CALDERON-ZYGMUND TYPE SINGULAR INTEGRAL OPERATORS AND THE SCHWARZ PROBLEM FOR HIGHER-ORDER COMPLEX PARTIAL DIFFERENTIAL EQUATIONS

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Abstract

In this paper, we investigate the solvability of a higher order elliptic differential equation with the Schwarz conditions on the unit disc of the complex plane.
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1 Introduction

This article is a short survey of the recent studies of the authors. The part related with the norm estimates of a class of a singular integral operators in $L^p(\mathbf{D})$ (with the exception of the compactness result) has been published [1]. The rest of this survey [2] is devoted to the investigation of the solutions of Schwarz problem in $W^{p,k}(\mathbf{D})$ for a general linear elliptic complex partial differential equations of higher order in the unit disc and will be published elsewhere.

We study this problem by transforming it into a singular integral equation. In the next section, the statements of the problems are given. In Section 3 we give a review of the integral operators appearing in the article. The solvability of the problems with homogeneous and nonhomogeneous Schwarz conditions are considered in Section 4 and Section 5.

2 Statements of the Problems

We consider the following problems.

Problem A. Find $w \in W^{p,k}(\mathbf{D})$ as a solution to the k-th order complex differential equation

$$\frac{\partial^k w}{\partial \bar{z}^k}+\sum_{j=1}^{k} q_{1j}(z)\frac{\partial^k w}{\partial \bar{z}^{k-j}\partial z^j}+\sum_{j=1}^{k} q_{2j}(z)\frac{\partial^k \overline{w}}{\partial z^{k-j}\partial \bar{z}^j}+$$

$$+\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\frac{\partial^l w}{\partial \bar{z}^{l-m}\partial z^m}+b_{ml}(z)\frac{\partial^l \overline{w}}{\partial z^{l-m}\partial \bar{z}^m}\right]=f(z) \text{ in } \mathbf{D} \tag{2.1}$$

satisfying the homogeneous Schwarz boundary conditions

$$\mathrm{Re}\frac{\partial^l w}{\partial \bar{z}^l}=0 \text{ on } \partial\mathbf{D}\ , \qquad \mathrm{Im}\frac{\partial^l w}{\partial \bar{z}^l}(0)=0\ ,\ 0\le l\le k-1\ , \tag{2.2}$$

where

$$a_{ml}, b_{ml}\in L^p(\mathbf{D}), f\in L^p(\mathbf{D}), \tag{2.3}$$

and q_{1j} and q_{2j}, $j=1,\ldots,k$, are measurable bounded functions satisfying

$$\sum_{j=1}^{k}(|q_{1j}(z)|+|q_{2j}(z)|)\le q_0<1\ . \tag{2.4}$$

Problem B. Find $w\in W^{p,k}(\mathbf{D})$ as a solution to the k-th order complex differential equation (2.1) in $\mathbf{D}$, subject to (2.3) and (2.4), satisfying the nonhomogeneous Schwarz boundary conditions

$$\mathrm{Re}\frac{\partial^l w}{\partial \bar{z}^l}=\gamma_l \text{ on } \partial\mathbf{D}\ , \qquad \mathrm{Im}\frac{\partial^l w}{\partial \bar{z}^l}(0)=c_l\ ,\ 0\le l\le k-1\ , \tag{2.5}$$

where $\gamma_l\in C(\partial\mathbf{D};\mathbf{R}), c_l\in\mathbf{R}, 0\le l\le k-1$.

We should note that Begehr [4] considered the higher order differential equation

$$\frac{\partial^k w}{\partial \bar{z}^k}+q_1(z)\frac{\partial^k w}{\partial \bar{z}^{k-1}\partial z}+q_2(z)\overline{\frac{\partial^k w}{\partial \bar{z}^{k-1}\partial z}}$$

$$+\sum_{l=0}^{k-1}\sum_{j=0}^{l}\left(a_{lj}(z)\frac{\partial^l w}{\partial \bar{z}^{l-j}\partial z^j}+b_{lj}(z)\overline{\frac{\partial^l w}{\partial \bar{z}^{l-j}\partial z^j}}\right)=f(z) \tag{2.6}$$

with the homogeneous Schwarz conditions and he proved the solvability of the problem and gave the form of the solution.

Problem A and Problem B are studied by reducing them into singular integral equations. The idea of this method is based on the work of I. N. Vekua [14] for generalized analytic functions. In the next section we give a survey of the operators that will be used in the singular integral equation.

3 Singular Integral operators

The Pompeiu operator

$$Tf(z) := -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{\zeta - z}\,, \qquad z \in \mathbf{C},$$

and its weak derivative with respect to z,

$$\Pi f := -\frac{1}{\pi}\int\int_D f(\zeta)\frac{d\xi d\eta}{(\zeta - z)^2}\,, \qquad z \in \mathbf{C},$$

are the main tools of Vekua's theory of generalized analytic functions. Investigations on complex first order partial differential equations are based on the properties of these operators, see [7],[11]. Iterating the Pompeiu integral operator T with itself and with its conjugate leads to a hierarchy of integral operators $T_{m,n}$ given in [9], related to the differential operator $\partial_{\bar z}^m \partial_z^n$. These operators are important in dealing with the differential equations of higher order. The Pompeiu operator can be modified in accordance with some boundary condition by adding a holomorphic function. For the unit disc $\mathbf{D} = \{|z| < 1\}$, the operator $\widetilde{T}_1$, given in [5] defined by

$$\widetilde{T}_1 f(z) := -\frac{1}{2\pi}\int\int_{\mathbf{D}}\left(\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1 + z\bar\zeta}{1 - z\bar\zeta}\right) d\xi d\eta\,,$$

satisfying

$$\frac{\partial \widetilde{T_1} f}{\partial \bar z} = f \quad \text{in } \mathbf{D}$$

and the Schwarz conditions

$$\mathrm{Re}\widetilde{T}_1 f = 0 \quad \text{on } \partial\mathbf{D} \text{ and } \quad \mathrm{Im}\widetilde{T}_1 f(0) = 0\,.$$

Since the operator $\widetilde{T}_1$ satisfies the Schwarz conditions naturally, it is in the center of interest when dealing with the Schwarz problem for complex partial differential equations. Iterating this operator with itself by the rule $\widetilde{T}_k f(z) = \widetilde{T}_1(\widetilde{T}_{k-1} f(z))$ generates the operators

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!}\int\int_{\mathbf{D}}(\overline{\zeta - z} + \zeta - z)^{k-1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta + z}{\zeta - z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1 + z\bar\zeta}{1 - z\bar\zeta}\right] d\xi d\eta$$

for $k \in \mathbf{N}$ with $\widetilde{T}_0 f(z) = f(z)$, see [9], [3], [8] and [4]. These operators satisfy

$$\frac{\partial^l}{\partial \bar z^l}\widetilde{T}_k f = \widetilde{T}_{k-l} f,\ 1 \le l \le k, \tag{3.7}$$

$$\operatorname{Re} \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f = 0 \text{ on } \partial \mathbf{D},\ 0 \le l \le k-1, \tag{3.8}$$

$$\operatorname{Im} \frac{\partial^l}{\partial \bar{z}^l} \widetilde{T}_k f(0) = 0\ ,\ 0 \le l \le k-1\ , \tag{3.9}$$

see [3],[4],[5],[8]. On the other hand, $\partial_z^l \widetilde{T}_k$ is a weakly singular integral operator for $0 \le l \le k-1$, while

$$\Pi_k f(z) := \frac{\partial^k}{\partial z^k} \widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbf{D}} \left[\left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} \right.$$

$$\left. + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} \right] d\xi d\eta \tag{3.10}$$

is a Calderon-Zygmund type strongly singular integral operator. These operators are investigated by decomposing them into two parts as $\Pi_k = T_{-k,k} + P_k$, where

$$T_{-k,k} f(z) = \frac{(-1)^k k}{\pi} \int\int \mathbf{D} \left(\frac{\overline{\zeta - z}}{\zeta - z} \right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2} d\xi d\eta, \tag{3.11}$$

which is investigated extensively in [9]. Thus the boundedness of Π_k relies on the operator

$$P_k f(z) = \frac{(-1)^k k}{\pi} \int\int_{\mathbf{D}} \left(\frac{\overline{\zeta - z} + \zeta - z}{1 - z\bar{\zeta}} \bar{\zeta} - 1 \right)^{k-1} \frac{\overline{f(\zeta)}}{(1 - z\bar{\zeta})^2} d\xi d\eta\ . \tag{3.12}$$

Theorem 3.1 (1) *Let f be a complex valued function in $L^p(\mathbf{D})$ where $1 < p < \infty$. Then $P_k f$ also belongs to $L^p(\mathbf{D})$, and there exists a constant $N(k,p) > 0$ depending on p and k such that*

$$\|P_k f\|_{L^p(\mathbf{D})} \le N(k,p) \|f\|_{L^p(\mathbf{D})}$$

holds.

Remark 1

$$\begin{aligned} \|\Pi_k f\|_{L^p(\mathbf{D})} &\le \|T_{-k,k} f\|_{L^p(\mathbf{D})} + \|P_k f\|_{L^p(\mathbf{D})} \\ &\le (M(p) + N(k,p)) \|f\|_{L^p(\mathbf{D})} \\ &= C(k,p) \|f\|_{L^p(\mathbf{D})}\ . \end{aligned}$$

In the proof of the boundedness of the operators P_k, Schur's theorem [15] and Forelli-Rudin estimates [12] are used. Using the same technique of the boundedness proof we have obtained

Theorem 3.2 (1) Π_k *is a bounded operator on* $L^2(\mathbf{D})$ *with norm less than or equal to*

$$1 + k\Gamma\left(\frac{1}{2}\right)\sum_{m=0}^{k-1}\left(\begin{array}{c} k-1 \\ m \end{array}\right)\frac{\Gamma(m+\frac{1}{2})}{\Gamma^2(\frac{m+2}{2})} \ .$$

The following theorem gives the compactness of P_k.

Theorem 3.3 (2) *The operators* $P_k : L^{p_1}(\mathbf{D}) \to L^{p_2}(\mathbf{D})$ *are compact if* $1 < p_2 < \dfrac{2p_1}{p_1+1} < \infty$.

4 Solvability of Problem A

Lemma 1 (2) *The Schwarz problem (2.1) and (2.2) is equivalent to the singular integral equation*

$$(I + \hat{\Pi} + \hat{K})g = f \ , \tag{4.13}$$

where $w = \widetilde{T}_k g$,

$$\hat{\Pi}g = \sum_{j=1}^{k}(q_{1j}\Pi_j g + q_{2j}\overline{\Pi_j g}) \tag{4.14}$$

and

$$\hat{K}g = \sum_{l=0}^{k-1}\sum_{m=0}^{l}\left(a_{ml}\frac{\partial^m \widetilde{T}_{k-l+m}g}{\partial z^m} + b_{ml}\frac{\partial^m \overline{\widetilde{T}_{k-l+m}g}}{\partial \bar{z}^m}\right) \ . \tag{4.15}$$

In this section the solvability of singular integral equation (4.13) is discussed in L^p spaces.

The case $p > 2$

First, we derive a condition on q_0 to make the operator $I + \hat{\Pi}$ invertible.

Lemma 2 (2) *If*

$$q_0 \max_{1\le j\le k} \|\Pi_j\|_{L^p(\mathbf{D})} < 1 \tag{4.16}$$

then the operator $I + \hat{\Pi}$ *is invertible for* $p > 1$.

The compactness of the operator $\hat{K}$ is stated by Begehr in [4] and is given below.

Theorem 4.1 *Under the conditions (2.3), $\hat{K}$ is a compact operator in $L^p(\mathbf{D})$ for $p > 2$.*

The following theorem gives the solvability of the Problem A.

Theorem 4.2 (2) *If condition (4.16) is satisfied, then equation (2.1) with the conditions (2.2) has a solution of the form $w = \widetilde{T}_k g$, where $g \in L^p(\mathbf{D})$, $p > 2$, is a solution of the singular integral equation (4.13).*

The case $0 < p - 2 < \epsilon$

For this case, we separate the strongly singular part $\hat{\Pi}$ of the singular integral equation (4.13) as

$$\hat{\Pi} = \hat{T} + \hat{P}$$

with

$$\hat{T}g = \sum_{j=1}^{k} (q_{1j} T_{-j,j} g + q_{2j} \overline{T_{-j,j} g})$$

and

$$\hat{P}g = \sum_{j=1}^{k} (q_{1j} P_j g + q_{2j} \overline{P_j g}) ,$$

where the operators $T_{-k,k}$ and P_k are defined in (3.11) and (3.12), respectively. Then the integral equation (4.13) is rewritten in the form

$$(I + \hat{T} + \hat{P} + \hat{K})g = f \; . \tag{4.17}$$

Now we can state the following lemma.

Lemma 3 (2) *The operator $I + \hat{T}$ is invertible in $L^p(\mathbf{D})$ for $0 < p - 2 < \epsilon$.*

Next theorem gives the solvability of (4.13) for p sufficiently close to 2, where K will denote compact operator.

Theorem 4.3 (2) *If the inequality*

$$q_0 \max_{1 \le j \le k} \|P_j\|_{L^p(\mathbf{D})} \|(I + \hat{T})^{-1} - K_1\|_{L^p(\mathbf{D})} < 1 \tag{4.18}$$

is satisfied for some $K_1 \in K(L^p(\mathbf{D}))$, $0 < p - 2 < \epsilon$, then equation (2.1) with the boundary conditions (2.2) has a solution of the form $w = \widetilde{T}_k g$, where $g \in L^p(\mathbf{D})$ is a solution of the singular integral equation (4.13).

5 Solvability of Problem B

We decompose a solution of the equation (2.1) with the conditions (2.5) as $w = w_1 + w_2$, where w_1 is the solution of the problem

$$\frac{\partial^k w_1}{\partial \bar{z}^k} = 0 \ , \quad \mathrm{Re}\frac{\partial^l w_1}{\partial \bar{z}^l} = \gamma_l \text{ on } \partial\mathbf{D} \ , \qquad \mathrm{Im}\frac{\partial^l w_1}{\partial \bar{z}^l}(0) = c_l \ , \ 0 \le l \le k-1 \ , \tag{5.19}$$

and w_2 is a solution of the problem

$$\frac{\partial^k w_2}{\partial \bar{z}^k} + Lw_2 = f - Lw_1 \ , \tag{5.20}$$

$$\mathrm{Re}\frac{\partial^l w_2}{\partial \bar{z}^l} = 0 \ \text{ on } \partial\mathbf{D} \ , \qquad \mathrm{Im}\frac{\partial^l w_2}{\partial \bar{z}^l}(0) = 0 \ , \ 0 \le l \le k-1 \ , \tag{5.21}$$

where

$$Lw_i := \sum_{j=1}^{k} q_{1j}(z)\frac{\partial^k w_i}{\partial \bar{z}^{k-j}\partial z^j} + \sum_{j=1}^{k} q_{2j}(z)\frac{\partial^k \overline{w}_i}{\partial \bar{z}^j \partial z^{k-j}} +$$
$$+ \sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\frac{\partial^l w_i}{\partial \bar{z}^{l-m}\partial z^m} + b_{ml}(z)\frac{\partial^l \overline{w}_i}{\partial \bar{z}^m \partial z^{l-m}}\right] \ , \quad i = 1, 2.$$

Now, let us consider problem (5.19) first. The unique solution is given as

$$w_1 = i\sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\bar{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbf{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta} \ ,$$

see for example [6],[10]. Then the problem (5.20), (5.21) is equivalent to the singular integral equation

$$(I + \hat{\Pi} + \hat{K})g_1 = \tilde{f} \tag{5.22}$$

where

$$\tilde{f} = f - Lw_1 \ .$$

Thus, if g_1 is a solution of the singular integral equation (5.22), then $w_2 = \widetilde{T}_k g_1$ is a solution of the problem (5.20) and (5.21). Therefore, the following theorems can be stated.

Theorem 5.1 (2) *If condition (4.16) is satisfied, then equation (2.1) with the conditions (2.5) has a solution of the form*

$$w = \widetilde{T}_k g_1 + i\sum_{l=0}^{k-1}\frac{c_l}{l!}\ (z+\bar{z})^l + \sum_{l=0}^{k-1}\frac{(-1)^l}{2\pi i l!}\int_{\partial\mathbf{D}}\gamma_l(\zeta)\frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l\frac{d\zeta}{\zeta} \ , \tag{5.23}$$

where $g_1 \in L^p(\mathbf{D})$, $p > 2$, is a solution of the singular integral equation (5.22).

Theorem 5.2 (2) *If the inequality*

$$q_0 \max_{1\leq j\leq k} \|P_j\|_{L^p(\mathbf{D})} \|(I+\hat{T})^{-1} - K_1\|_{L^p(\mathbf{D})} < 1$$

is satisfied for some $K_1 \in K(L^p)$, $0 < p-2 < \epsilon$, then equation (2.1) with the boundary conditions (2.5) has a solution of the form

$$w = \widetilde{T}_k g_1 + i\sum_{l=0}^{k-1} \frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial \mathbf{D}} \gamma_l(\zeta) \frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}\ , \tag{5.24}$$

where $g_1 \in L^p(\mathbf{D})$ is a solution of the singular integral equation (5.22).

Now, we consider the differential equation given in [4] with the nonhomogeneous Schwarz conditions.

Theorem 5.3 (2) *The Schwarz problem*

$$Re \frac{\partial^l w}{\partial \overline{z}^l} = \gamma_l \quad on\ \partial\mathbf{D}, \qquad Im \frac{\partial^l w}{\partial \overline{z}^l}(0) = c_l, \quad 0 \leq l \leq k-1,$$

for the k-th order equation (2.6) is solvable for given $a_{ij}, b_{ij} \in L^p(\mathbf{D}), f \in L^p(\mathbf{D})$, q_1, q_2 measurable satisfying $|q_1(z)| + |q_2(z)| \leq q_0 < 1$, $q_0\|\Pi_1\|_{L^p(\mathbf{D})} < 1$ and
$\gamma_l \in C(\partial\mathbf{D}; \mathbf{R}), c_l \in \mathbf{R}, 0 \leq l \leq k-1$. Any solution is given in the form

$$w = \widetilde{T}_k g + i\sum_{l=0}^{k-1} \frac{c_l}{l!}\ (z+\overline{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial \mathbf{D}} \gamma_l(\zeta) \frac{\zeta+z}{\zeta-z}\ (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}\ ,$$

where $g \in L^p(\mathbf{D})$ is a solution of the singular integral equation

$$\begin{aligned} g + \quad & q_1(z)\Pi_1 g \quad + q_2(z)\overline{\Pi_1 g} + \\ + \quad & \sum_{l=0}^{k-1}\sum_{j=0}^{l} \left(a_{lj}(z) \frac{\partial^j \widetilde{T}_{k-l+j} g}{\partial z^j} + b_{lj}(z) \overline{\frac{\partial^j \widetilde{T}_{k-l+j} g}{\partial z^j}} \right) = \tilde{f}\ , \end{aligned}$$

in which

$$\begin{aligned} \tilde{f} = f &- q_1(z) \frac{\partial^k f_1}{\partial \overline{z}^{k-1} \partial z} - q_2(z) \frac{\partial^k \overline{f_1}}{\partial z^{k-1} \partial \overline{z}} \\ &- \sum_{l=0}^{k-1}\sum_{j=0}^{l} \left(a_{lj}(z) \frac{\partial^l f_1}{\partial \overline{z}^{l-j} \partial z^j} + b_{lj}(z) \overline{\frac{\partial^l f_1}{\partial \overline{z}^{l-j} \partial z^j}} \right) \\ &- \sum_{l=0}^{k-1}\sum_{j=0}^{l} (a_{lj}(z) + b_{lj}(z)) i \sum_{n=l}^{k-1} \frac{c_n}{(n-l)!} (z+\overline{z})^{n-l} \end{aligned}$$

with

$$f_1(z) = \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial \mathbf{D}} \gamma_l(\zeta) \frac{\zeta + z}{\zeta - z} \, (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta} \, .$$

References

1. Ü. Aksoy and A. O. Çelebi, *Norm estimates of a class of Calderon-Zygmund type Strongly Singular Integral Operators*, Integral Transforms Spec. Funct. 18 (2007), no. 1-2, 87-93

2. Ü. Aksoy and A. O. Çelebi, *Schwarz problem for higher-order complex elliptic partial differential equations*, Integral Transforms Spec. Funct. 19 (2008), no. 5-6, 413-428

3. H. Begehr, *Complex analytic methods for partial differential equations.* An Introductory Text. World Scientific, Singapore etc., 1994.

4. H. Begehr, *Iteration of the Pompeiu integral operator and complex higher order equations*, General Math.,7 (1999), 3–23.

5. H. Begehr, *Pompeiu operators in complex, hypercomplex, and clifford analysis*, Rev. Roumaine Math. Pures, Appl.,**46**(2001), I, 1-11.

6. H. Begehr, *Boundary value problems in complex analysis II*, Boletin de la Asosiación Matemática Venezolana, **XII** (2005), 223-241.

7. H. Begehr and R.P. Gilbert, *Transformations, transmutations and Kernel functions.* I, II. Longman, Harlow, 1992, 1993.

8. H. Begehr and G.N. Hile, *Higher order Cauchy-Pompeiu operator theory for complex and hypercomplex analysis*, Eds. E. Ramirez de Arellano et al. Contemp. Math. 212 (1998), 41-49.

9. H. Begehr and G.N. Hile, *A Hierarchy of integral operators*, Rocky Mountain J. Math., **27**(1997), 669-706.

10. H. Begehr and D. Schmersau, *The Schwarz problem for polyanalytic function*, Z. Anal. Anwendungen 24 (2005), no. 2, 341–351

11. A. Dzhuraev, *Methods of singular integral equations*, Longman, Harlow, 1992.

12. F. Forelli and W. Rudin, *Projections on spaces of holomorphic functions in unit balls*, Indiana Univ. Math. J., **24**(1974), 593-602.

13. H. A. Schwarz, *Zur integration der partiellen differentialgleichung* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = 0$, J. Reine Angew. Math., **74**(1872), 218-253.

14. I. N. Vekua, *Generalized analytic functions*, Pergamon Press, Oxford, 1962.

15. K. Zhu, *Operator theory in function spaces*, Marcel Dekker, New York, 1990.