ON THE CONVERGENCE OF JACOBI TYPE ITERATIVE METHOD FOR THE SYSTEM OF I.VEKUA EQUATIONS FOR A SPHERICAL SHELL

D. Gulua, [1] D. Rogava [2], J. Rogava [3]

[1] Georgian Technical University
77, Kostava st., 0175, Tbilisi, Georgia
[2] Sokhumi State University
9, Jikia st., 0186, Tbilisi, Georgia
[3] I.Vekua Institute of Applied Mathematics
of Tbilisi State University
2, University st., 0143, Tbilisi, Georgia

(Received: 12.09.07; accepted: 12.04.08)

Abstract

Jacobi type iterative process is proposed for approximate solution of Dirichlet boundary-value problem for the system of I.Vekua equations of a spherical shell. The essence of the process consists in that at each iteration step are solved three independent equations with operator $a_1\partial^2_{xx} + a_2\partial^2_{yy} - a_3 I$, where a_1, a_2 and a_3 are positive constants. It is proved that this iterative process converges with geometric progression rate, if ratio of shell thickness to sphere radius satisfies a certain condition.

Key words and phrases: Spherical shell, iteration, split, operator.

AMS subject classification: 65M12, 65M15, 65M55.

Let us consider the system of I.Vekua equations for a spherical shell (see [1]):

$$(A_0 + A_1)u(x,y) = \frac{2(1+\sigma)}{E} f(x,y), \quad (x,y) \in]-1,1[\times]-1,1[, \tag{1}$$

with the Dirichlet boundary conditions:

$$u\,|_{\partial\Omega} = 0, \quad \partial\Omega\ :\ |x| = |y| = 1, \tag{2}$$

where

$$A_0 + A_1 =$$

$$= -\begin{pmatrix} \frac{2(1-\sigma)}{1-2\sigma}\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \varepsilon^2 & 0 & 0 \\ 0 & \frac{\partial^2}{\partial x^2} + \frac{2(1-\sigma)}{1-2\sigma}\frac{\partial^2}{\partial y^2} - \varepsilon^2 & 0 \\ 0 & 0 & \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - 4\varepsilon^2\frac{1-\sigma}{1-2\sigma} \end{pmatrix}$$

$$-\begin{pmatrix} 0 & \frac{1}{1-2\sigma}\frac{\partial^2}{\partial x\partial y} & -\varepsilon\frac{3-2\sigma}{1-2\sigma}\frac{\partial}{\partial x} \\ \frac{1}{1-2\sigma}\frac{\partial^2}{\partial x\partial y} & 0 & -\varepsilon\frac{3-2\sigma}{1-2\sigma}\frac{\partial}{\partial y} \\ \varepsilon\frac{3-2\sigma}{1-2\sigma}\frac{\partial}{\partial x} & \varepsilon\frac{2\sigma}{1-2\sigma}\frac{\partial}{\partial y} & 0 \end{pmatrix},$$

where $f = (f_1, f_2, f_3)^T$ is the known continuous vector function, $u = (u_1, u_2, u_3)^T$ is the unknown twice continuously differentiable vector function, $\varepsilon = 2R^{-1}h$, h is the shell half-thickness, R is the sphere radius, σ is Poisson's coefficient, E is Young's modulus.

We introduce the following spaces:

$L_2(\Omega)$ is the space of square-integrable functions in the domain Ω (Hilbert space);

$H = [L_2(\Omega)]^3$ is the Hilbert space with the scalar product

$$((u, v)) = (u_1, v_1) + (u_2, v_2) + (u_3, v_3),$$

and the norm

$$\|u\| = (\|u_1\|^2_{L_2} + \|u_2\|^2_{L_2} + \|u_3\|^2_{L_2})^{\frac{1}{2}},$$

where $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$ are the vector functions with components from $L_2(\Omega)$; $(.,.)$ and $\|\cdot\|_{L_2}$ are respectively the scalar product and the norm in the Hilbert space $L_2(\Omega)$;

$C^2(\overline{\Omega})$ is the space of twice continuously differentiable functions in the closure of the domain Ω ;

$[C^2(\overline{\Omega})]^3$ is the space of twice continuously differentiable vector functions in the domain $\overline{\Omega}$;

The definition domain of the operator A_0 is represented as follows:

$$D(A_0) = \{u \in [C^2\overline{\Omega}]^3 : u|_{\partial\Omega} = 0\}.$$

As is known, A_0 is a symmetric and positive definite operator (see [2], [3]). A_1 is a symmetric operator.

Instead of equation (1) we consider the equation:

$$(\widetilde{A_0} + \widetilde{A_1})u = f, \quad f \in H, \tag{3}$$

where $\widetilde{A_0}$ is the extension of the operator A_0 to the self-conjugate operator, and $\widetilde{A_1}$ is the closure of the operator A_0.

The following theorem is true.

Theorem. *Assume the following condition is fulfilled:*

$$\varepsilon \leq \frac{(1-2\sigma)\pi}{4} \cdot \sqrt{\frac{1-q_1}{1-\sigma}},$$

where $0 < q_1 < 1$. *Then the iterative process*

$$\widetilde{A_0} u_n = \widetilde{A_1} u_{n-1} + f \quad (n = 1, 2, \ldots) \tag{4}$$

is convergent for any initial vector $u_0 \in D(\widetilde{A_0})$ *and the following estimate is valid:*

$$\|\widetilde{A_0}^{1/2} u_* - \widetilde{A_0}^{1/2} u_n\| \leq q^n \|\widetilde{A_0}^{1/2} u_* - \widetilde{A_0}^{1/2} u_0\|, \tag{5}$$

where u_* *is an exact solution,* $q = (1+\lambda_1)^{-1}$,

$$\lambda_1 = max\left(\frac{\sqrt{2((1-q_1)\pi^2 + 2\varepsilon^2)}}{2\varepsilon c} - 1,\ 2(1-2\sigma)\sqrt{q_1}\right), \quad c = \frac{3-2\sigma}{1-2\sigma}.$$

To prove the theorem we need the following lemma.

Lemma. *The inequality*

$$((A_0 u, u)) + (1+\lambda_1)((A_1 u, u)) \geq 0, \quad \forall u \in D(A_0) \tag{6}$$

is valid.

Proof. We have:

$$A_0 u \pm (1-\lambda_1) A_0 u =$$

$$= -\begin{pmatrix} a\partial^2_{xx} + \partial^2_{yy} - \varepsilon^2 & 0 & 0 \\ 0 & \partial^2_{xx} + a\partial^2_{yy} - \varepsilon^2 & 0 \\ 0 & 0 & \partial^2_{xx} + \partial^2_{yy} - 4b\varepsilon^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\pm(1+\lambda_1) \begin{pmatrix} 0 & b\partial^2_{xy} & -\varepsilon c\partial_x \\ b\partial^2_{xy} & 0 & -\varepsilon c\partial_y \\ \varepsilon c\partial_x & \varepsilon c\partial_y & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$= -\begin{pmatrix} a\partial^2_{xx} u_1 + \partial^2_{yy} u_1 - \varepsilon^2 u_1 \pm (1+\lambda_1) b\partial^2_{xy} u_2 \mp (1+\lambda_1)\varepsilon c\partial_x u_3 \\ \partial^2_{xx} u_2 + a\partial^2_{yy} u_2 - \varepsilon^2 u_2 \pm (1+\lambda_1) b\partial^2_{xy} u_1 \mp (1+\lambda_1)\varepsilon c\partial_y u_3 \\ \partial^2_{xx} u_3 + \partial^2_{yy} u_3 - 4b\varepsilon^2 u_3 \pm (1+\lambda_1)\varepsilon c\partial_x u_1 \pm (1+\lambda_1)\varepsilon c\partial_y u_2 \end{pmatrix},$$

where

$$\partial_x = \frac{\partial}{\partial x}, \quad \partial_{xy}^2 = \frac{\partial^2}{\partial x \partial y}, \quad a = \frac{2(1-\sigma)}{1-2\sigma}, \quad b = \frac{1}{1-2\sigma}.$$

Let us calculate the scalar product in the left-hand side of the inequality (6). Obviously we have:

$$\begin{gathered}((A_0 u, u)) \pm (1+\lambda_1)((A_1 u, u)) \\ = -[a(\partial_{xx}^2 u_1, u_1) + (\partial_{yy}^2 u_1, u_1) - \varepsilon^2 (u_1, u_1) \\ \pm(1+\lambda_1) b(\partial_{xy}^2 u_2, u_1) \mp (1+\lambda_1)\varepsilon c(\partial_x u_3, u_1) \\ +(\partial_{xx}^2 u_2, u_2) + a(\partial_{yy}^2 u_2, u_2) - \varepsilon^2 (u_2, u_2) \\ \pm(1+\lambda_1) b(\partial_{xy}^2 u_1, u_2) \mp (1+\lambda_1)\varepsilon c(\partial_y u_3, u_2) \\ +(\partial_{xx}^2 u_3, u_3) + (\partial_{yy}^2 u_3, u_3) - 4b\varepsilon^2 (u_3, u_3) \\ \pm(1+\lambda_1)\varepsilon c(\partial_x u_1, u_3) \pm (1+\lambda_1)\varepsilon c(\partial_y u_2, u_3)].\end{gathered}$$

If $u \in D(A_0)$, then, using formula of integration by parts, we obtain:

$$\begin{gathered}((A_0 u, u)) \pm (1+\lambda_1)((A_1 u, u)) \\ = (a\|\partial_x u_1\|_{L_2}^2 + \|\partial_y u_1\|_{L_2}^2) + (\|\partial_x u_2\|_{L_2}^2 + a\|\partial_y u_2\|_{L_2}^2) \\ +(\|\partial_x u_3\|_{L_2}^2 + \|\partial_y u_3\|_{L_2}^2) + \varepsilon^2(\|u_1\|_{L_2}^2 + \|u_2\|_{L_2}^2 + 4b\|u_3\|_{L_2}^2) \\ \pm 2b(1+\lambda_1)(\partial_x u_1, \partial_y u_2) \pm 2(1+\lambda_1)\varepsilon c\cdot(\partial_x u_3, u_1) \pm 2(1+\lambda_1)\varepsilon c\cdot(\partial_y u_3, u_2). \quad (7)\end{gathered}$$

If we take into account that $a = b + 1$ and $(\partial_x u_1, \partial_y u_2) = (\partial_y u_1, \partial_x u_2)$, then from (7) we obtain:

$$\begin{gathered}((A_0 u, u)) \pm (1+\lambda_1)((A_1 u, u)) \\ = b(\|\partial_x u_1 \pm \partial_y u_2\|_{L_2}^2 + 4\varepsilon^2 \|u_3\|_{L_2}^2) \\ +[\|\partial_x u_3\|_{L_2}^2 + \varepsilon^2 \|u_1\|_{L_2}^2 \pm (1+\lambda_1) 2\varepsilon c \cdot (\partial_x u_3, u_1)] \\ +[\|\partial_y u_3\|_{L_2}^2 + \varepsilon^2 \|u_2\|_{L_2}^2 \pm (1+\lambda_1) 2\varepsilon c \cdot (\partial_y u_3, u_2)] \\ +[q_1(\|\partial_x u_1\|_{L_2}^2 + \|\partial_y u_2\|_{L_2}^2) \pm b\lambda_1(\partial_x u_1, \partial_y u_2)] \\ +[q_1(\|\partial_y u_1\|_{L_2}^2 + \|\partial_x u_2\|_{L_2}^2) \pm b\lambda_1(\partial_y u_1, \partial_x u_1)] \\ +(1-q_1)(\|\partial_x u_1\|_{L_2}^2 + \|\partial_y u_2\|_{L_2}^2) \\ +(1-q_1)(\|\partial_x u_2\|_{L_2}^2 + \|\partial_y u_2\|_{L_2}^2), \quad (8)\end{gathered}$$

where $0 < q_1 < 1$.

As is known, the inequality

$$\|\partial_x u_i\|^2_{L_2} + \|\partial_y u_i\|^2_{L_2} \geq \frac{\pi^2}{2}\|u_i\|^2_{L_2}, \quad i = 1, 2, 3 \tag{9}$$

is valid (see [3], p. 195).

According to Schwartz inequality and inequality (9), from (8) we obtain:

$$((A_0 u, u)) \pm (1 + \lambda_1)((A_1 u, u)) \geq b(\|\partial_x u_1 \pm \partial_y u_2\|^2_{L_2} + 4\varepsilon^2\|u_3\|^2_{L_2})$$

$$+\Big[\|\partial_x u_3\|^2_{L_2} + ((1 - q_1)\frac{\pi^2}{2} + \varepsilon^2)\|u_1\|^2_{L_2} - (1 + \lambda_1)2\varepsilon c \cdot \|\partial_x u_3\|_{L_2} \cdot \|u_1\|_{L_2}\Big]$$

$$+\Big[\|\partial_y u_3\|^2_{L_2} + ((1 - q_1)\frac{\pi^2}{2} + \varepsilon^2)\|u_2\|^2_{L_2} - (1 + \lambda_1)2\varepsilon c \cdot \|\partial_y u_3\|_{L_2} \cdot \|u_2\|_{L_2}\Big]$$

$$+\Big[q_1(\|\partial_x u_1\|^2_{L_2} + |\partial_y u_2\|^2_{L_2}) - b\lambda_1\|\partial_x u_1\|_{L_2} \cdot \|\partial_y u_2\|_{L_2}\Big]$$

$$+\Big[q_1(\|\partial_y u_1\|^2_{L_2} + |\partial_x u_2\|^2_{L_2}) - b\lambda_1\|\partial_y u_1\|_{L_2} \cdot \|\partial_x u_2\|_{L_2}\Big]. \tag{10}$$

It is obvious that on the right-hand side of inequality (10) the expressions in the square brackets are non-negative if the inequalities

$$(1 + \lambda_1)^2(\varepsilon c)^2 - ((1 - q_1)\frac{\pi^2}{2} + \varepsilon^2) \leq 0,$$

$$(b\lambda_1)^2 - 4q_1 \leq 0.$$

are satisfied.

The latter inequality holds when:

$$\lambda_1 \leq \frac{\sqrt{2((1 - q_1)\pi^2 + 2\varepsilon^2)} - 2\varepsilon c}{2\varepsilon c},$$

$$\lambda_1 \leq \frac{2\sqrt{q_1}}{b} = 2(1 - 2\sigma)\sqrt{q_1}.$$

In addition, it is obvious that the condition should be fulfilled:

$$2\varepsilon c \leq \sqrt{2((1 - q_1)\pi^2 + 2\varepsilon^2)},$$

from here

$$2\varepsilon^2(c - 1)(c + 1) \leq (1 - q_1)\pi^2.$$

If we make substitution $c = 2b + 1$, we obtain:

$$\varepsilon \leq \sqrt{\frac{(1 - q_1)\pi^2}{8b(b + 1)}} = \frac{(1 - 2\sigma)\pi}{4} \cdot \sqrt{\frac{1 - q_1}{1 - \sigma}}.$$

Therefore, if parameters ε and λ_1 satisfy conditions of the Theorem, then from (10) it follows:

$$((A_0u,u)) \pm (1+\lambda_1)((A_1u,u)) \geq b\|\partial_x u_1 + \partial_y u_2\|_{L_2}^2 + 4b\varepsilon^2\|u_3\|_{L_2}^2.$$

which clearly implies inequality (6).□

Proof of Theorem. Formula (4) clearly implies the dependence:

$$v_n = Sv_{n-1} + \widetilde{A_0}^{-1/2}f, \tag{11}$$

where $v_n = \widetilde{A_0}^{-1/2}u_n,\ \ S = -\widetilde{A_0}^{-1/2}\widetilde{A_1}\widetilde{A_0}^{-1/2}$.

From the **lemma** proved above it follows that:

$$|((\widetilde{A_1}u,u))| \leq \frac{1}{1+\lambda_1}((\widetilde{A_0}u,u)), \quad \forall u \in D(\widetilde{A_0}),$$

or, which is the same, that

$$-\frac{1}{1+\lambda_1}((\widetilde{A_0}u,u)) \leq ((\widetilde{A_1}u,u)) \leq \frac{1}{1+\lambda_1}((\widetilde{A_0}u,u)), \quad u \in D(A_0).$$

Hence we obtain:

$$-\frac{1}{1+\lambda_1}((v,v)) \leq ((Sv,v)) \leq \frac{1}{1+\lambda_1}((v,v)),$$

where $v = \widetilde{A_0}^{1/2}u$.

This implies that (see for example [4])

$$\|\widetilde{S}\| \leq \frac{1}{1+\lambda_1} < 1,$$

where $\widetilde{S}$ is an extension of S.

From this it obviously follows that $v_n = \widetilde{A_0}^{1/2}u_n$ is fundamental. Since $\widetilde{A_0}$ is positive definite, in turn it follows that the sequence u_n is fundamental.

Assume that $u_n \to u_*$. We will show that u_* is a solution of equation (3).

Let us note that operator $A_0 + A_1$ is positive definite (see [5]). Obviously, from here it follows that equation (3) has unique solution. In the work [6], for approximate solution of equation (3), it is proposed iterative process of type (4), where the first addend (the main operator) consists of operator of plane elasticity theory and the Laplacian, perturbed by operator $(-\gamma_0\varepsilon^2 I)$ (I is an identity operator, γ_0 a positive constant), and the second addend consists of the first order derivative with regards to spatial variables ($\pm\varepsilon\gamma_1$) with multiplier (γ_1 is also positive constant). It is proved that this iterative process is convergent and the limit vector satisfies equation (3). From this fact (taking into account uniqueness) it follows that u_* will be solution of equation (3). As regards to estimate (5), it is obtained in the usual way.

References

1. Vekua I. N. On the construction of approximate solutions of equations of the shallow spherical shell. Int. J. Solid Structures, 5 (1969), 991-1003.

2. Fichera G. Existence theorems in the theory of elasticity. Mir, Moscow (1974), 160 pp. (in Russian).

3. Rektoris K. Variational methods in mathematical physics and engineering. Moscow, Mir (1970) (in Russian).

4. Kato T., Perturbation theory for linear operators. Die grundlehren der mathematischen wissenschaften, Band 132 *Springer-Verlag New York, Inc., New York (1966) 592 pp.*

5. Galdava R., Rogava D., Rogava J. On the stability and convergence of a weighted three-layer semidiscrete scheme for I.Vekua's equations of a spherical shell. Proceedings of I.Vekua Institute of Applied Mathematics, vol.54-55 (2004-2005), 22-54.

6. Rogava D. On the Convergence of an Iteration Method for the System of I.Vekua's Equations with a Split Operator for a Spherical Shell. Bulletin of Georgian Academy of Sciences, 173, no. 1 (2006), 49-52.