GENERAL REPRESENTATION OF SOLUTIONS OF THE EQUATION OF LINEAR MULTIVELOCITY NEUTRON TRANSPORT THEORY

D. Shulaia

I. Vekua Institute of Applied Mathematics,
Iv. Javakhishvili Tbilisi State University
0143 University Street 2, Tbilisi, Georgia
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Abstract

The paper presents the general procedure of solving of the equations of linear multivelocity neutron transport theory in plane geometry. Elementary solutions are found and then it is proved that the general solutions can be formed by the superposition of elementary solutions. As an application the Green's function for a uniform infinite medium is constructed.

Key words and phrases: Transport equation, elementary solutions, complete set.
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1 Introduction

We wish to present here a method for solving the transport equations describing multivelocity neutron diffusion. The motivation is manifold (see [2]). First, conventional methods of treatment, such as converting to integral equations, become extremely complicated for anything but a single uniform medium. Moreover, the solutions obtained for the integral equations are usually expressed as contour integrals. These are put in a tractable form for numerical transformation only after many transformation. It is desirable to find the transformed forms directly. Secondly, it may be hoped that an alternate approach will throw light on the general subject and suggest new methods for corresponded of inverse and nonlinear problems. Thirdly, the usual methods of obtaining rigorous solutions of particular problems are quite varied and seem to have no common bases, especially to elementary approach familiar in the treatment of partial differential equations in which variables are separated and solutions expanded in normal modes seems lacking.

The last remark contains the essence of the method to be discussed. It is suggested by Van Kampen's work [1] on the related problems of plasma oscillations. In 1955 N.G. Van Kampen made two important points. First he noted that in problems involving the transport equation it is sufficient that admissible solutions be distributions in the sense of Schwartz. Secondly he has shown that for the particular problem of a plasma these eigen-distributions are complete. In 1960 K.M.Case showed that similar (but more comprehensive) completeness properties hold for various one velocity neutron diffusion problems. In addition an orthogonality property is found which simplifies the expansion [2].In 1973 starting from Van Kampen's and K.Case's observations that it is sufficient that "solutions" be distributions, there were found by us the elementary solutions of the homogeneous equation of multivelocity theory [4].

Attention here is restricted to the linear boundary value problems of multivelocity neutron diffusion in plane geometry. In Section 2 solutions of homogeneous form of Equations are discussed. As an application the Green's function for a uniform infinite medium is constructed in Section 3. For the reader's convenience our paper is written accordingly to the paper of K.Case [2].

2 Elementary Solutions of the Homogeneous Equation

We consider the problems of multivelocity neutron diffusion. We look for solutions of the homogeneous equation there. The idea (as in approach of Case) is to construct special solutions appropriate to various boundary conditions in terms of superpositions of the elementary solutions.

For illustration, we will restrict to the simplest form of multivelocity transport equation

$$\mu\frac{\partial\Psi}{\partial x}+\Psi=\int_{E_1}^{E_2}\int_{-1}^{+1}K(E,E')\Psi(x,\mu',E')d\mu'dE', \tag{1}$$

$$x\in(-\infty,+\infty),\ \mu\in(-1,+1),\ E\in[E_1,E_2],$$

where K is the real valued continuous, symmetric function.

Translational invariance suggests trying

$$\Psi(x,\mu,E)=\exp(-x/\nu)\phi_\nu(\mu,E)$$

here ν is a parameter. With this assumption, Eq.(1) becomes

$$(\nu-\mu)\phi_\nu(\mu,E)=\nu\int_{E_1}^{E_2}\int_{-1}^{+1}K\phi_\nu d\mu'dE'. \tag{2}$$

It is seen that when $\nu\in[-1,1]$ this equation admits continuous solution only the zero.

Let us denote

$$M(E)=\int_{E_1}^{E_2}\int_{-1}^{+1}K\phi_\nu d\mu' dE',$$

then when $\nu\notin[-1,1]$ Eq.(2) leads to

$$M(E)=\lambda\int_{E_1}^{E_2}K(E,E')M(E')dE', \tag{3}$$

where

$$\lambda=\nu\ln\frac{\nu+1}{\nu-1}\equiv\rho(\nu).$$

From this point the conventional argument runs as follows: Solving Eq.(2) gives

$$\phi_\nu(\mu,E)=\frac{\nu M(E)}{\nu-\mu}$$

but ν is defined from $\rho(\nu)=\lambda_i$ where λ_i is the eigenvalue of (3). The pair roots $\pm\nu_i$ occur. The argument has given the usual solutions of the original homogeneous transport equation

$$\Psi(x,\mu,E)=\sum_k a_{\pm\nu_k}\phi_{\pm\nu_k}(\mu,E)\exp(\mp x/\nu_k),$$

(Here $a_{\pm\nu_k}$ are constants.) However, there are other solutions of Eq.(1). We had found the class of continuous solutions which have the form

$$\Psi(x,\mu,E)=\int_{E_1}^{E_2}\int_{-1}^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu)d\nu d\zeta$$

where

$$\phi_{\nu,(\zeta)}(\mu,E)=\mathbf{P}\frac{\nu K(E,\zeta)}{\nu-\mu}+\big(\delta(\zeta-E)-\int_{-1}^{+1}\frac{\nu K(E,\zeta)}{\nu-\mu'}d\mu'\big)\delta(\nu-\mu), \tag{4}$$

$$\nu\in(-1,+1),\quad \zeta\in[E_1,E_2],$$

is the solutions of Eq.(2) in Van Kampen sense. (Here A is the arbitrary continuous function satisfying the certain conditions, $\mathbf{P}$ indicates that principal values is to be understood, and δ is the distribution (Dirac function). Note that

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\phi_{\nu,(\zeta)}(\mu,E)d\mu dE=1. \tag{5}$$

To summarize: There are for (2) discrete solutions and a continuum of solutions given by Eq.(4).

The usefulness of these functions arises from the facts that they are both orthogonal and complete. This can be stated in the form of theorems.

Theorem 1:

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_\nu\phi_{\nu'}d\mu dE=0,\quad \nu\neq\nu', \tag{6}$$

moreover

$$N_{\nu_k}=\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{\nu_k}^2 d\mu dE\neq 0. \tag{7}$$

Let **D** be the class of continuous in domain $(-1,+1)\times[E_1,E_2]$ functions $\psi(\mu,E)$ satisfying the H^* conditions with respect to μ (Muskhelishvili class)[3].

Theorem 2: *The set of functions* $\{\phi_{\pm\nu_k},\quad \phi_{\nu,(\zeta)}\}$ *is complete for functions* $\psi\in\mathbf{D}$.

It is to be shown that one can express ψ in the form

$$\psi=\sum_k a_{\pm\nu_k}\phi_{\pm\nu_k}+\int_{E_1}^{E_2}\int_{-1}^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}d\nu d\zeta. \tag{8}$$

If the expansion is possible, the coefficients in discrete term are readily found using Theorem 1. In particular, it follows from Eqs. (6) and (7) that

$$a_{\pm\nu_k}=\frac{1}{N_{\pm\nu_k}}\int_{E_1}^{E_2}\int_{-1}^{+1}\mu\phi_{\pm\nu_k}\psi d\mu dE. \tag{9}$$

Hence it is sufficient to show that given any ψ the function

$$\psi'=\psi-\sum_k a_{\pm\nu_k}\phi_{\pm\nu_k} \tag{10}$$

(with $a_{\pm\nu_k}$ give by (9)) can be written as

$$\psi'=\int_{E_1}^{E_2}\int_{-1}^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}d\nu d\zeta.$$

With (4) this becomes

$$\psi'(\mu,E)=A(\mu,E)-\int_{E_1}^{E_2}\int_{-1}^{+1}\frac{\mu K(E,\zeta)}{\mu-\mu'}d\mu' A(\mu,\zeta)d\zeta$$

$$+\int_{E_1}^{E_2}\int_{-1}^{+1}\frac{\nu K(E,\zeta)}{\nu-\mu}A(\nu,\zeta)d\nu d\zeta. \tag{11}$$

To prove completeness the existence of a solution of this singular integral equation must be demonstrated. Such problems have been treated by us [5-6]. The essential point is to relate the functions which occur to the boundary values of functions of a complex variable. The properties which can be ascribed to these functions serve to determine them completely.

The basic result for Eq.(11) is the following:

Theorem 3. *The equation (11) is solvable if and only if $\psi' \in \mathbf{D}$ satisfies the conditions*

$$\int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_{\pm\nu_i}\psi' d\mu dE = 0. \tag{12}$$

Provided these conditions are satisfied, the equation (11) has one and only one solution $A \in \mathbf{D}$.

From (10) it follows that

$$\int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_{\pm\nu_i}\psi d\mu dE$$

$$-\sum_k \frac{1}{N_{\pm\nu_k}}\int_{E_1}^{E_2}\int_{-1}^{+1} \mu'\phi_{\pm\nu_k}\psi d\mu' dE' \int_{E_1}^{E_2}\int_{-1}^{+1} \mu\phi_{\pm\nu_i}\phi_{\pm\nu_k} d\mu dE = 0.$$

(Here the orthogonality and normalization properties of the ϕ_{ν_k} have been used.) Thus, the question of Eq. (12) is answered affirmatively.

For determination of the factor for function of the continuum spectrum it will be necessary to do more drudgery, since functions of the continuum spectrum are not a function with integrable square and not orthogonal whit respect to ζ.

Assume

$$S(\nu,\zeta_0,\zeta) = 2\int_{-1}^{+1} \frac{\nu K(\zeta_0,\zeta)}{\nu-\mu} d\mu - \pi^2\nu^2 \int_{E_1}^{E_2} K(\zeta_0,E)K(\zeta,E)dE$$

$$-\int_{E_1}^{E_2}\int_{-1}^{+1} \frac{\nu K(\zeta_0,E)}{\nu-\mu} d\mu \int_{-1}^{+1} \frac{\nu K(\zeta,E)}{\nu-\mu} d\mu dE$$

$$\nu \in (-1,1) \quad \zeta_0,\zeta \in [E_1,E_2].$$

We can prove that

Lemma: *The regular second kind integral equation*

$$p(\nu,\zeta_0,\zeta) - \int_{E_1}^{E_2} S(\nu,\zeta',\zeta)p(\nu,\zeta_0,\zeta')d\zeta' = S(\nu,\zeta_0,\zeta)$$

for any $\nu \in (-1,1)$ and $\zeta_0 \in [E_1,E_2]$ has unique continuous solution.

Denote by

$$\tilde{\phi}_{\nu,(\zeta)}(\mu, E) = \phi_{\nu,(\zeta)}(\mu, E) + \int_{E_1}^{E_2} p(\nu, \zeta, \zeta')\phi_{\nu,(\zeta')}d\zeta'.$$

Now multiply both sides of (8) by $\mu\tilde{\phi}_{\nu,(\zeta)}(\mu, E)$ and integrate over μ and E, then using well known Bertrand-Poincaré formula and taking into account Lemma and also the property of orthogonality, we obtain

$$A(\nu, \zeta) = \int_{E_1}^{E_2} \int_{-1}^{+1} \mu\tilde{\phi}_{\nu,(\zeta)}(\mu, E)\psi(\mu, E)d\mu dE.$$

Based on Theorem 2, the following theorem is proved

Theorem 4: *Every differentiable with respect to x continuous solution Ψ of (1) can be represent in the form*

$$\Psi(x, \mu, E) = \sum_k c_{\pm\nu_k}\phi_{\pm\nu_k} \exp(-x/\nu_k) + \int_{E_1}^{E_2} \int_{-1}^{+1} C(\nu, \zeta)\phi_{\nu,(\zeta)} \exp(-x/\nu)d\nu d\zeta,$$

where $c_{\pm\nu_k}$ are constants and the arbitrary functions $c(\nu, \zeta) \in D$.

3 Green's Function for a Uniform Infinite Medium

As an illustration of the applicability of the results of the preceding section, the Green's function for the transport equation will be constructed. The Green's function Ψ_g satisfies the equation

$$\mu\frac{\partial\Psi_g}{\partial x} + \Psi_g = \int_{E_1}^{E_2} \int_{-1}^{+1} K(E, E')\Psi_g(x, \mu', E')d\mu' dE' + \frac{1}{4\pi}\delta(x)\delta(\mu - \mu_0)\delta(E - E_0). \tag{13}$$

$$\mu, \mu_0 \in (-1, 1), \quad E, E_0 \in [E_1, E].$$

Integrating across the plane x shows that Ψ_g satisfies the homogeneous equation for $x \neq 0$ and the jump condition

$$\mu(\Psi_g(0^+, \mu, E) - \Psi_g(0^-, \mu, E)) = \frac{1}{4\pi}\delta(\mu - \mu_0)\delta(E - E_0). \tag{14}$$

Let us look for the solution Ψ_g which vanishes as $\mid x \mid \to \infty$. It is sufficient to expand Ψ_g in the form

$$\Psi_g(x, \mu, E) = \sum_k a_{+\nu_k} \exp(-x/+\nu_k)\phi_{+\nu_k}(\mu, E)$$

$$+ \int_{E_1}^{E_2} \int_0^{+1} A(\nu, \zeta)\phi_{\nu,(\zeta)}(\mu, E) \exp(-x/\nu)d\nu dE, \qquad x > 0, \tag{15}$$

or

$$\Psi_g(x,\mu,E) = -\sum_k a_{-\nu_k}\exp(-x/-\nu_k)\phi_{-\nu_k}(\mu,E)$$

$$-\int_{E_1}^{E_2}\int_{-1}^{0} A(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu)d\nu dE, \qquad x<0. \tag{16}$$

Condition (14) then gives an integral equation to determine the expansion coefficients. It is

$$\frac{1}{4\pi}\delta(\mu-\mu_0)\delta(E-E_0) = \mu\sum_k a_{\pm\nu_k}\phi_{\pm\nu_k} + \mu\int_{E_1}^{E_2}\int_{-1}^{+1} A(\nu,\zeta)\phi_{\nu,(\zeta)}d\nu d\zeta. \tag{17}$$

The solution obtained using the orthogonality relations is

$$a_{\pm\nu_k} = \frac{1}{N_{\pm\nu_k}}\int_{E_1}^{E_2}\int_{-1}^{+1}\frac{\phi_{\pm\nu_k}\delta(\mu-\mu_0)\delta(E-E_0)d\mu dE}{4\pi} = \frac{1}{4\pi}\frac{\phi_{\pm\nu_k}(\mu_0,E_0)}{N_{\pm\nu_k}}$$

and

$$A(\nu,\zeta) = \frac{1}{4\pi}\tilde{\phi}_{\nu,(\zeta)}(\mu_0,E_0).$$

Hence Ψ_g can be written in the typical normal mode expansion

$$\Psi_g = \frac{1}{4\pi}\sum_k \frac{\phi_{+\nu_k}(\mu_0,E_0)\phi_{+\nu_k}(\mu,E)\exp(-x/+\nu_k)}{N_{+\nu_k}}$$

$$+\frac{1}{4\pi}\int_{E_1}^{E_2}\int_{0}^{+1}\tilde{\phi}_{\nu,(\zeta)}(\mu_0,E_0)\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu)d\nu d\zeta. \quad (x>0)$$

$$= -\frac{1}{4\pi}\sum_k \frac{\phi_{-\nu_k}(\mu_0,E_0)\phi_{-\nu_k}(\mu,E)\exp(x/+\nu_k)}{N_{-\nu_k}}$$

$$-\frac{1}{4\pi}\int_{E_1}^{E_2}\int_{-1}^{0}\tilde{\phi}_{\nu,(\zeta)}(\mu_0,E_0)\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu)d\nu d\zeta. \quad (x<0).$$

For angular and energetic density Ψ_g we have

$$\Psi_o(x,\mu,E) = \int_{E_1}^{E_2}\int_{-1}^{1}\Psi_g(x,\mu,E,\mu_0,E_0)d\mu_0 dE_0 \tag{18}$$

$$= \frac{1}{4\pi}\sum_k \frac{\phi_{+\nu_k}(\mu,E)\exp(-\mid x\mid/+\nu_k)}{N_{+\nu_k}}$$

$$+\frac{1}{4\pi}\int_{E_1}^{E_2}\int_{0}^{+1}(1+\int_{E_1}^{E_2}p(\nu,\zeta,\zeta')d\zeta')\phi_{\nu,(\zeta)}(\mu,E)\exp(-\mid x\mid/\nu)d\nu d\zeta,$$

where the normalization of Eq.(5) has been used. To find the neutron density it is only necessary to integrate (18) over all μ and E. This yields

$$\rho_0(x) = \frac{1}{4\pi}\sum_k \frac{\phi_{+\nu_k}(\mu, E)\exp(-\mid x \mid / + \nu_k)}{N_{+\nu_k}}$$

$$+\frac{1}{4\pi}\int_{E_1}^{E_2}\int_0^{+1}(1+\int_{E_1}^{E_2} p(\nu,\zeta,\zeta')d\zeta')\exp(-\mid x \mid /\nu)d\nu d\zeta.$$

It is worth noting that the method is well suited to finding the asymptotic behavior of solutions.

4 Partial Range Completeness

The elementary solutions found in Section 2 have a much more general completeness property than is indicated by Theorem 2. But to this end beforehand must be investigate the inhomogeneous equation corresponding to Eqs.(2)

$$(\omega - \mu)\varphi_\omega = \omega\int_{E_1}^{E_2}\int_{-1}^{+1} K\varphi_\omega d\mu' dE' + f. \tag{19}$$

We are able to prove the results concerning for questions of solvability and solution of this equation. Namely

Theorem 5: *If* $\omega \notin [-1, 1]$ *then Eq.* (19) *has a unique solution* $\varphi \in \mathbf{D}$ *for any* $f \in \mathbf{D}$. *The solution of this equation is given by*

$$\varphi_\omega = \sum_k c_{\pm\nu_k}\phi_{\pm\nu_k} + \int_{E_1}^{E_2}\int_{-1}^{+1} C(\nu,\zeta)\phi_{\nu,(\zeta)}d\nu d\zeta, \tag{20}$$

where

$$c_{\pm\nu_k} = \frac{\pm\nu_k}{\pm\nu_k - \omega}\frac{1}{N_{\pm\nu_k}}\int_{E_1}^{E_2}\int_{-1}^{1}\phi_{\pm\nu_k} f d\mu dE,$$

$$C(\nu,\zeta) = \frac{\nu}{\omega-\nu}\int_{E_1}^{E_2}\int_{-1}^{+1}\tilde{\phi}_{\nu,(\zeta)} f d\mu dE$$

Theorem 6: *Let* $\omega = \nu_{k_0}$ *be an eigenvalue. Then Eq.(19) is solvable, if and only if the function* f *satisfies the conditions*

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\phi_{\nu_{k_0}} f d\mu dE = 0.$$

Provided these conditions are satisfied, then solutions of (19) may be written as

$$\varphi_\omega = c\phi_{\nu_{k_0}} + \sum_{k\neq k_0} c_{\pm\nu_k}\phi_{\pm\nu_k} + \int_{E_1}^{E_2}\int_{-1}^{+1} C(\nu,\zeta)\phi_{\nu,(\zeta)}d\nu d\zeta,$$

where c is arbitrary constant.

Theorem 7: *If $\omega = \omega_0 \in (-1, 1)$ Eq.(19) is solvable, if and only if the function f satisfies the conditions*

$$\int_{E_1}^{E_2} \int_{-1}^{+1} \tilde{\phi}_{\omega_0,(\zeta)} f d\mu dE = 0$$

for every $\zeta \in [E_1, E_2]$. Provided these conditions are satisfied, Eq.(19) has one and only one solution and this solution may be given by (20).

Several results on completeness can be derived from the preceding theorems.

Let $\mathbf{D}_+$ be the class of continuous in domain $(0, +1) \times [E_1, E_2]$ functions $\psi(\mu, E)$ satisfying the H^* conditions with respect to μ.

Theorem 7: *The set of functions $\{\phi_{+\nu_k}, \phi_{\nu,(\zeta)}\}$ where $\nu \in (0, 1)$, $\zeta \in [E_1, E_2]$ is complete for functions $\psi \in \mathbf{D}_+$.*

This theorem means that the expansion

$$\psi = \sum_k a_{+\nu_k} \phi_{+\nu_k} + \int_{E_1}^{E_2} \int_0^{+1} A(\nu, \zeta) \phi_{\nu,(\zeta)} d\nu d\zeta \tag{21}$$

is possible. The coefficients can be also found from the certain regular integral equation of the second kind .

5 Applications of the Half-Range Completeness Relations

The first of our applications is to the albedo problem. A plane parallel beam is incident at $x = 0$ on the half space $0 \leq x < \infty$. The problem is to find a solution Ψ_a of Eq.(1) in this region subject to the conditions:

$$(a) \quad \Psi_a(0, \mu, E) = \delta(\mu - \mu_0)\delta(E - E_0) \quad \mu, \mu_0 > 0, \quad E, E_0 \in [E_1, E_2];$$

$$(b) \qquad \lim_{|x| \to \infty} \Psi_a(x, \mu, E) = 0.$$

A general solution of (1) subject to condition (b) is

$$\Psi_a(x, \mu, E) = \sum_k a_{+\nu_k} \phi_{+\nu_k}(\mu, E) \exp(-x/ + \nu_k)$$

$$+ \int_{E_1}^{E_2} \int_0^{+1} A(\nu, \zeta) \phi_{\nu,(\zeta)}(\mu, E) \exp(-x/\nu) d\nu d\zeta.$$

Condition (a) gives the equation

$$\delta(\mu-\mu_0)\delta(E-E_0)=\sum_k a_{+\nu_k}\phi_{+\nu_k}(\mu,E)+\int_{E_1}^{E_2}\int_0^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)d\nu d\zeta,$$

$$\mu>0,\quad E\in[E_1,E_2].$$

The coefficients $a_{+\nu_k}$ and $A(\nu,\zeta)$ is given by the completeness proof above.

Secondly, we consider the Milne problem. A solution Ψ_o of Eq.(1) is required in the region $0\le x<\infty$ subject to the conditions:

$(a)\quad \Psi_o(0,\mu,E)=0,\quad \mu\ge 0,\quad E\in[E_1,E_2];$
$(b)\quad \lim_{x\to\infty}(\Psi_o(x,\mu,E)-\phi_{-\nu_{k_0}}\exp(-x/-\nu_{k_0}))=0.$

The general solution of Eq.(1) subject to condition (b) can be written as

$$\Psi_o(x,\mu,E)=\phi_{-\nu_{k_0}}(\mu,E)\exp(-x/-\nu_{k_0})+\sum_k a_{+\nu_k}\phi_{+\nu_k}(\mu,E)\exp(-x/+\nu_k)$$

$$+\int_{E_1}^{E_2}\int_0^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu)d\nu d\zeta.$$

Condition (a) then requires that

$$-\phi_{-\nu_{k_0}}(\mu,E)=\sum_k a_{+\nu_k}\phi_{+\nu_k}(\mu,E)+\int_{E_1}^{E_2}\int_0^{+1}A(\nu,\zeta)\phi_{\nu,(\zeta)}(\mu,E)d\nu d\zeta.$$

Again the solution has been found above. All that is needed is to put $\psi(\mu,E)=-\phi_{-\nu_{k_0}}(\mu,E)$ in the formulas for definitions of coefficients.

A generalization of the Milne problem suggests itself. We ask for a solution $\Psi_\nu(x,\mu,E)$ of Eq.(1) subject to condition (a) and

$$(b')\quad \lim_{x\to\infty}(\Psi_\nu(x,\mu,E)-\phi_{\nu,(\zeta)}(\mu,E)\exp(-x/\nu))=0\quad(-1\le\nu\le 0).$$

The solution of this Milne problem is obviously obtained just as that of the original one. Thus, instead of a single solution of the half-space problem with zero incoming flux, we have one discrete solution plus a continuum of solutions. This set of solutions determines the Green's function for a half space. Thus suppose it is required to solve the equation

$$\mu\frac{\partial\Phi_g}{\partial x}+\Phi_g=\int_{E_1}^{E_2}\int_{-1}^{+1}K\Phi_g d\mu' dE'+\frac{1}{4\pi}\delta(x-x_0)\delta(\mu-\mu_0)\delta(E-E_0)\quad(22)$$

in the region $0\le x<\infty$ subject to the boundary conditions

$$\Phi_g(0,\mu,E)=0,\quad \mu\ge 0,\quad E\in[E_1,E_2],\qquad(23)$$

and

$$\lim_{x\to\infty} \Phi_g(x,\mu,E) = 0. \tag{24}$$

We write down the solution and then check the properties.

$$\Phi_g(x,\mu,E) = \Psi_g(x - x_0,\mu,E)$$

$$-\frac{1}{4\pi}\sum_k \frac{\phi_{-\nu_k}(\mu_0,E_0)}{N_{-\nu_k}}(\Psi_{-\nu_k}(x,\mu,E) - \phi_{-\nu_k}(\mu,E)\exp(-x/-\nu_k))\exp(x_0/-\nu_k)$$

$$-\frac{1}{4\pi}\int_{E_1}^{E_2}\int_{-1}^{0}\tilde{\phi}_{\nu,(\zeta)}(\mu_0,E_0)(\Psi_{\nu,(\zeta)}(x,\mu,E)$$

$$-\phi_{-\nu}(\mu,E)\exp(-x/\nu))\exp(x_0/\nu)d\nu d\zeta. \tag{25}$$

Ψ_g is the infinite medium Green's function given by Eq.(15,16) and hence satisfies Eq.(22). The remainder is a solution of the homogeneous equation. Hence Eq.(22) is satisfied. Condition (24) is fulfilled since

$$\Psi_{-\nu_k}(x,\mu,E) - \phi_{-\nu_k}(\mu,E)\exp(-x/(-\nu_k))$$

and

$$\Psi_{\nu,(\zeta)}(x,\mu,E) - \phi_{-\nu}(\mu,E)\exp(-x/\nu)$$

are combinations of decreasing exponentials. Finally, at $x = 0$ the expression becomes

$$\Phi_g(0,\mu,E) = -\frac{1}{4\pi}\sum_k \frac{\phi_{-\nu_k}(\mu_0,E_0)}{N_{-\nu_k}}\Psi_{-\nu_k}(0,\mu,E)\exp(x_0/(-\nu_k))$$

$$-\frac{1}{4\pi}\int_{E_1}^{E_2}\int_{-1}^{0}\tilde{\phi}_{\nu,(\zeta)}(\mu_0,E_0)\Psi_{\nu,(\zeta)}(0,\mu,E)\exp(x_0/\nu)d\nu d\zeta,$$

and each term vanishes for $\mu \geq 0, E \in [E_1,E_2]$.

6 Time-Dependent Problems

The method is readily generalizable to time-dependent problems. Consider the simplest time-dependent homogeneous multi-velocity transport equation

$$\frac{\partial\Psi}{\partial t} + \mu\frac{\partial\Psi}{\partial x} + \Psi = \int_{E_1}^{E_2}\int_{-1}^{+1} K\Psi d\mu' dE'. \tag{26}$$

Look for solutions of the form

$$\Psi(t,x,\mu,E) = \exp(\mathbf{i}kx)\exp(-(1+\mathbf{i}\alpha k)t)\phi_{\alpha,k}(\mu,E). \tag{27}$$

(k is any fixed real number). The permissible values of α are to be determined.) With this assumption Eq.(26) becomes

$$(\alpha - \mu)\phi_{\alpha,k}(\mu, E) = \frac{\mathbf{i}}{k}\int_{E_1}^{E_2}\int_{-1}^{+1} K(E, E')\phi_{\alpha,k}(\mu', E')d\mu' dE'.$$

This practically same equation that was received for ϕ_ν in stationary case. Thus, results of the previous chapters possible carry in this case practically without change. Since a generalization of this eigenvalue problem has been discussed elsewhere, it is sufficient to state the conclusions.

For each k there is a continuum of solutions $\phi_{\alpha,(\zeta),k}(\mu, E)$ with $-1 \le \alpha \le 1$. These are

$$\phi_{\alpha,(\zeta),k}(\mu, E) = \frac{\mathbf{i}}{k}\mathbf{P}\frac{K(E,\zeta)}{\alpha - \mu}$$

$$+(\delta(\zeta - E) - \frac{\mathbf{i}}{k}\int_{-1}^{+1}\frac{K(E,\zeta)}{\alpha - \mu'}d\mu')\delta(\alpha - \mu),$$

$$\zeta \in [E_1, E_2].$$

For any k the enumerated solutions are complete for functions $\psi(\mu, , E)$ defined in the range $-1 \le \mu \le 1, \quad E \in [E_1, E_2]$. They are all normalized so that

$$\int_{E_1}^{E_2}\int_{-1}^{+1}\phi_{\alpha,(\zeta),k}(\mu, E)d\mu dE = 1. \tag{28}$$

As an application we will solve the initial value problem for a uniform infinite medium. It is sufficient to consider an initial distribution which is

$$\Psi_i(0, x, \mu, E) = \delta(x - x_0)\delta(\mu - \mu_0)\delta(E - E_0).$$

Expand in the complete set of functions $\exp(\mathbf{i}kx)\phi_{\alpha,(\zeta),k}(\mu, E)$, that is,

$$\Psi_i(0, x, \mu, E) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \exp(\mathbf{i}kx)\int_S \phi_{\alpha,(\zeta),k}(\mu, E)A(\alpha, k)d\alpha.$$

(Here $\int_S d\alpha$ means adding the discrete term to an integral over the continuum.) We readily find the expansion coefficients using the bi-orthogonality properties. The distribution at a time t is then (using the time dependence of the eigenfunctions indicated by Eq. (27))

$$\Psi_i(t, x, \mu, E)$$

$$= \frac{\exp(-t)}{2\pi}\int_{-\infty}^{\infty}\exp(\mathbf{i}k(x - x_0))dk\int_S \bar{\phi}_{\alpha,(\zeta),k}(\mu_0, E_0)\phi_{\alpha,(\zeta),k}(\mu, E)\exp(\mathbf{i}\alpha kt)d\alpha. \tag{29}$$

The neutron density ρ due to an initial distribution which is located at x_0 but uniform in velocity directions is particularly simple. All that is needed is to integrate (29) with respect to μ_0, E_0 and μ, E and remember Eq.(28). The result is

$$\rho(x,t) = \frac{\exp(-t)}{2\pi} \int_{-\infty}^{\infty} \exp(\mathbf{i}k(x-x_0))dk \int_S \frac{\exp(\mathbf{i}\alpha kt)}{N_{\alpha,k}} d\alpha,$$

where $N_{\alpha,k}$ is the coefficient of normalization of eigenfunctions.

Finally, the time-dependent Green's function for a uniform infinite medium, i.e., the function which satisfies nonhomogeneous equation corresponding (26) with nonhomogeneous term $\delta(x-x_0)\delta(\mu-\mu_0)\delta(E-E_0)\delta(t-t_0)$, is trivially expressible as

$$G = \begin{cases} \Psi_i(t,x,\mu,E), & t > t_0, \\ 0, & t < t_0. \end{cases}$$

Conclusion

It has been seen that a varied set of neutron transport problems can be treated in a uniform manner with the present method. The approach is the analogy of Case approach and also of the classical separation of variables method for partial differential equations.

It should be emphasized that while the illustrations have all been relatively simple problems, but the structure of the solutions for more complicated situations will be similar.

The simplifications utilized in this paper serve primarily to permit certain integrals to be found explicitly. No particular limitations of principle seem to have been made. In complicated problems where the explicit functions may not be readily evaluated, the present method may serve as a basis of approximation.

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