THE FOURTH ORDER OF ACCURACY CRANC-NICKOLSON TYPE DECOMPOSITION SCHEME FOR MULTIDIMENSIONAL EVOLUTION PROBLEM

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Abstract

In the given work the evolution problem is discussed with the self-adjoint positive-definite operator, which can be represented as a sum of addends $m \geq 2$ (we call such case as multi-dimensional). In addition, each addend is also self-adjoint and positive-definite operator. For this problem there is constructed the fourth order accuracy rational decomposition scheme of sequential type. The explicit a priori estimate is obtained for the error of approximate solution.

Key words and phrases: Decomposition method, operator split, semigroup, Trotter formula, Cauchy abstract problem, rational approximation.

AMS subject classification: 65M12, 65M15, 65M55.

Introduction

Decomposition method is a general method for obtaining economical schemes for the solution to multidimensional problems of mathematical physics. This method allows to construct approximate solution to multidimensional problem of mathematical physics by means of combination of solutions to the corresponding one dimensional problems, numerical realization of which obviously takes less resources. Working in this direction have begun in the sixties of the twentieth century da continues intensively nowadays. Note, that decomposition schemes considered in the works, which were published till eighties of the last century, are of the first and second order of accuracy (for example, see [1-3] and the references therein).

Important results with regard to construction of higher order decomposition schemes are obtained in the works [4-7]. Automatically stable decomposition schemes with the third and fourth accuracy order are constructed and investigated in the works [8-12].

In the present work there is constructed the fourth order rational decomposition scheme of sequential type. The explicit a priori estimate is obtained for the error of approximate solution.

1 Statement of the problem and main result

Let us consider the Cauchy abstract problem in H Hilbert space:

$$\frac{du(t)}{dt}+Au(t)=f(t),\quad t>0,\quad u(0)=\varphi, \tag{1.1}$$

where A is a self-adjoint (generally unbounded), positive- definite operator with the definition domain $D(A)$, which is everywhere dense in H, $\overline{D(A)}=H,\quad A=A^*$ and

$$(Au,u)\geq a\left\|u\right\|^2,\quad \forall u\in D(A),\quad a=const>0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined correspondingly the norm and scalar product in H; φ is a given vector from H; $u(t)$ is a continuous and continuously differentiable, searched function with values in H; $f(t)\in C^1([0;\infty);H)$.

Let $A=A_1+...+A_m$, where $A_1,...,A_m$ are also self-adjoint positive-definite operators. The solution of the problem (1.1) is given by the following formula ([13]):

$$u(t)=U(t,A)\varphi+\int\limits_0^t U(t-s,A)f(s)ds, \tag{1.2}$$

where $U(t,A)=exp(-tA)$ is a strongly continuous semigroup.

As it is well-known, the main principle of decomposition method is splitting of the semigroup $U(t,A)$ by means of semigroups $U(t,A_j)$ $(j=1,...,m)$. In [12] there is constructed the following decomposition formula with the local accuracy of fifth order:

$$\begin{aligned}V(t) &= T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right)T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right)\\ &\quad\times T\left(t,\frac{\alpha}{4}\right)\overline{T}\left(t,\frac{\alpha}{4}\right)T\left(t,\frac{\overline{\alpha}}{4}\right)\overline{T}\left(t,\frac{\overline{\alpha}}{4}\right),\end{aligned} \tag{1.3}$$

where $\alpha=\frac{1}{2}\pm i\frac{1}{2\sqrt{3}}\quad\left(i=\sqrt{-1}\right)$ and where

$$\begin{aligned}T(t,\alpha) &= U(t,\alpha A_1)...U(t,\alpha A_{m-1})U(t,\alpha A_m),\\ \overline{T}(t,\alpha) &= U(t,\alpha A_m)...U(t,\alpha A_2)U(t,\alpha A_1).\end{aligned}$$

In [12] we have constructed the following semigroup approximation with the local accuracy of the fifth order:

$$W(t,A) = \left(I - \frac{\alpha}{2}tA\right)\left(I + \frac{\overline{\alpha}}{2}tA\right)^{-1}\left(I - \frac{\overline{\alpha}}{2}tA\right)\left(I + \frac{\alpha}{2}tA\right)^{-1}. \quad (1.4)$$

The approximation defined by formula (1.4) in the scalar case represents the Pade approximation for exponential function [15].

On the basis of formulas (1.3) and (1.4) we can construct the following decomposition formula:

$$\begin{aligned}\widetilde{V}(t) = {} & S\left(t, \frac{\overline{\alpha}}{4}\right)\overline{S}\left(t, \frac{\overline{\alpha}}{4}\right)S\left(t, \frac{\alpha}{4}\right)\overline{S}\left(t, \frac{\alpha}{4}\right) \\ & \times S\left(t, \frac{\alpha}{4}\right)\overline{S}\left(t, \frac{\alpha}{4}\right)S\left(t, \frac{\overline{\alpha}}{4}\right)\overline{S}\left(t, \frac{\overline{\alpha}}{4}\right). \end{aligned} \quad (1.5)$$

where

$$\begin{aligned} S(t,\alpha) &= W(t,\alpha A_1)\ldots W(t,\alpha A_{m-1})\,W(t,\alpha A_m), \\ \overline{S}(t,\alpha) &= W(t,\alpha A_m)\ldots W(t,\alpha A_2)\,W(t,\alpha A_1). \end{aligned}$$

In the present work, on the basis of formula (1.5), a decomposition scheme with the fourth order accuracy will be constructed for the solution of problem (1.1).

Let us introduce the following grid domain:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 0, 1, ..., \tau > 0\}.$$

According to formula (1.2), we have:

$$u(t_k) = U(\tau, A)u(t_{k-1}) + \int_{t_{k-1}}^{t_k} U(t_k - s, A)f(s)ds.$$

Let us use Simpson's formula and rewrite this formula in the following form:

$$\begin{aligned} u(t_k) &= U(\tau, A)\,u(t_{k-1}) + \frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right)f\left(t_{k-1/2}\right)\right. \\ &\quad \left. + U(\tau, A)\,f(t_{k-1})\right) + R_{5,k}(\tau), \\ u(t_0) &= \varphi, \quad k = 1, 2, \ldots, \end{aligned} \quad (1.6)$$

where $R_{5,k}(\tau)$ is a reminder term of Simpson's formula.

On the basis of formula (1.6), let us construct the following scheme:

$$
\begin{aligned}
u_k &= \widetilde{V}(\tau)\, u_{k-1} \\
&\quad + \frac{\tau}{6}\left(f(t_k) + 4\widetilde{V}\left(\frac{\tau}{2}\right) f\left(t_{k-1/2}\right) + \widetilde{V}(\tau)\, f(t_{k-1})\right), \qquad (1.7)\\
u_0 &= \varphi, \quad k = 1, 2, \ldots .
\end{aligned}
$$

Let us carry out calculation of the scheme (1.7) by the following algorithm:

$$
u_k = u_k^{(0)} + \frac{2\tau}{3} u_k^{(1)} + \frac{\tau}{6} f(t_k),
$$

where $u_{k,0}$ (in case of $m = 2$) is calculated by the scheme:

$$
\begin{aligned}
u_{k-8/9}^{(0)} &= W\left(\tau, \frac{\overline{\alpha}}{4} A_1\right)\left(u_{k-1} + \frac{\tau}{6} f(t_{k-1})\right) \quad u_{k-7/9}^{(0)} = W\left(\tau, \frac{\overline{\alpha}}{2} A_2\right) u_{k-8/9}^{(0)},\\
u_{k-6/9}^{(0)} &= W\left(\tau, \frac{1}{4} A_1\right) u_{k-7/9}^{(0)}, \quad u_{k-5/9}^{(0)} = W\left(\tau, \frac{\alpha}{2} A_2\right) u_{k-6/9}^{(0)},\\
u_{k-4/9}^{(0)} &= W\left(\tau, \frac{\alpha}{2} A_1\right) u_{k-5/9}^{(0)}, \quad u_{k-3.9}^{(0)} = W\left(\tau, \frac{\alpha}{2} A_2\right) u_{k-4/9}^{(0)},\\
u_{k-2/9}^{(0)} &= W\left(\tau, \frac{1}{4} A_1\right) u_{k-3/9}^{(0)}, \quad u_{k-1/9}^{(0)} = W\left(\tau, \frac{\overline{\alpha}}{2} A_2\right) u_{k-2/9}^{(0)},\\
u_k^{(0)} &= W\left(\tau, \frac{\alpha}{2} A_1\right) u_{k-1/9}^{(0)}, \quad u_0 = \varphi + \frac{\tau}{6} f(0), \qquad (1.8)
\end{aligned}
$$

and $u_{k,1}$ - by the scheme:

$$
\begin{aligned}
u_{k-8/9}^{(1)} &= W\left(\tau, \frac{\overline{\alpha}}{8} A_1\right) f\left(t_{k-1/2}\right) \quad u_{k-7/9}^{(1)} = W\left(\tau, \frac{\overline{\alpha}}{4} A_2\right) u_{k-8/9}^{(1)},\\
u_{k-6/9}^{(1)} &= W\left(\tau, \frac{1}{8} A_1\right) u_{k-7/9}^{(1)}, \quad u_{k-5/9}^{(1)} = W\left(\tau, \frac{\alpha}{4} A_2\right) u_{k-6/9}^{(1)},\\
u_{k-4/9}^{(1)} &= W\left(\tau, \frac{\alpha}{4} A_1\right) u_{k-5/9}^{(1)}, \quad u_{k-3.9}^{(1)} = W\left(\tau, \frac{\alpha}{4} A_2\right) u_{k-4/9}^{(1)},\\
u_{k-2/9}^{(1)} &= W\left(\tau, \frac{1}{8} A_1\right) u_{k-3/9}^{(1)}, \quad u_{k-1/9}^{(1)} = W\left(\tau, \frac{\overline{\alpha}}{4} A_2\right) u_{k-2/9}^{(1)},\\
u_k^{(1)} &= W\left(\tau, \frac{\alpha}{4} A_1\right) u_{k-1/9}^{(1)}, \qquad (1.9)
\end{aligned}
$$

Let us introduce the following notation:

$$
\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D(A);\\
\|\varphi\|_{A^2} &= \left\|A_1^2\varphi\right\| + \left\|A_2^2\varphi\right\| + \|A_1A_2\varphi\| + \|A_2A_1\varphi\|, \quad \varphi \in D\left(A^2\right),
\end{aligned}
$$

where $\|\cdot\|$ is a norm in H. $\|\varphi\|_{A^s}, \quad (s = 3, 4, 5)$ is defined analogously.

The following theorem takes place (below everywhere c denotes a positive constant).

Theorem. *Let the following conditions be satisfied:*

(a) $A_1, ..., A_m$ *are self-adjoint positive- definite operators and* $A = A_1 + ... + A_m$ *is also self-adjoint operator on* $D(A) = D(A_1) \cap ... \cap D(A_m)$*;*

(b) $U(s, A)\varphi \in D(A^5)$ *for any* $s \geq 0$*;*

(c) $f(t) \in C^4([0,\infty); X)$*;* $f(t) \in D(A^4)$, $f'(t) \in D(A^3)$, $f''(t) \in D(A^2)$, $f'''(t) \in D(A)$ *and* $U(s, A) f(t) \in D(A^4)$ *for any fixed* t *and* s $(t, s \geq 0)$.

Then the following estimate holds:

$$\begin{aligned}\|u(t_k) - u_k\| \leq\ & c t_k \tau^4 \left(\sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^5} + t_k \sup_{s,t\in[0,t_k]} \|U(s,A)f(t)\|_{A^5} \right. \\ & + \sup_{t\in[0,t_k]} \|f(t)\|_{A^4} + \sup_{t\in[0,t_k]} \left\|f'(t)\right\|_{A^3} + \sup_{t\in[0,t_k]} \left\|f''(t)\right\|_{A^2} \\ & \left. + \sup_{t\in[0,t_k]} \left\|f'''(t)\right\|_{A} + \sup_{t\in[0,t_k]} \left\|f^{(IV)}(t)\right\| \right).\end{aligned}$$

For proving this theorem we need the following lemma.

Lemma. [see [12]] *Let the following conditions be satisfied:*

(a) The operator A *satisfies the conditions of Theorem;*

(b) $f(t) \in C^4([0,\infty); X)$*, and* $f(t) \in D(A^4)$, $f^{(k)}(t) \in D(A^{4-k})$ $(k = 1, 2, 3)$ *for every fixed* $t \geq 0$.

Then the following estimate holds

$$\|R_{5,k}(\tau)\| \leq c\tau^5 \sum_{i=0}^{4} \max_{s\in[t_{k-1},t_k]} \left\|f^{(i)}(s)\right\|_{A^{4-i}}, \tag{1.10}$$

where

$$\begin{aligned}R_{5,k}(\tau) =\ & \int_{t_{k-1}}^{t_k} U(t_k - s, A) f(s)\, ds \\ & -\frac{\tau}{6}\left(f(t_k) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right)\right. \\ & + U(\tau, A) f(t_{k-1})) \end{aligned} \tag{1.11}$$

and where $f^{(0)}(s) = f(s)$.

Proof of the theorem. Let us return to the proof of the theorem.

Let us write formula (1.6) in the following form:

$$u(t_k) = U^k(\tau, A)\varphi + \sum_{i=1}^{k} U^{k-i}(\tau, A)\left(F_i^{(1)} + R_{5,k}(\tau)\right), \tag{1.12}$$

where

$$F_k^{(1)} = \frac{\tau}{6}\left(f\left(t_k\right) + 4U\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + U\left(\tau, A\right) f\left(t_{k-1}\right)\right). \tag{1.13}$$

Analogously let us present u_k as follows:

$$u_k = \widetilde{V}^k(\tau)\varphi + \sum_{i=1}^{k} \widetilde{V}^{k-i}(\tau) F_i^{(2)}, \tag{1.14}$$

where

$$F_i^{(2)} = \frac{\tau}{6}\left(f\left(t_k\right) + 4\widetilde{V}\left(\frac{\tau}{2}, A\right) f\left(t_{k-1/2}\right) + \widetilde{V}\left(\tau, A\right) f\left(t_{k-1}\right)\right). \tag{1.15}$$

Equalities (1.12) and (1.14) yield:

$$\begin{aligned} u(t_k) - u_k &= \left[U^k(\tau, A) - \widetilde{V}^k(\tau)\right]\varphi \\ &\quad + \sum_{i=0}^{k}\left[U^{k-i}(\tau, A) F_i^{(1)} - \widetilde{V}^{k-i}(\tau) F_i^{(2)}\right] \\ &\quad + \sum_{i=0}^{k} U^{k-i}(\tau, A) R_{k,5}\left(\tau\right) \\ &= \left[U^k(\tau, A) - \widetilde{V}^k(\tau)\right]\varphi + \sum_{i=1}^{k}\left[\left(U^{k-i}(\tau, A) - \widetilde{V}^{k-i}(\tau)\right) F_i^{(1)}\right. \\ &\quad \left. + \widetilde{V}^{k-i}(\tau)\left(F_i^{(1)} - F_i^{(2)}\right)\right] + \sum_{i=0}^{k} U^{k-i}(\tau, A) R_{5,k}\left(\tau\right). \end{aligned} \tag{1.16}$$

From formulas (1.13) and (1.15) we have:

$$\begin{aligned} F_k^{(1)} - F_k^{(2)} &= \frac{\tau}{6}\left(4\left(U\left(\frac{\tau}{2}, A\right) - \widetilde{V}\left(\frac{\tau}{2}\right)\right) f\left(t_{k-1/2}\right)\right. \\ &\quad \left. + \left(U\left(\tau, A\right) - \widetilde{V}\left(\tau, A\right)\right) f\left(t_{k-1}\right)\right). \end{aligned} \tag{1.17}$$

To estimate this difference, we need to estimate $W\left(\tau, \frac{\alpha}{4}A\right)$ and $W\left(\tau, \frac{\overline{\alpha}}{4}A\right)$. As is known, when the argument represents a self-adjoint bounded operator, the norm of the operator polynomial is equal to the C-norm of the corresponding scalar polynomial on the spectrum (see, e.g., [14] Chapter VII)

$$\left\| W\left(\tau, \frac{\alpha}{4}A\right)\right\| \leq \max_{x\geq 0}\left| \frac{1 - \frac{\alpha^2}{8}\tau x}{1 + \frac{\alpha^2}{8}\tau x} \frac{1 - \frac{\alpha\overline{\alpha}}{8}\tau x}{1 + \frac{\alpha\overline{\alpha}}{8}\tau x}\right|$$

$$
\begin{aligned}
&= \max_{x\geq 0}\left(\left|\frac{1-\frac{\alpha^2}{8}\tau x}{1+\frac{\alpha^2}{8}\tau x}\right|\left|\frac{1-\frac{1}{24}\tau x}{1+\frac{1}{24}\tau x}\right|\right) \\
&\leq \max_{x\geq 0}\left(\frac{\left(1-\frac{\alpha^2}{8}\tau x\right)\left(1-\frac{\overline{\alpha}^2}{8}\tau x\right)}{\left(1+\frac{\alpha^2}{8}\tau x\right)\left(1+\frac{\overline{\alpha}^2}{8}\tau x\right)}\right)^{1/2} \\
&= \max_{x\geq 0}\left(\frac{1-\frac{\alpha^2+\overline{\alpha}^2}{8}\tau x+\frac{\alpha^2+\overline{\alpha}^2}{64}\tau^2 x^2}{1+\frac{\alpha^2+\overline{\alpha}^2}{8}\tau x+\frac{\alpha^2+\overline{\alpha}^2}{64}\tau^2 x^2}\right)^{1/2} \\
&= \max_{x\geq 0}\left(\frac{1-\frac{1}{24}\tau x+\frac{1}{576}\tau^2 x^2}{1+\frac{1}{24}\tau x+\frac{1}{576}\tau^2 x^2}\right)^{1/2} \leq 1. \qquad (1.18)
\end{aligned}
$$

Analogously we obtain:

$$
\left\|W\left(\tau,\frac{\overline{\alpha}}{4}A\right)\right\| \leq 1. \qquad (1.19)
$$

From (1.18) and (1.19) we have:

$$
\left\|\widetilde{V}\left(\tau\right)\right\| \leq 1. \qquad (1.20)
$$

If we take into consideration that $W(\tau, A)$ approximates the semigroup $U(\tau, A)$ with fifth order local accuracy with regard to τ, with account of inequalities (1.18) and (1.19), than, analogously to the estimate for (1.3) (see [12]), we obtain:

$$
\left\|\left(U\left(\tau,A\right)-\widetilde{V}\left(\tau\right)\right)\varphi\right\| \leq c\tau^s\left\|\varphi\right\|_{A^s}, \quad \varphi\in D\left(A^s\right), \quad s=4,5. \qquad (1.21)
$$

With account of this inequality, from (1.17) follows:

$$
\left\|F_k^{(1)}-F_k^{(2)}\right\| \leq c\tau^5 \sup_{t\in[t_{k-1},t_k]}\left\|f(t)\right\|_{A^4}. \qquad (1.22)
$$

According to (1.21) we have:

$$
\begin{aligned}
&\left\|\sum_{i=1}^{k}\left(U^{k-i}(\tau,A)-\widetilde{V}^{k-i}(\tau)\right)F_i^{(1)}\right\| \\
\leq\ & ct_k^2\tau^4 \sup_{s,t\in[0,t_k]}\left\|U(s,A)f\left(t\right)\right\|_{A^5}. \qquad (1.23)
\end{aligned}
$$

From equality (1.16) according to inequalities (1.22), (1.23), (1.21), (1.20) and the condition *(a)* of the Theorem we obtain sought estimation ■

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