as $\tau \to 0$, $h \to 0$ and the following estimate is true*

$$\|u^j - U^j\|_h \le C(\tau + h), \quad j = 1, 2, \dots, N.$$

We now comment on the numerical implementation of the discrete problem (3.1)-(3.3). Note that (3.1) can be rewritten as:

$$\frac{1}{\tau} u_i^{j+1} - A\left(\mathbf{u}^{j+1}\right) \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2} - f_i^j - \frac{1}{\tau} u_i^j = 0,$$
$$i = 1, \dots, M-1.$$

where

$$A\left(\mathbf{u}^{j+1}\right) = 1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \left(\frac{u_\ell^k - u_{\ell-1}^k}{h}\right)^2.$$

This system can be written in matrix form

$$\mathbf{H}\left(\mathbf{u}^{j+1}\right) \equiv \mathbf{G}\left(\mathbf{u}^{j+1}\right) - \frac{1}{\tau}\mathbf{u}^j - \mathbf{f}^j = 0.$$

The vector $\mathbf{u}$ containing all the unknowns $u_1, \dots, u_{M-1}$ at the level indicated. The vector $\mathbf{G}$ is given by

$$\mathbf{G}\left(\mathbf{u}^{j+1}\right) = \mathbf{T}\left(\mathbf{u}^{j+1}\right) \mathbf{u}^{j+1},$$

where the matrix $\mathbf{T}$ is symmetric and tridiagonal with elements

$$\mathbf{T}_{ir} = \begin{cases} \dfrac{1}{\tau} + 2\dfrac{A}{h^2}, & r = i, \\ \\ -\dfrac{A}{h^2}, & r = i \pm 1. \end{cases}$$

Newton's method for the system is given by

$$\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(n)} \left(\mathbf{u}^{j+1}\Big|^{(n+1)} - \mathbf{u}^{j+1}\Big|^{(n)}\right) = -\mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|^{(n)}.$$

The elements of the matrix $\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)$ require the derivative of A. The elements are:

$$\nabla \mathbf{H}\left(\mathbf{u}^{j+1}\right)\Big|_{i\,r} = \begin{cases} \dfrac{1}{\tau} + \dfrac{2}{h^2} A\left(\mathbf{u}^{j+1}\right) - \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_i^{j+1}} \delta_i^{j+1}, & r = i, \\ \\ -\delta_i^{j+1} \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_r^{j+1}} - \dfrac{1}{h^2} A\left(\mathbf{u}^{j+1}\right), & r = i \pm 1, \\ \\ -\delta_i^{j+1} \dfrac{\partial A\left(\mathbf{u}^{j+1}\right)}{\partial u_r^{j+1}}, & \text{otherwise}, \end{cases}$$

where

$$\delta_i^{j+1} = \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{h^2}.$$

To evaluate the partial derivatives, we use

$$\begin{aligned} \frac{\partial A}{\partial u_r^{j+1}} &= \frac{\partial}{\partial u_r^{j+1}} \left[1 + \tau h \sum_{\ell=1}^{M} \sum_{k=1}^{j+1} \left(\frac{u_\ell^k - u_{\ell-1}^k}{h} \right)^2 \right] = \\ &= \frac{\partial}{\partial u_r^{j+1}} \left[R + \tau h \left(\frac{u_r^{j+1} - u_{r-1}^{j+1}}{h} \right)^2 + \tau h \left(\frac{u_{r+1}^{j+1} - u_r^{j+1}}{h} \right)^2 \right] = \\ &= 2\tau h \frac{u_r^{j+1} - u_{r-1}^{j+1}}{h} \cdot \frac{1}{h} + 2\tau h \frac{u_{r+1}^{j+1} - u_r^{j+1}}{h} \cdot \left(-\frac{1}{h} \right) = \\ &= -2\tau h \frac{u_{r+1}^{j+1} - 2u_r^{j+1} + u_{r-1}^{j+1}}{h^2}. \end{aligned}$$

Note that we incorporated into the constant R all the terms that are independent of u_r^{j+1}.

So, we have the nonlinear system of equations $H_i\left(u_1^{j+1}, \dots, u_{M-1}^{j+1}\right) = 0$, $i = 1, 2, \dots, M-1$.

As it is known [12], if H_i are three times continuously differentiable in a region containing the solution $\xi_1, \dots, \xi_{M-1}$ and the Jacobian does not vanish in that region, then Newton's method converges at least quadratically.

The Jacobian is the matrix ∇H computed above. The term $\dfrac{1}{\tau}$ on diagonal ensures that the Jacobian doesn't vanish. The differentiability is guaranteed, since ∇H is quadratic. Therefore, Newton's method, for our problem converges at least quadratically.

In our first numerical experiment we have chosen the right hand side of equation (3.1) so that the exact solution of problem (2.1) is given by

$$W(x,t) = x(1-x)\cos t,$$

which satisfy homogeneous boundary conditions ($\psi = 0$).

The parameters used are $M = 100$ which dictates $h = 0.01$. Since the method is implicit we can use $\tau = h$ and we took 100 time steps. In the Figure 1 we plotted the numerical solution and the exact solutions at $t = 0.5$ and $t = 1.0$ (Fig. 1). As it is visible from these pictures, the numerical and exact solutions are almost identical.

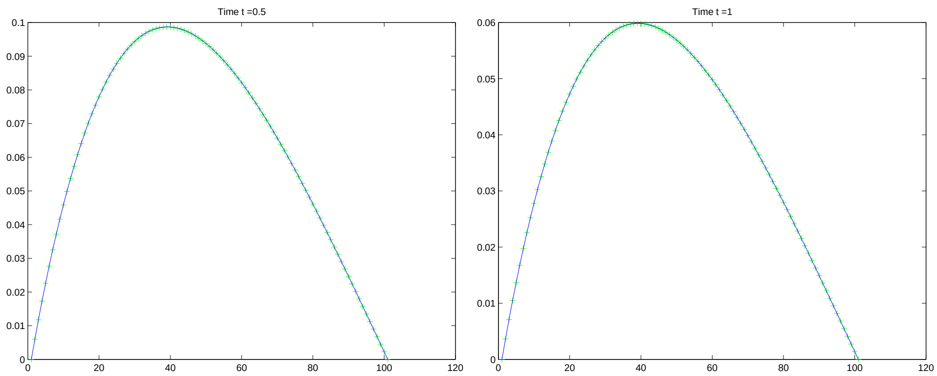

Figure 1: The solution at $t = 0.5$ and $t = 1$. The exact solution is solid line and the numerical solution is marked by $+$.

In the second experiment we have taken zero right hand side and initial data given by

$$W_0(x) = x(1-x)\cos(4\pi x).$$

The parameters M, h, τ are as before. In Figure 2, we plotted the initial data and the numerical solution at four different times. It is clear that the numerical solution is approaching zero for all x.

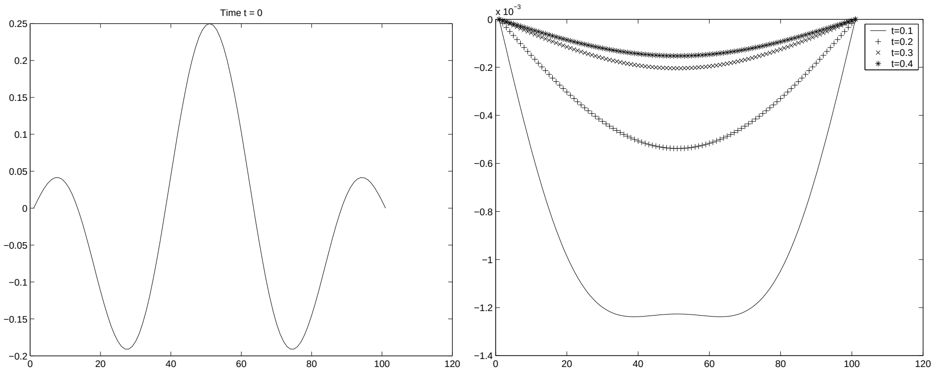

Figure 2: The initial data and the numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for homogeneous boundary conditions.

The numerical experiments for problem with nonhomogeneous boundary condition on part of lateral boundary was carried out as well. For our next experiment we have taken zero right hand side and initial data given by

$$W_0(x) = x(1-x)\cos(4\pi x) + 0.001x.$$

In this case, we know (Theorem 2.2) that the solution will approach to the steady-state solution, which in this case is $W(x) = 0.001x$. The parameters M, h, τ are as before. In Figure 3, we plotted the initial data and the numerical solution at four different times. It is clear that the numerical solution is approaching steady-state solution for all x in this case too.

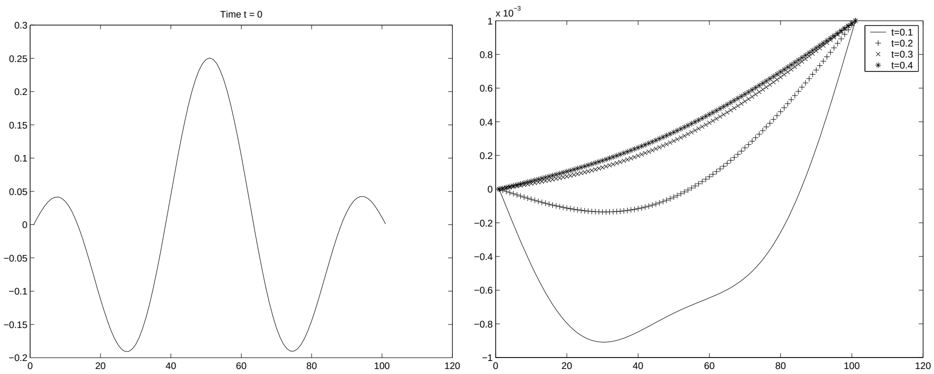

Figure 3: The initial data and the numerical solution at $t = 0.1, 0.2, 0.3, 0.4$ for nonhomogeneous boundary condition on part of lateral boundary.

We have experimented with several other initial data for both inialboundary value problem (2.1). In all cases we noticed that numerical solutions are approaching steady-state solution as it is shown in theoretical researches.

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