LARGE TIME BEHAVIOR OF SOLUTIONS AND DIFFERENCE SCHEMES TO NONLINEAR INTEGRO-DIFFERENTIAL SYSTEM ASSOCIATED WITH THE PENETRATION OF A MAGNETIC FIELD INTO A SUBSTANCE

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Abstract

Large time behavior of solutions and finite difference approximation of a nonlinear system of integro-differential equations associated with the penetration of a magnetic field into a substance are studied. Two initial-boundary value problems are investigated - the first with homogeneous conditions on whole boundary and the second with nonhomogeneous boundary data on one side of lateral boundary. The rates of convergence are also established. The convergence properties of the corresponding finite difference schemes are also given.

Key words and phrases: System of nonlinear integro-differential equations, large time behavior, finite difference scheme.

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1 Introduction

The nonlinear differential and integro-differential equations and their systems describe various processes in physics, economics, chemistry, technology and so on. Study of qualitative and structural properties of the solutions of initial-boundary value problems for such models are very important. It is doubtless that construction and investigation of corresponding discrete analogues and the study of numerical algorithms are significant as well.

One type of integro-differential system arises for mathematical modelling of the process of penetrating of magnetic field into the substance. In a quasistationary case the corresponding system of Maxwell's equations has the form [1]:

$$\frac{\partial H}{\partial t} = -rot(\nu_m rot H), \tag{1.1}$$

$$c_\nu \frac{\partial \theta}{\partial t} = \nu_m \left(rotH\right)^2, \tag{1.2}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field, θ is temperature, c_ν and ν_m characterize the thermal heat capacity and electroconductivity of the substance. System (1.1) defines the process of diffusion of the magnetic field and equation (1.2) – change of the temperature at the expense of Joule heating without taking into account the heat conductivity.

If c_ν and ν_m depend on temperature θ, i.e., $c_\nu = c_\nu(\theta)$, $\nu_m = \nu_m(\theta)$, then the system (1.1), (1.2) can be rewritten in the following form [2]:

$$\frac{\partial H}{\partial t} = -rot \left[a \left(\int_0^t |rotH|^2 \, d\tau \right) rotH \right], \tag{1.3}$$

where function $a = a(S)$ is defined for $S \in [0, \infty)$.

Note that the system (1.3) is complex. Equations and systems of type (1.3) still yield to the investigation for special cases. The model of (1.3) type was intensively studied by many authors and a large amount of literature is devoted to its investigation (see, for example, [2-19]).

The existence, uniqueness and asymptotic behavior of the solutions of the initial-boundary value problems for the equations of type (1.3) are studied in the works [2-13,17,18]. The existence theorems, that are proved in [2-4], [10] are based on Galerkin's method and compactness arguments as in [20, 21] for nonlinear problems.

If the magnetic field has the form $H = (0, U, V)$ and $U = U(x,t)$, $V = V(x,t)$, then we have

$$rot(a(S)rotH) = \left(0,\ -\frac{\partial}{\partial x}\left(a(S)\frac{\partial U}{\partial x}\right),\ -\frac{\partial}{\partial x}\left(a(S)\frac{\partial V}{\partial x}\right)\right).$$

So, from (1.3) we obtain the following system of nonlinear integro-differential equations:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right], \tag{1.4}$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau. \tag{1.5}$$

In [7] some generalization of the system of type (1.4),(1.5) is proposed. In particular, assuming the temperature of the considered body to be constant throughout the material, i.e. depending on time, but independent of the space coordinates, the process of penetration of the magnetic field into the material is modelled by the following averaged integro-differential model

$$\frac{\partial U}{\partial t} = a(S)\frac{\partial^2 U}{\partial x^2}, \quad \frac{\partial V}{\partial t} = a(S)\frac{\partial^2 V}{\partial x^2}, \tag{1.6}$$

where

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] dx d\tau. \tag{1.7}$$

Our aim is to study long time behavior of solutions of the first boundary value problems for the system (1.4),(1.5) and (1.6),(1.7) with zero conditions in whole lateral boundary as well as the problem with non zero conditions on one side of lateral boundary. The results presented show the difference between stabilization character of solutions in these two cases.

The corresponding difference schemes are also given. The difference schemes for (1.4),(1.5) and (1.6),(1.7) type equations are studied in [13-16].

2 Problem with nonhomogeneous Dirichlet conditions on one side of the lateral boundary

Consider the following problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right],$$

$$(x,t) \in Q = (0,1) \times (0,\infty), \tag{2.1}$$

$$U(0,t) = 0, \quad U(1,t) = \psi_1, \quad V(0,t) = 0, \quad V(1,t) = \psi_2, \quad t \geq 0, \tag{2.2}$$

$$U(x,0) = U_0(x), \quad V(x,0) = V_0(x), \quad x \in [0,1], \tag{2.3}$$

where

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] d\tau, \tag{2.4}$$

or

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2 \right] dx d\tau, \tag{2.5}$$

$\psi_1 = const \geq 0$, $\psi_2 = const \geq 0$.

In this paper everywhere we assume that $a(S) = (1+S)^p$. Restrictions for the p will be concretized in the statements.

It should be noted that boundary conditions (2.2) are used by taking into account the physical problems considered in [19].

The following statement takes place [11].

Theorem 2.1. *Suppose that* $0 < p \leq 1$, U_0, $V_0 \in H^2(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$. *Then for the solution of the problem (2.1)-(2.4) the following asymptotic relations hold as* $t \to \infty$:

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| \leq C\left(t^{-1-p}\right), \quad \left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \leq C\left(t^{-1-p}\right),$$

$$\left|\frac{\partial U(x,t)}{\partial t}\right| \leq C\left(t^{-1}\right), \quad \left|\frac{\partial V(x,t)}{\partial t}\right| \leq C\left(t^{-1}\right),$$

uniformly in x *on* $[0,1]$.

Everywhere in this paper we use usual inner-product $L_2(0,1)$, the corresponding norm and Sobolev spaces $H^k(0,1)$ and $H_0^k(0,1)$. As to symbols C, as well as C_i and c, in Sections 2 and 3, they denote various positive constants, independent of t.

A series of lemmas is necessary to prove Theorem 2.1. We assume that conditions of the Theorem 2.1 hold.

Lemma 2.1. *For the solution of the problem (2.1)-(2.4) the following estimates are true:*

$$\int_0^t \int_0^1 \left(\frac{\partial U}{\partial \tau}\right)^2 dx d\tau \leq C, \quad \int_0^t \int_0^1 \left(\frac{\partial V}{\partial \tau}\right)^2 dx d\tau \leq C.$$

Lemma 2.2. *For the function* S *the following estimates hold:*

$$c\varphi^{\frac{1}{1+2p}}(t) \leq 1 + S(x,t) \leq C\varphi^{\frac{1}{1+2p}}(t),$$

where

$$\varphi(t) = 1 + \int_0^t \int_0^1 \left(\sigma_1^2 + \sigma_2^2\right) dx d\tau \tag{2.6}$$

and $\sigma_1 = (1+S)^p \partial U/\partial x$, $\quad \sigma_2 = (1+S)^p \partial V/\partial x$.

Lemma 2.3. *The following inequalities are true:*

$$c\varphi^{\frac{2p}{1+2p}}(t) \leq \int_0^1 \left(\sigma_1^2(x,t) + \sigma_2^2(x,t)\right) dx \leq C\varphi^{\frac{2p}{1+2p}}(t).$$

Lemma 2.4. *The derivatives* $\partial U/\partial t$ *and* $\partial V/\partial t$ *satisfy the inequality*

$$\int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx \leq C\varphi^{-\frac{2}{1+2p}}(t).$$

Lemma 2.5. *For $\partial S/\partial x$ the following inequality is true*

$$\int_0^1 \left|\frac{\partial S}{\partial x}\right| dx \le C\varphi^{-\frac{p}{1+2p}}(t).$$

It is not difficult to show that at $p > 0$ Lemmas 2.2, 2.3, and 2.4 are also true for the solution of the problem (2.1)-(2.3), (2.5). From these lemmas, according to the scheme used in [11], we get analogous theorem for the problem (2.1)-(2.3), (2.5) (see [12]).

Theorem 2.2. *Suppose that $p > 0$, $U_0, V_0 \in H^2(0,1)$, $U_0(0) = V_0(0) = 0$, $U_0(1) = \psi_1$, $V_0(1) = \psi_2$, $\psi_1^2 + \psi_2^2 \neq 0$. Then for the solution of the problem (2.1)-(2.3), (2.5) the following asymptotic relations hold as $t \to \infty$:*

$$\left|\frac{\partial U(x,t)}{\partial x} - \psi_1\right| \le C\left(t^{-1-p}\right), \quad \left|\frac{\partial V(x,t)}{\partial x} - \psi_2\right| \le C\left(t^{-1-p}\right),$$

$$\left|\frac{\partial U(x,t)}{\partial t}\right| \le C\left(t^{-1}\right), \quad \left|\frac{\partial V(x,t)}{\partial t}\right| \le C\left(t^{-1}\right),$$

uniformly in x on $[0,1]$.

Note that to obtain results given in this section, the scheme similar to [22], in which the adiabatic shearing of incompressible fluids with temperature-dependent viscosity is studied, was used.

3 Problem with homogeneous Dirichlet boundary conditions

Consider the following initial-boundary value problem:

$$\frac{\partial U}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial U}{\partial x}\right], \quad \frac{\partial V}{\partial t} = \frac{\partial}{\partial x}\left[a(S)\frac{\partial V}{\partial x}\right], \quad (x,t) \in Q, \tag{3.1}$$

$$U(0,t) = U(1,t) = V(0,t) = V(1,t) = 0, \quad t \ge 0, \tag{3.2}$$

$$U(x,0) = U_0(x), \; V(x,0) = V_0(x), \quad x \in [0,1], \tag{3.3}$$

where again

$$S(x,t) = \int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau, \tag{3.4}$$

or

$$S(t) = \int_0^t \int_0^1 \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx d\tau. \tag{3.5}$$

It is easy to verify the following statement.

Lemma 3.1. *If $p > 0$, then for the solution of the problems (3.1)-(3.4) and (3.1)-(3.3), (3.5) the following estimate is true*

$$\|U\| + \|V\| \leq C \exp(-t).$$

Therefore, Lemma 3.1 gives exponential stabilization of the solution of the problems (3.1)-(3.4) and (3.1)-(3.3), (3.5) in the norm of the space $L_2(0,1)$. As it was shown in [12, 18], the stabilization takes place in the norm of the space $H^1(0,1)$ as well. In particular, the following statement holds.

Theorem 3.1. *Assume that $U_0, V_0 \in H^2(0,1) \cap H_0^1(0,1)$. If $0 < p \leq 1$ then for the solution of the problem (3.1)-(3.4), and if $p > 0$ then for the solution of the problem (3.1)-(3.3), (3.5), the following estimate is true as $t \to \infty$*

$$\left\|\frac{\partial U}{\partial x}\right\| + \left\|\frac{\partial U}{\partial t}\right\| + \left\|\frac{\partial V}{\partial x}\right\| + \left\|\frac{\partial V}{\partial t}\right\| \leq C \exp\left(-\frac{t}{2}\right).$$

Let us strengthen the Theorem 3.1. Namely, let us show that stabilization can be achieved in the stronger norm.

The main result of this section has the form

Theorem 3.2. *Suppose that $U_0, V_0 \in H^2(0,1) \cap H_0^1(0,1)$. If $0 < p \leq 1$ then for the solution of the problem (3.1)-(3.4), and if $p > 0$ then for the solution of the problem (3.1)-(3.3), (3.5), the following estimates hold as $t \to \infty$:*

$$\left|\frac{\partial U(x,t)}{\partial x}\right| \leq C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial x}\right| \leq C \exp\left(-\frac{t}{2}\right),$$

$$\left|\frac{\partial U(x,t)}{\partial t}\right| \leq C \exp\left(-\frac{t}{2}\right), \quad \left|\frac{\partial V(x,t)}{\partial t}\right| \leq C \exp\left(-\frac{t}{2}\right),$$

uniformly in x on $[0,1]$.

Theorem 3.1 helps us to deduce that Lemma 2.2 holds also for the solution of the problem (3.1)-(3.4) and (3.1)-(3.3), (3.5). Therefore, using this lemma, (2.6) and again Theorem 3.1, we obtain

$$\frac{d\varphi(t)}{dt} = \int_0^1 (1+S)^{2p} \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx \leq C \varphi^{\frac{2p}{1+2p}}(t) \exp(-t).$$

After integrating this inequality, taking into account (2.6), we arrive at

$$1 \leq \varphi(t) \leq C.$$

From this, keeping in mind Lemma 2.2, we get

$$1 \le 1 + S(x,t) \le C. \tag{3.6}$$

From (3.6) and Theorem 3.1, by taking into account identities

$$\sigma_1^2(x,t) = \int_0^1 \sigma_1^2(y,t)dy + \int_0^1 \int_y^x \frac{\partial \sigma_1^2(\xi,t)}{\partial \xi} d\xi dy =$$
$$= \int_0^1 \sigma_1^2(y,t)dy + 2\int_0^1 \int_y^x \sigma_1(\xi,t)\frac{\partial U(\xi,t)}{\partial t} d\xi dy,$$
$$\sigma_2^2(x,t) = \int_0^1 \sigma_2^2(y,t)dy + \int_0^1 \int_y^x \frac{\partial \sigma_2^2(\xi,t)}{\partial \xi} d\xi dy =$$
$$= \int_0^1 \sigma_2^2(y,t)dy + 2\int_0^1 \int_y^x \sigma_2(\xi,t)\frac{\partial V(\xi,t)}{\partial t} d\xi dy,$$

we get

$$\sigma_1^2(x,t) + \sigma_2^2(x,t) \le 2\int_0^1 (1+S)^{2p} \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] dx +$$
$$+ \int_0^1 \left[\left(\frac{\partial U}{\partial t}\right)^2 + \left(\frac{\partial V}{\partial t}\right)^2\right] dx \le C\exp(-t). \tag{3.7}$$

At last, if we remind definition of σ_1 and σ_2, from (3.7), validity of the first part of the Theorem 3.2 will be obvious.

Now let us estimate derivatives $\partial U/\partial t$ and $\partial V/\partial t$. For this, differentiate the first equation of the system (3.1) with respect to t

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left[\frac{\partial (1+S)^p}{\partial t}\frac{\partial U}{\partial x} + (1+S)^p \frac{\partial^2 U}{\partial t \partial x}\right] = 0. \tag{3.8}$$

Multiplying (3.8) by $\partial U/\partial t$ and carrying out integration by parts gives

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\frac{\partial U}{\partial t}\right)^2 dx + \int_0^1 (1+S)^p \left(\frac{\partial^2 U}{\partial t \partial x}\right)^2 dx +$$
$$+p\int_0^1 (1+S)^{p-1} \left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial t \partial x} dx + \tag{3.9}$$

$$+p\int\limits_0^1 (1+S)^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2\frac{\partial^2 U}{\partial t\partial x}dx=0.$$

Identity (3.9) yields

$$\frac{d}{dt}\int\limits_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\int\limits_0^1 (1+S)^p\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx\le$$

$$\le 2p^2\int\limits_0^1 (1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^6 dx+ \tag{3.10}$$

$$+2p^2\int\limits_0^1 (1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial V}{\partial x}\right)^4 dx.$$

Let us multiply (3.10) scalarly by $\exp(2t)$ and integrate it on $(0,t)$. Using (3.6), Theorem 3.1, and the first part of the Theorem 3.2, after simple transformations we get

$$\int\limits_0^t \exp(2\tau)\frac{d}{d\tau}\int\limits_0^1\left(\frac{\partial U}{\partial \tau}\right)^2 dxd\tau+\int\limits_0^t \exp(2\tau)\int\limits_0^1 (1+S)^p\left(\frac{\partial^2 U}{\partial x\partial \tau}\right)^2 dxd\tau\le$$

$$\le 2p^2\int\limits_0^t \exp(2\tau)\int\limits_0^1 (1+S)^{p-2}\left(\frac{\partial U}{\partial x}\right)^2\left[\left(\frac{\partial U}{\partial x}\right)^4+\left(\frac{\partial V}{\partial x}\right)^4\right]dxd\tau,$$

$$\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial x\partial \tau}\right)^2 dxd\tau\le-\exp(2t)\int\limits_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx+\int\limits_0^1\left(\frac{\partial U}{\partial t}\right)^2 dx\Bigg|_{t=0}+$$

$$+2\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial U}{\partial \tau}\right)^2 dxd\tau+C\int\limits_0^t \exp(-\tau)d\tau,$$

or

$$\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial x\partial \tau}\right)^2 dxd\tau\le C\exp(t). \tag{3.11}$$

Similarly,

$$\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 V}{\partial x\partial \tau}\right)^2 dxd\tau\le C\exp(t). \tag{3.12}$$

Multiplying (3.8) scalarly by $\exp(2t)\partial^2U/\partial t^2$, using the first part of the Theorem 3.2, a priori estimates (3.6), (3.11), (3.12), we get

$$\int_0^t \exp(2\tau)\int_0^1 \left(\frac{\partial^2 U}{\partial\tau^2}\right)^2 dxd\tau + \frac{1}{2}\int_0^t\int_0^1 \exp(2\tau)(1+S)^p \frac{\partial}{\partial\tau}\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau +$$

$$+p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial}{\partial\tau}\left(\frac{\partial^2 U}{\partial\tau\partial x}\right) dxd\tau +$$

$$+p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial}{\partial\tau}\left(\frac{\partial^2 U}{\partial\tau\partial x}\right) dxd\tau = 0.$$

$$\frac{\exp(2t)}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx \le \frac{1}{2}\int_0^1 \left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx\Bigg|_{t=0} +$$

$$+\int_0^t\int_0^1 \exp(2\tau)(1+S)^p \left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau +$$

$$+\frac{p}{2}\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right]\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau -$$

$$-p\exp(2t)\int_0^1 (1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial t\partial x}dx + p\int_0^1 \left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial t\partial x}dx\Bigg|_{t=0} +$$

$$+2p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^3 \frac{\partial^2 U}{\partial\tau\partial x}dxd\tau +$$

$$+p(p-1)\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-2}\left[\left(\frac{\partial U}{\partial x}\right)^5 + \left(\frac{\partial U}{\partial x}\right)^3\left(\frac{\partial V}{\partial x}\right)^2\right]\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau +$$

$$+3p\int_0^t\int_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial U}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau -$$

$$-p\exp(2t)\int_0^1 (1+S)^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t\partial x}dx + p\int_0^1 \frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2 \frac{\partial^2 U}{\partial t\partial x}dx\Bigg|_{t=0} +$$

$$+2p\int\limits_0^t\int\limits_0^1 \exp(2\tau)(1+S)^{p-1}\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau+$$

$$+p(p-1)\int\limits_0^t\int\limits_0^1 \exp(2\tau)(1+S)^{p-2}\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]\frac{\partial U}{\partial x}\left(\frac{\partial V}{\partial x}\right)^2\frac{\partial^2 U}{\partial\tau\partial x}dxd\tau+$$

$$+p\int\limits_0^t\int\limits_0^1 \exp(2\tau)(1+S)^{p-1}\left(\frac{\partial V}{\partial x}\right)^2\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+$$

$$+2p\int\limits_0^t\int\limits_0^1 \exp(2\tau)(1+S)^{p-1}\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\frac{\partial^2 U}{\partial\tau\partial x}\frac{\partial^2 V}{\partial\tau\partial x}dxd\tau\le$$

$$\le C_1+C_2\exp(t)+C_3\int\limits_0^t \exp(2\tau)\exp(-\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+$$

$$+\frac{\exp(2t)}{8}\int\limits_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx+C_4\exp(-t)+$$

$$+\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_5\int\limits_0^t\exp(-\tau)d\tau+$$

$$+\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_6\int\limits_0^t\exp(-3\tau)d\tau+$$

$$+C_7\int\limits_0^t \exp(2\tau)\exp(-\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+\frac{\exp(2t)}{8}\int\limits_0^1\left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx+$$

$$+C_8\exp(-t)+\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_9\int\limits_0^t\exp(-\tau)d\tau+$$

$$+\int\limits_0^t \exp(2\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_{10}\int\limits_0^t\exp(-3\tau)d\tau+$$

$$+C_{11}\int\limits_0^t \exp(2\tau)\exp(-\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+C_{12}\int\limits_0^t\exp(\tau)\int\limits_0^1\left(\frac{\partial^2 U}{\partial\tau\partial x}\right)^2 dxd\tau+$$

$$+C_{12}\int_0^t \exp(\tau)\int_0^1 \left(\frac{\partial^2 V}{\partial\tau\partial x}\right)^2 dx d\tau \le \frac{\exp(2t)}{4}\int_0^1 \left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx + C_{13}\exp(t).$$

i.e.,

$$\int_0^1 \left(\frac{\partial^2 U}{\partial t\partial x}\right)^2 dx \le C\exp(-t). \tag{3.13}$$

Analogously,

$$\int_0^1 \left(\frac{\partial^2 V}{\partial t\partial x}\right)^2 dx \le C\exp(-t). \tag{3.14}$$

Using Theorem 3.1 from (3.13) and (3.14), taking into account the relation

$$\frac{\partial U(x,t)}{\partial t} = \int_0^1 \frac{\partial U(y,t)}{\partial t} dy + \int_0^1 \int_y^x \frac{\partial^2 U(\xi,t)}{\partial t\partial \xi} d\xi dy,$$

we prove the second part of the Theorem 3.2.

Results of Theorems 2.1, 2.2, 3.1, and 3.2 show the difference between stabilization character of solutions with homogeneous and nonhomogeneous boundary conditions.

4 Finite difference schemes and numerical solution

Now, assume that $p = 1$ and rewrite systems (2.1),(2.4) and (2.1),(2.5) in the following forms

$$\begin{aligned}
\frac{\partial U}{\partial t} &= \frac{\partial}{\partial x}\left\{\left(1+\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial U}{\partial x}\right\}, \\
\frac{\partial V}{\partial t} &= \frac{\partial}{\partial x}\left\{\left(1+\int_0^t \left[\left(\frac{\partial U}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial x}\right)^2\right] d\tau\right)\frac{\partial V}{\partial x}\right\}
\end{aligned} \tag{4.1}$$

and

$$
\begin{aligned}
\frac{\partial U}{\partial t} &= \left(1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial^2 U}{\partial x^2},\\
\frac{\partial V}{\partial t} &= \left(1+\int_0^t\int_0^1\left[\left(\frac{\partial U}{\partial x}\right)^2+\left(\frac{\partial V}{\partial x}\right)^2\right]dxd\tau\right)\frac{\partial^2 V}{\partial x^2}.
\end{aligned} \tag{4.2}
$$

For the systems (4.1) and (4.2) let us consider the following initial-boundary value problem:

$$U(0,t)=U(1,t)=V(0,t)=V(1,t)=0, \quad t\geq 0, \tag{4.3}$$

$$U(x,0)=U_0(x), \quad V(x,0)=V_0(x), \quad x\in[0,1], \tag{4.4}$$

On $[0,1]\times[0,T]$ let us introduce a grid with mesh points denoted by $(x_i,t_j)=(ih,j\tau)$, where $i=0,1,...,M$; $j=0,1,...,N$, with $h=1/M$, $\tau=T/N$. The initial line is denoted by $j=0$. The discrete approximation at (x_i,t_j) is denoted by u_i^j, v_i^j and the exact solutions to problems (4.1), (4.3), (4.4) and (4.2)-(4.4) by U_i^j, V_i^j. We will use the following known notation:

$$r_{t,i}^j=\frac{r_i^{j+1}-r_i^j}{\tau}, \quad r_{\bar t,i}^j=r_{t,i}^{j-1}=\frac{r_i^j-r_i^{j-1}}{\tau}.$$

For problem (4.1), (4.3), (4.4) let us consider the finite difference scheme:

$$
\begin{gathered}
\frac{u_i^{j+1}-u_i^j}{\tau}-\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar x,i}^k)^2+(v_{\bar x,i}^k)^2\right]\right)u_{\bar x,i}^{j+1}\right\}_x=0,\\
\frac{v_i^{j+1}-v_i^j}{\tau}-\left\{\left(1+\tau\sum_{k=1}^{j+1}\left[(u_{\bar x,i}^k)^2+(v_{\bar x,i}^k)^2\right]\right)v_{\bar x,i}^{j+1}\right\}_x=0,\\
i=1,2,...,M-1;\ \ j=0,1,...,N-1,\\
u_0^j=u_M^j=v_0^j=v_M^j=0,\ \ j=0,1,...,N,\\
u_i^0=U_{0,i},\ ,v_i^0=V_{0,i},\ \ i=0,1,...,M.
\end{gathered} \tag{4.5}
$$

and the corresponding scheme for averaged problem (4.2)-(4.4):

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\left(1+\tau h\sum_{i=1}^{M}\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)u_{\bar{x}x,i}^{j+1}=0,$$

$$\frac{v_i^{j+1}-v_i^j}{\tau}-\left(1+\tau h\sum_{i=1}^{M}\sum_{k=1}^{j+1}\left[(u_{\bar{x},i}^k)^2+(v_{\bar{x},i}^k)^2\right]\right)v_{\bar{x}x,i}^{j+1}=0,$$
$$i=1,2,...,M-1;\;\; j=0,1,...,N-1, \tag{4.6}$$

$$u_0^j=u_M^j=v_0^j=v_M^j=0,\;\; j=0,1,...,N,$$

$$u_i^0=U_{0,i},\;\; ,v_i^0=V_{0,i},\;\; i=0,1,...,M.$$

Theorem 4.1. *If problems (4.1),(4.3),(4.4) and (4.2)-(4.4) have sufficiently smooth solution* $U=U(x,t)$, $V=V(x,t)$, *then the solution* $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $v^j=(v_1^j,v_2^j,\dots,v_{M-1}^j)$, $j=1,2,\dots,N$ *of the difference schemes (4.5) and (4.6) tend to the solution of continuous problems* $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $V^j=(V_1^j,V_2^j,\dots,V_{M-1}^j)$, $j=1,2,\dots,N$, *correspondingly as* $\tau\to 0$, $h\to 0$ *and the following estimates are true*

$$\|u^j-U^j\|_h\le C(\tau+h),\quad \|v^j-V^j\|_h\le C(\tau+h),\quad j=1,2,\dots,N.$$

Note that in Theorem 4.1 C is independent of h and τ.

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