ON THE CONSTRUCTION OF ANALYTIC AND NUMERICAL SOLUTIONS OF SOME PLANE BOUNDARY VALUE PROBLEMS OF THE ELASTIC MIXTURE THEORY

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Abstract

We consider a variant of the mixture theory of two isotropic elastic rigid materials. In the case of plane deformation, analytic solutions of the following static problems are obtained: the Flaman problem (concentrated force is applied to a point of the half-plane boundary), the Kelvin problem (concentrated force is applied to a point of a half-plane) and the problems where on a plane segment, partial displacements of two mixture components undergo a constant discontinuity and they are continuous everywhere except the considered segment. The obtained singular solutions are used in constructing numerical solutions of various boundary value problems of the mixture theory by the boundary element methods, namely by the method of fictitious loads and the method of displacements discontinuity.

Key words and phrases: Binary mixture, Flaman Problem, Kelvin problem, boundary element method, displacement discontinuity, fictitious load.

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1. A system of equilibrium equations in the complex form. The considered model of an elastic binary mixture is called the Green-Naghdi-Steel model [1], [2]. In this case, the elastic medium is characterized by two displacement vectors and two stress and strain tensors corresponding to two mixture components.

Let $z = x + iy$ be a point of a complex plane. In the case of plane deformation, a system of equilibrium equations in terms of displacement vector components $u_x = (u'_x, u''_x)^T$, $u_y = \left(u'_y, u''_y\right)^T$ has the form

$$4A\left(u_+\right)_{,z\bar{z}} + 2B\theta_{,\bar{z}} + f_+ = 0, \tag{1.1}$$

where

$$\bar{z} = x - iy, \qquad (\cdot)_{,z} = \frac{1}{2}\left[(\cdot)_{,x} - i\,(\cdot)_{,y}\right], \qquad (\cdot)_{,\bar{z}} = \frac{1}{2}\left[(\cdot)_{,x} + i\,(\cdot)_{,y}\right],$$

$$(.)_{,x} = \frac{\partial}{\partial x}, \qquad (.)_{,y} = \frac{\partial}{\partial y}; \qquad u_+ = u_x + iu_y, \qquad \theta = (u_+)_{,z} + (\bar{u}_+)_{,\bar{z}};$$

$$A = \begin{pmatrix} \mu_1 - \lambda_5 & \mu_3 + \lambda_5 \\ \mu_3 + \lambda_5 & \mu_2 - \lambda_5 \end{pmatrix},$$

$$B = \begin{pmatrix} \lambda_1 + \lambda_5 + \mu_1 - \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_3 - \lambda_5 + \mu_3 - \dfrac{\alpha_2 \rho_1}{\rho} \\ \\ \lambda_4 - \lambda_5 + \mu_3 - \dfrac{\alpha_2 \rho_2}{\rho} & \lambda_2 + \lambda_5 + \mu_2 - \dfrac{\alpha_2 \rho_1}{\rho} \end{pmatrix};$$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \mu_1, \mu_2, \mu_3$ are the elasticity constants, $\alpha_2 = \mu_3 - \mu_4$; ρ_1, ρ_2 are the densities of the mixture components, $\rho = \rho_1 + \rho_2$; $f_+ = f_x + if_y$, $f_x = (f_x', f_x'')^T$, $f_y = \left(f_y', f_y''\right)^T$ are the mass force components.

A general solution of the homogeneous system (1.1) is given by an analogue of the Kolosov-Muskhelishvili formula [3], [4]

$$2u_+ = A^* \varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \tag{1.2}$$

where $A^* = I + 2B^{-1}A$, I is a 2×2 unit matrix; $\varphi(z) = (\varphi_1(z), \varphi_2(z))^T$, $\psi(z) = (\psi_1(z), \psi_2(z))^T$ are the matrix columns consisting of arbitrary analytic functions.

The stress vector components $\sigma_{\alpha\beta} = \left(\sigma'_{\alpha\beta}, \sigma''_{\alpha\beta}\right)^T$, where α and β denote x or y, are expressed through the introduced functions as follows:

$$\begin{aligned} &\sigma_{yy} - \sigma_{xx} + i(\sigma_{xy} + \sigma_{yx}) = 2M\left[\overline{z}\Phi'(z) + \Psi(z)\right], \\ &\sigma_{xx} + \sigma_{yy} + i(\sigma_{xy} - \sigma_{yx}) = 2\left[(A - \lambda_5 SA^*)\Phi(z) + M\overline{\Phi(z)}\right], \end{aligned} \tag{1.3}$$

where $\Phi(z) = (\varphi_1'(z), \varphi_2'(z))^T$, $\Psi(z) = (\psi_1'(z), \psi_2'(z))^T$.

$$M = \begin{pmatrix} \mu_1 & \mu_3 \\ \mu_3 & \mu_2 \end{pmatrix}, \qquad S = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

2. Statement of the problems. Problem 2.1. Let S be a half-plane $y < 0$ filled with a binary mixture, L be its boundary. It is assumed that the external loads are given on the boundary L (t is a boundary point affix)

$$\sigma_{xx} = T(t) = (T_1(t), T_2(t))^T, \quad \sigma_{yy} = N(t) = (N_1(t), N_2(t))^T, \tag{2.1}$$

where $T_\alpha(t)$, $N_\alpha(t)$, $\alpha = 1{,}2$, are functions of the Hölder class including a point at infinity; besides,

$$T_\alpha(t) = O\left(\frac{1}{t}\right), \qquad N_\alpha(t) = O\left(\frac{1}{t}\right), \quad \alpha = 1,2.$$

It is assumed that stresses and rotations tend to zero as $z \to \infty$.

Problem 2.2. In this problem we have an infinite domain with a circular hole of radius R and center at the origin. It is assumed that the polar components of the stress tensor ($z = re^{i\vartheta}$) are given on the hole contour:

$$\sigma_{rr} - i\sigma_{r\vartheta} = -\frac{1}{2\pi R}\left(F_x - iF_y\right)e^{i\vartheta}, \tag{2.2}$$

where $F_x = (F_x', F_x'')^T$, $F_y = \left(F_y', F_y''\right)^T$ are the matrix columns consisting of constant values.

Problem 2.3. Assume that we have an infinite solid medium filled with the considered binary mixture and having a crack on a segment $|x| \leq a$, $y = 0$. A point of this segment is denoted by t. The surface of the crack lying on the positive side of the axis $y = 0$ is denoted by y=0+, and the surface lying on the negative side of the axis $y = 0$ by $y = 0$-. Let on the two sides of the segment, the displacements be given in the following manner:

$$\begin{aligned}(u_x + iu_y)^{\pm} = (u_x + iu_y)(t, 0_{\pm}) = \Big\{\pm\tfrac{I}{2} + \tfrac{i}{2\pi}\left(I - 2\left(A^* + I\right)^{-1}\right) \\ \times \ln\left|\tfrac{t-a}{t+a}\right|\Big\}\left(D_x + D_y\right),\end{aligned} \tag{2.3}$$

where $D_x = (D_x', D_x'')^T$, $D_y = \left(D_y', D_y''\right)^T$ are the matrix columns consisting of constant positive values.

From the boundary conditions (2.3) we see that when passing from one side of the segment onto the other side, displacements experience a given constant change D_x , D_y .

In problems (2.1) and (2.3) it is also assumed that stresses and rotations at infinity are equal to zero.

3. Solution of the stated problems. We will first deal with the boundary value problem (1.1), (2.1). With formulas (1.3) taken into account, this problem is reduced to the following boundary value problem: define the holomorphic functions $\Phi_\alpha(z)$, $\Psi_\alpha(z)$, α = 1,2, in the domain S, which satisfy the boundary condition

$$M\overline{\Phi(t)} + (A - \lambda_5 SA^*)\Phi(t) + Mt\Phi'(t) + M\overline{\Psi(\mathrm{t})} = N(t) - iT(t). \tag{3.1}$$

Taking the basic properties of Cauchy type integrals into account, from the boundary conditions (3.1) we find the desired holomorphic functions

$$\Phi(z) = -\frac{M^{-1}\overline{A}}{2\pi i}\int\limits_L \frac{N-iT}{t-z}dt,$$

$$\Psi(z) = -\frac{M^{-1}}{2\pi i}\int\limits_L \frac{N+iT}{t-z}dt + \frac{M^{-1}\overline{A}}{2\pi i}\left(\int\limits_L \frac{N-iT}{t-z}dt \right. \tag{3.2}$$

$$\left. +z\int\limits_L \frac{N-iT}{(t-z)^2}dt\right),$$

where we have introduced the notation $\overline{A} = M(A-\lambda_5 SA^*)^{-1}$.

The corresponding displacements and stresses are defined by the substitution of formulas (3.2) into equalities (1.2) and (1.3), respectively. For instance, for the stress components σ_{xx} and σ_{xy} we obtain the formulas

$$\sigma_{xx} = -\frac{1}{\pi}\int\limits_L \left\{2\overline{A}y(x-t)N(t) + \left[\left(I+\overline{A}\right)(x-t)^2 + \left(I-\overline{A}\right)y^2\right]T(t)\right\}$$

$$\times\frac{(x-t)\,dt}{r^4},$$

$$\sigma_{xy} = -\frac{1}{\pi}\int\limits_L \left\{\left[\left(I+\overline{A}\right)y^2 + \left(I-\overline{A}\right)(x-t)^2\right]N(t) + 2\overline{A}y(x-t)T(t)\right\}$$

$$\times\frac{(x-t)\,dt}{r^4},$$

where $r^2 = (x-t)^2 + y^2$.

As an example we can consider the case in which only the normally distributed force is applied to a part of the boundary, while the rest of the boundary is stress-free. From the solution of the latter boundary value problem we can obtain a solution of the Flaman problem for a binary mixture when the concentrated force $P_y = \left(P'_y, P''_y\right)^T$ is applied to a boundary point $t = 0$. Stresses in a half-plane $y \le 0$ are defined by the following expressions:

$$\sigma_{xx} = -\frac{2}{\pi}\left[\frac{x^2y}{(x^2+y^2)^2}\overline{A}\right]P_y, \quad \sigma_{yy} = -\frac{y}{\pi}\left[\frac{\left(I+\overline{A}\right)y^2 + \left(I-\overline{A}\right)x^2}{(x^2+y^2)^2}\right]P_y,$$

$$\sigma_{yx} = -\frac{2}{\pi}\left[\frac{y^2x}{(x^2+y^2)^2}\overline{A}\right]P_y, \quad \sigma_{xy} = -\frac{x}{\pi}\left[\frac{\left(I+\overline{A}\right)y^2 + \left(I-\overline{A}\right)x^2}{(x^2+y^2)^2}\right]P_y.$$

The displacements associated with these stresses are given by the formulas

$$u_x = \frac{1}{2\pi}\left\{\frac{1}{2}\left[A^*M^{-1}\overline{A} - M^{-1}\right]\left(\arctan\frac{y}{x} + \frac{\pi}{2}\right) + M^{-1}\overline{A}\frac{xy}{x^2+y^2}\right\}P_y,$$

$$u_y = \frac{1}{2\pi}\left\{-\frac{1}{2}\left[A^*M^{-1}\overline{A} + M^{-1}\right]\ln\left(\frac{x^2+y^2}{m^2}\right)^{1/2} + M^{-1}\overline{A}\frac{y^2}{x^2+y^2}\right\}P_y,$$

where m is an arbitrary constant.

Let us now proceed to the consideration of the boundary value problem (1.1), (2.2). This problem is reduced to the following boundary value problem for the holomorphic functions Φ_α, Ψ_α, $\alpha = 1{,}2$, in an infinite domain with a circular hole

$$M\Phi(z) + (A - \lambda_5 SA^*)\overline{\Phi}(z) - M\left[\overline{z}\Phi'(z) + \Psi(z)\right]e^{2i\vartheta}$$

$$= -\frac{1}{2\pi R}(F_x - iF_y)e^{i\vartheta} = 0 \quad, r = R,$$

whose solution has the form

$$\Phi(z) = \frac{a_1}{z}, \qquad \Psi(z) = \frac{b_1}{z} + \frac{b_3}{z^3}.$$

where $a_1 = -\dfrac{1}{2\pi}(I + A^*)^{-1}A^{-1}(F_x + iF_y)$, $\quad b_1 = -A^*\overline{a}_1, \quad b_3 = 2R^2a_1.$

Assuming that $R \to 0$ and the stresses σ_{rx}, σ_{ry} increase infinitely so that the principal vector remains invariable, we obtain

$$\Phi(z) = \frac{a_1}{z}, \qquad \Psi(z) = \frac{b_1}{z} = -\frac{A^*}{z}\overline{a_1}. \tag{3.3}$$

From formula (1.2), taking into account the expressions obtained for $\Phi(z)$ and $\Psi(z)$, we have

$$u_+ = -\tfrac{1}{4\pi}A^*(I + A^*)^{-1}A^{-1}(F_x + iF_y)\ln z\overline{z} + \tfrac{1}{4\pi}(I + A^*)^{-1}A^{-1}$$

$$\times (F_x - iF_y)\frac{z}{\overline{z}}.$$

Separating in the latter formula the real and imaginary parts and rejecting the summands corresponding to rigid displacement, we obtain

$$u_x = (A^*G - xG_{,x})F_x + (-yG_{,x})F_y,$$
$$u_y = (-xG_{,y})F_x + (A^*G - yG_{,y})F_y,$$

where

$$G(x,y) = -\frac{1}{2\pi}(I + A^*)^{-1} A^{-1} \ln\left(x^2 + y^2\right)^{1/2}.$$

The substitution of values (3.3) into formulas (1.3) gives us the expressions for stresses:

$$\sigma_{xx} = \left[(\Lambda + 2M)(A^* - I)G_{,x} - 2MxG_{,xx}\right]F_x + \left[\Lambda(A^* - I)G_{,y}\right.$$

$$\left.-2MyG_{,xx}\right]F_y,$$

$$\sigma_{yy} = \left[\Lambda(A^* - I)G_{,x} - 2MxG_{,yy}\right]F_x + \left[(\Lambda + 2M)(A^* - I)G_{,y}\right.$$

$$\left.-2MyG_{,yy}\right]F_y,$$

$$\sigma_{xy} = \left[A_0G_{,y} - 2MxG_{,xy}\right]F_x + \left[B_0G_{,x} - 2MyG_{,xy}\right]F_y,$$

$$\sigma_{yx} = \left[B_0G_{,y} - 2MxG_{,xy}\right]F_x + \left[A_0G_{,x} - 2MyG_{,xy}\right]F_y,$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 - \dfrac{\alpha_2\rho_2}{\rho} & \lambda_3 - \dfrac{\alpha_2\rho_1}{\rho} \\ \lambda_4 + \dfrac{\alpha_2\rho_2}{\rho} & \lambda_2 + \dfrac{\alpha_2\rho_1}{\rho} \end{pmatrix},$$

$$A_0 = (B - \Lambda)A^* - A, \quad B_0 = AA^* - B + \Lambda.$$

Thus we have obtained the solution of the Kelvin problem for a binary mixture.

Let us now consider the boundary value problem (1.1), (2.3). It is reduced to a boundary value problem for four functions Φ_α, Ψ_α, $\alpha = 1{,}2$, which are holomorphic all over the plane except the ends of a segment $|x| \leq a$, $y = 0$ and vanishing at infinity. We write the solution of the stated problem omitting details of its derivation:

$$\Phi(z) = \frac{1}{\pi}(A^* + I)^{-1}(-D_y + iD_x)\left(\frac{1}{z-a} - \frac{1}{z+a}\right),$$

$$\Psi(z) = \frac{1}{\pi}(A^* + I)^{-1}\left[(-D_y - iD_x)\left(\frac{1}{z-a} - \frac{1}{z+a}\right)\right. \tag{3.4}$$

$$\left. + (-D_y + iD_x)\left(\frac{a}{(z-a)^2} + \frac{a}{(z+a)^2}\right)\right].$$

Integrating these expressions, we obtain respectively the values of $\varphi(z)$ and $\psi(z)$. Substituting the functions $\varphi(z)$ and $\psi(z)$ into formula (1.2) and separating the real and imaginary parts, we obtain the following expressions for displacements:

$$u_x = \left[\frac{1}{2}\left(A^* + I\right)F_{,y} - IyF_{,xx}\right]D_x + \left[-\frac{1}{2}\left(A^* - I\right)F_{,x} - IyF_{,xy}\right]D_y,$$

$$u_x = \left[\frac{1}{2}\left(A^* - I\right)F_{,x} - IyF_{,xy}\right]D_x + \left[\frac{1}{2}\left(A^* + I\right)F_{,y} - IyF_{,yy}\right]D_y.$$

By the substitution of formulas (3.4) into (1.3) we find the expressions for stresses:

$$\sigma_{xx} = \left\{\left[4M - \lambda_5 S\left(A^* + I\right)\right]F_{,xy} + 2MyF_{,xyy}\right\}D_x$$

$$+\left\{\left[2M - \lambda_5 S\left(A^* + I\right)\right]F_{,yy} + 2MyF_{,yyy}\right\}D_y,$$

$$\sigma_{yy} = \left\{-\lambda_5 S\left(A^* + I\right)F_{,xy} - 2MyF_{,xyy}\right\}D_x$$

$$+\left\{\left[2M - \lambda_5 S\left(A^* + I\right)\right]F_{,yy} - 2MyF_{,yyy}\right\}D_y,$$

$$\sigma_{yx} = \left\{\left[2M - \lambda_5 S\left(A^* + I\right)\right]F_{,yy} + 2MyF_{,yyy}\right\}D_x$$

$$+\left\{\lambda_5 S\left(A^* + I\right)F_{,xy} - 2MyF_{,xyy}\right\}D_y,$$

$$\sigma_{xy} = \left\{\left[2M + \lambda_5 S\left(A^* + I\right)\right]F_{,yy} + 2MyF_{,yyy}\right\}D_x$$

$$+\left\{-\lambda_5 S\left(A^* + I\right)F_{,xy} - 2MyF_{,xyy}\right\}D_y,$$

where $F(x,y)$ denotes the function (2×2 matrix)

$$F(x,y) = -\frac{1}{\pi}\left(A^* + I\right)^{-1}\left[y\left(\arctan\frac{y}{x-a} - \arctan\frac{y}{x+a}\right)\right.$$

$$\left. -(x-a)\ln\sqrt{(x-a)^2 + y^2} + (x+a)\ln\sqrt{(x+a)^2 + y^2}\right].$$

In the case of a classical elastic medium, the solution of the corresponding problem was obtained by Crouch [5]. Therefore we call the solution obtained by us the solution of the Crouch problem for a binary mixture.

4. Numerical solutions of some boundary value problems. Let us first consider some applications of the Flaman problem. Using the superposition principle, this solution can be generalized for a more complicated distribution of stresses on the half-plane boundary. An elementary case is when concentrated forces $F_y^1, F_y^2, ..., F_y^k$ are applied at the boundary points

$\xi^1, \xi^2, ..., \xi^k$. "Moving" the above-given solution so that it would correspond to the point of load application and performing summation, we obtain the solution of our problem. For instance, for the displacement component uy we have

$$u_y = \sum_{i=1}^{k} Y\left(x - \xi_i, y\right) F_y^i,$$

where

$$Y\left(x - \xi^i, y\right) = \frac{1}{2\pi}\left\{-\frac{1}{2}\left[A^* M^{-1}\overline{A} + M^{-1}\right]\left(\ln\sqrt{\left(x - \xi^i\right)^2 + y^2}\right.\right.$$
$$\left.\left. - \ln\left|m - \xi^i\right|\right) + M^{-1}\overline{A}\frac{y^2}{\left(x - \xi^i\right)^2 + y^2}\right\}, \qquad i = 1, 2, ..., k.$$

In the case of continuous load distribution

$$\sigma_{yx} = 0, \quad -\infty \leq x \leq \infty, \qquad y = 0\ ,$$

$$\sigma_{yy} = \begin{cases} P_y\left(x\right), & b_1 \leq x \leq b_2, \qquad\qquad y = 0, \\ 0, & \text{at other points for } y = 0, \end{cases}$$

for u_y we obtain the formula

$$\begin{aligned} u_y = \frac{1}{2\pi}\int_{b_1}^{b_2}\left\{-\frac{1}{2}\left[A^* M^{-1}\overline{A} + M^{-1}\right]\left(\ln\sqrt{\left(x - \xi\right)^2 + y^2}\right.\right. \\ \left.\left. - \ln\left|m - \xi\right|\right) + M^{-1}\overline{A}\frac{y^2}{\left(x - \xi\right)^2 + y^2}\right\} P_y\left(\xi\right) d\xi. \end{aligned} \tag{4.1}$$

Analogous formulas are obtained for the other displacement and stress components.

For most distributions $P_y(x)$, integrals of form (4.1) have no analytic expressions. But if we consider a discrete approximation of the real distribution of stresses on the boundary, then the problem can be solved numerically. For this, the loaded part is splitted into N segments, the so-called boundary elements. Normal stresses on each boundary element are assumed to have constant values. Using the results obtained above, the solution of the problem is defined as a sum of N individual solutions.

Let us consider the so-called problem of a rigid die. In the case of a half-plane, for the elastic mixture we consider the following mixed boundary value problem:

$$u_y = -u_0 = -\left(u_0', u_0''\right)^T, \quad |x| \leq b, \qquad y = 0,$$

$$\sigma_{xy} = 0 \quad |x| < \infty, \quad y = 0,$$

$$\sigma_{yy} = 0 \quad |x| > b \quad y = 0,$$

where $u_0'>0$, $u_0''>0$ are constants.

This is the problem on impression of a rigid die with the "lubricated" contact into a half-plane. It can be formulated as follows: find a distribution of normal stresses σ_{yy} on $|x| \le b$, $y = 0$, such that normal displacements on this part would be equal to $-u_0'$, $-u_0''$. .

A numerical solution of the rigid die problem is given by the following system of $2N$ linear equations with $2N$ unknowns

$$u_y^i = -u_0 = \sum_{i=1}^{N} B^{ij} T_y^j, \qquad i = 1, 2, ..., N, \tag{4.2}$$

where u_y^i is the displacement on the part $\left|x - x^i\right| \le a$, y=0; T_y^j is the sought constant normal stress in the j-th boundary element; B^{ij} are the influence coefficients,

$$B^{ij} = -\frac{1}{4\pi}\left(A^* M^{-1}\overline{A} + M^{-1}\right)\left\{\left(x^i - x^j + a\right)\ln\left|x^i - x^j + a\right|\right.$$
$$- \left(x^i - x^j - a\right)\ln\left|x^i - x^j - a\right| + \left(m - x^j - a\right)\ln\left(m - x^j - a\right)$$
$$\left. - \left(m - x^j + a\right)\ln\left(m - x^j + a\right)\right\}.$$

System (4.2) is solved by means of the standard methods of numerical analysis.

The solution of the Kelvin problem underlies the numerical solution of various boundary value problems by the so-called method of fictitious loads. Domains may be both finite and infinite. In that case, the boundary is approximated by N segments adjoining each other and it is assumed that the constant normal and tangential stresses P_s^i and P_n^i (fictitious loads) are acting on each of N segments. Using a singular solution and coordinate transformation formulas, we calculate the real stresses σ_s^i, σ_n^i at the mid-point of each segment, $i = 1, ..., N$. As a result we obtain the following system of algebraic equations [6], [7]:

$$\left.\begin{array}{l} \sigma_s^i = \sum\limits_{j=1}^{N} A_{ss}^{ij} P_s^j + \sum\limits_{j=1}^{N} A_{sn}^{ij} P_n^j \\[2ex] \sigma_n^i = \sum\limits_{j=1}^{N} A_{ns}^{ij} P_s^j + \sum\limits_{j=1}^{N} A_{nn}^{ij} P_n^j \end{array}\right\}, \qquad i = 1, 2, ..., N,$$

where A_{ss}^{ij},..., A_{nn}^{ij} are the boundary coefficients of stress influence. They are calculated a priori. After finding the fictitious loads and substituting them into the respective formulas, we can define stresses and displacements at any point of the body except the point lying inside a circle with center at the midpoint of a boundary element and radius equal to the length of this element, certainly not counting its midpoint.

As an example we will consider several simple problems and compare the numerical solution with the analytic one.

5. Examples. Below, using the method of boundary elements we give the solution of two boundary value problems for an elastic body consisting of a binary mixture. The first of them is an external problem for an infinite domain with a circular hole when the contour is stress-free, while unilateral tensile stresses are acting at infinity. The second problem concerns a circular semi-ring when stresses are given on two opposite semicircles, and the symmetry and antisymmetry conditions on two opposite segments [8].

Problem 1. This is a boundary value problem, i.e. a search for elastic equilibrium in the domain $\Omega = \{r_1 < r < \infty, 0 < \alpha < 2\pi\}$ with the following boundary conditions:

$$\text{when} \quad r = r_1\text{:}\ \sigma_{rr} = (0,0)^T, \quad \sigma_{r\alpha} = (0,0)^T,$$

$$\text{when} \quad r \to \infty.\text{:}\ \sigma_{xx} = p = (p', p'')^T, \quad \sigma_{yy} = \sigma_{xy} = \sigma_{yx} = (0,0)^T.$$

Since the problem has two axes of symmetry, the numerical solution is obtained if a quarter of the circular boundary is divided into 50 elements and $\lambda_1 = 0.1$; $\lambda_2 = 0.2$; $\lambda_3 = 0.3$; $\lambda_4 = 0.4$; $\lambda_5 = 0.5$; $\mu_1 = 0.6$; $\mu_2 = 0.7$; $\mu_3 = 0.8$; $\rho_1 = 0.15$; $\rho_2 = 0.25$; $p'/E' = 10^{-3}$, $p''/E'' = 15 \cdot 10^{-4}$, $r_1 = 16$; $0 < \alpha < 2\pi$.

An analytic solution for stresses along the circular hole boundary has the form [9]

$$\text{when} \ \ r = r_1\text{:} \quad \sigma_{\alpha\alpha} = \left\{ \mathrm{I} - \left[\mathrm{I} + \mathrm{M} \left(\mathrm{A} - \lambda_5 S \mathrm{A}^* \right)^{-1} \right] \cos 2\alpha \right\} p,$$

$$\sigma_{\alpha r} = -\left\{ \left[\mathrm{I} - \mathrm{M} \left(\mathrm{A} - \lambda_5 S \mathrm{A}^* \right)^{-1} \right] \sin 2\alpha \right\} p,$$

where the angle α is counted from the x-axis. These functions are shown in Fig. 1 together with numerical results.

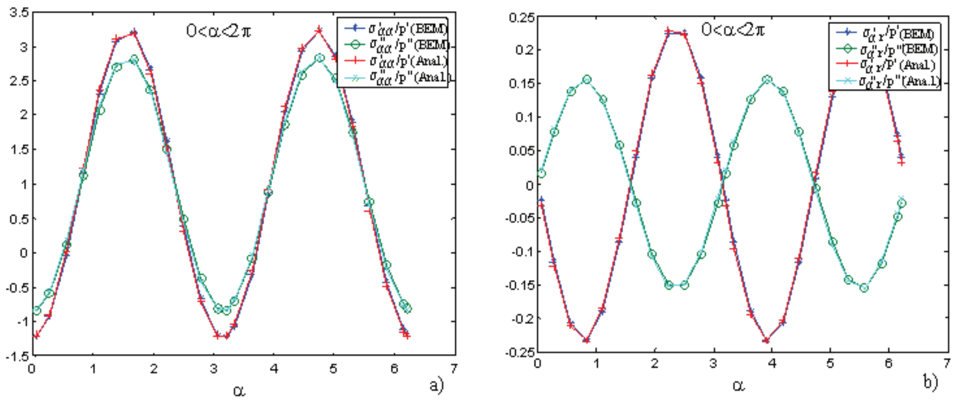

Fig.1. a) Tangential ($\sigma_{\alpha\alpha}$) and b) shearing ($\sigma_{\alpha r}$) stresses on the hole boundary.

Problem 2. The elastic equilibrium of this problem is considered in the domain

$\Omega = \{r_1 < r < r_2, 0 < \alpha < \pi\}$ with the following boundary conditions:

$$a)\quad \text{when}\ \ r = r_1:\ \ \sigma_{rr} = \left(p' \cos\frac{\alpha}{2}, p'' \cos\frac{\alpha}{2}\right)^T, \ \ \sigma_{r\alpha} = (0,0)^T,$$

$$b)\quad \text{when}\ \ r = r_2:\ \ \sigma_{rr} = (0,0)^T, \ \ \sigma_{r\alpha} = (0,0)^T,$$

$$c)\quad \text{when}\ \ \alpha = 0:\ \ v = (0,0)^T, \quad \sigma_{r\alpha} = (0,0)^T,$$

$$d)\quad \text{when}\ \ \alpha = \pi.:\ \ u = (0,0)^T, \quad \sigma_{\alpha\alpha} = (0,0)^T.$$

The problem is solved by the method of boundary elements. Given the data $\lambda_1 = 0.1$; $\lambda_2 = 0.2$; $\lambda_3 = 0.3$; $\lambda_4 = 0.4$; $\lambda_5 = 0.5$; $\mu_1 = 0.6$; $\mu_2 = 0.7$; $\mu_3 = 0.8$; $\rho_1 = 0.15$; $\rho_2 = 0.25$; $p'/E' = 10^{-3}$, $p''/E'' = 15 \cdot 10^{-4}$, $r_1 = 16$; $r_2 = 4$, we obtain the stress values at the characteristic points of the considered domain. The semi-circles of radii $r = r_1$ and $r = r_2$ are divided into 180 equal arcs, while the linear parts of the boundary are divided into 40 equal segments. Fig. 2 shows the diagrams for the stresses $\sigma_{\alpha\alpha}/p = (\sigma'_{\alpha\alpha}/p', \sigma''_{\alpha\alpha}/p'')^T$, $\sigma_{\alpha r}/p = (\sigma'_{\alpha r}/p', \sigma''_{\alpha r}/p'')^T$, $\sigma_{r\alpha}/p = (\sigma'_{r\alpha}/p', \sigma''_{r\alpha}/p'')^T$, $\sigma_{rr}/p = (\sigma'_{rr}/p', \sigma''_{rr}/p'')^T$ when $r_1 < r < r_2$, $\alpha = \pi/3$.

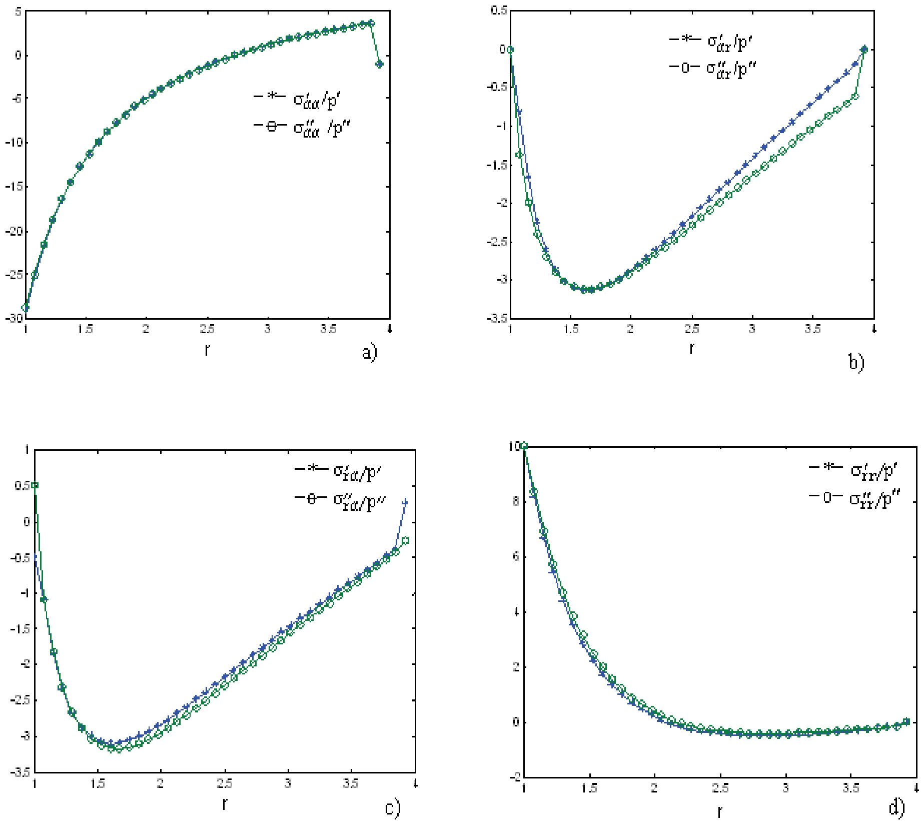

Fig.2. a) Tangential ($\sigma_{\alpha\alpha}$), b),c) shearing ($\sigma_{\alpha r}$ and $\sigma_{r\alpha}$) and d) normal (σ_{rr}) stresses in the semi-ring along $\alpha = \pi/3$.

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