THE BOUNDARY VALUE PROBLEMS OF MAGNETOELASTICITY DYNAMICS FOR IDEAL ELECTROCONDUCTIVE ISOTROPIC CIRCLE

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Abstract

Using the Laplace transform, the formulated problems are reduced to auxiliary problems for pseudooscillation equations. We have general representations of the solutions of these equations by means of metaharmonics functions. Representations of metaharmonics functions are used for the circle. Problems of pseudooscillations are solved approximately, deviations are estimated. Conditions providing the use of Laplaces inverted theorem are determined.

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Let us consider the case when strong, permanent magnetic $H^0 = (0, 0, H_3^0)$ field acts perpendicularly to elastic flatness of K^+ $(x_1^2 + x_2^2 \leq R^2)$ circle. In case of flat bulge, equations of dynamics are of the following type [1]:

$$\mu \Delta u + (\lambda + \mu + \alpha_0) \operatorname{grad} \operatorname{div} u - \rho \partial_t^2 u = 0, \qquad (1)$$

$$h(x,t) = \operatorname{rot}[u \times H^0], \quad e(x,t) = -\mu_e [\partial_t u \times H^0], \qquad (2)$$

$$j = \operatorname{rot} h, \quad x \in K^+,$$

where $\alpha_0 = \mu_e (H_3^0)^2$, $h = (0, 0, h_3)$, $h_3 = -H_3^0 \operatorname{div} u$, j-is electric current density, h and e-accordingly are fields of magnetic and electric intensities. $u(x,t) = (u_1, u_2)$-is a vector of displacement;

We consider problems for equations (1): Let us find such regular solution $u(x,t)$ of equation (1), which satisfies the initial conditions:

$$u(x,0) = 0, \quad \partial_t u(x,0) = 0 \qquad (3)$$

and one of the following conditions of K^+ circle on L circumference:

$$u_n^+(z,t) = f^{(1)}(z,t), \quad [Mu]_s^+ = f^{(2)}(z,t) - \text{problem (A)},$$

$$u_s^+(z,t) = f^{(1)}(z,t), \quad [Mu]_n^+ = f^{(2)}(z,t) - \text{problem (B)},$$

where $f^{(1)} = (f_1^{(1)}, f_2^{(1)}, 0)$ and $f^{(2)} = (f_1^{(2)}, f_2^{(2)}, 0)$-are functions given at boundary; $Mu = T(\partial_x, n)u + Q(h)$-is a vector of magnetoelasticity stress, $T(\partial_x, n)$-is a vector of elastic stress, $Q(h) = \frac{1}{2}\mu_e[H^0 \times [h \times n]]$, $n = (n_1, n_2, 0)$ is an exterior normal and $s = (-n_2, n_1, 0)$-is a tangent. Its obvious from (2) that if we know u vector components values, than we can determine h and e vectors and j.

Using the Laplace transform relative to time the formulated problems are reduced to auxiliary problems for pseudooscillation equations:

$$\mu\Delta\widetilde{u} + (\lambda + \mu + \alpha_0)\,\mathrm{grad}\,\mathrm{div}\,\widetilde{u} - \rho\tau^2\widetilde{u} = 0, \tag{4}$$
$$\widetilde{h}(x,t) = \mathrm{rot}[\widetilde{u} \times H^0], \quad \widetilde{e}(x,t) = -\mu_e\tau[\widetilde{u} \times H^0], \quad x \in K^+$$

and on the L circumference we will have:

$$(A)_\tau : \widetilde{u}_n^+(z,\tau) = \widetilde{f}^{(1)}(z,\tau), \quad \{M\widetilde{u}(z,\tau)\}_s^+ = \widetilde{f}^{(2)}(z,\tau); \tag{5}$$
$$(B)_\tau : \widetilde{u}_s^+(z,\tau) = \widetilde{f}^{(1)}(z,\tau), \quad \{M\widetilde{u}(z,\tau)\}_n^+ = \widetilde{f}^{(2)}(z,\tau); \tag{6}$$

If we act with div operation on equation (4) we will obtain:

$$(\Delta + \omega_1^2)\varphi_1 = 0, \tag{7}$$

where

$$\omega_1^2 = \frac{\rho\tau^2}{\lambda + 2\mu + \alpha_0}, \quad \varphi_1 = \mathrm{div}\,\widetilde{u} \tag{8}$$

By acting with rot on the (4) we obtain:

$$(\Delta + \omega_2^2)\varphi_2 = 0, \tag{9}$$

where

$$\omega_2^2 = -\frac{\rho\tau^2}{\mu}, \quad \varphi_2 = \mathrm{rot}\,\widetilde{u}. \tag{10}$$

From the (4) we obtain: $\widetilde{u} = C_1\,\mathrm{grad}\,\mathrm{div}\,\widetilde{u} - C_2\,\mathrm{rot}\,\mathrm{rot}\,\widetilde{u}$, where

$$C_1 = \frac{\lambda + 2\mu + \alpha_0}{\rho\tau^2}, \quad C_2 = -\frac{\mu}{\rho\tau^2}. \tag{11}$$

By using the (8) and the (10):

$$\widetilde{u} = C_1\,\mathrm{grad}\,\varphi_1 + C_2\,\mathrm{rot}\,\varphi_2, \tag{12}$$

where φ_1 and φ_2 are metaharmonics functions: they satisfy (7) and (9) equations. By verifying we are convinced that the (12) satisfies equation (4).

We know that in K^+ circle [2] $\varphi_k(z,\tau)$ metaharmonics function is represented in the following way:

$$\varphi_k(z,t)=\sum_{m=1}^{\infty} J_m(\omega_k r)(A_{mk}\cos m\psi + B_{mk}\sin m\psi), \tag{13}$$

where $x=(r,\psi)$, J_m-is Bessel's function with complex argument, A_{mk}, B_{mk}-are the constants to be found, $k=1,2$.

$(A)_\tau$ and $(B)_\tau$ problems have the only solution [3]. Let us use the known formulae:

$$\partial_1 = n_1\partial_r - \frac{n_2}{r}\partial_\psi, \quad \partial_2 = n_2\partial_r - \frac{n_1}{r}\partial_\psi. \tag{14}$$

From the (12) we obtain:

$$\widetilde{u}_n = C_1\partial_r\varphi_1 + \frac{C_2}{r}\partial_\psi\varphi_2, \quad \widetilde{u}_s = \frac{C_1}{r}\partial_\psi\varphi_1 - C_2\partial_r\varphi_2, \tag{15}$$

$$\{M(\widetilde{u})\}_n = \Big(\lambda + 2\mu + \frac{1}{2}\alpha_0\Big)\partial_r\widetilde{u}_n - \frac{1}{r}\Big(\lambda + \frac{1}{2}\alpha_0\Big)\partial_\psi\widetilde{u}_s,$$

$$\{M(\widetilde{u})\}_s = \mu\Big(\partial_r\widetilde{u}_s + \frac{1}{r}\partial_\psi\widetilde{u}_n\Big), \tag{16}$$

where $\alpha_0=\mu_e(H_3^0)^2$, $\widetilde{u}_n$ and $\widetilde{u}_s$ are determined by (15) formulae.

$(A)_\tau$ problem. In the (5) decompose the functions $\widetilde{f}^{(1)}$ (analogously $\widetilde{f}^{2}$) in Fourier series (e.g. [4]) and separate the particular sum from these series:

$$\widetilde{f}^{(k)}(z,\tau) = \frac{1}{2}\alpha_{0k}(\tau) + \tag{17}$$

$$+\sum_{m=1}^{m_0}[\alpha_{mk}(\tau)\cos m\psi + \beta_{mk}(\tau)\sin m\psi] + M_{m_0k}(z,\tau), \quad k=1,2,$$

where

$$\alpha_{mk}(\tau) = \frac{1}{\pi}\int_0^{2\pi}\widetilde{f}^{(k)}(\varphi,\tau)\cos m\varphi d\varphi$$

$$\beta_{mk}(\tau) = \frac{1}{\pi}\int_0^{2\pi}\widetilde{f}^{(k)}(\varphi,\tau)\sin m\varphi d\varphi, \quad 0\le\varphi\le 2\pi,$$

$M_{m_0k}(z,\tau) = \sum\limits_{m=m_0+1}^{\infty}[\alpha_{mk}(\tau)\cos m\psi + \beta_{mk}(\tau)\sin m\psi]$-is an error which, while $\widetilde{f}^{(k)}\in C^2(L)$, will be estimated (e.g. [4]):

$$|M_{m_0}(z,\tau)| < \frac{C'}{m_0^{\frac{3}{2}}}, \quad C' = \text{const}. \tag{18}$$

$M_{m_0}(z,\tau) \equiv \max|M_{m_0k}(z,\tau)|$, m_0-is a sufficiently big number. For the big m ($m = m_0+1,\ldots$) because of the M_{m_0k} error slightness we can remove it.

Insert representation (13) in the (15), the (16). Regarding the (17) and passing on boundary, when $r \to R$. For each m towards the A_{mk} and B_{mk} values ($k = 1,2$) we obtain a system of algebraic equations:

$$C_1\omega_1 J_0'(\omega_1 R)A_{01} = \frac{1}{2}\alpha_{01}, \quad -C_2\mu\omega_2^2 J_0''(\omega_2 R)A_{02} = \frac{1}{2}\alpha_{02}, \tag{19}$$

$$C_1\omega_1 J_m'(\omega_1 R)\begin{pmatrix} A_{m1} \\ B_{m1}\end{pmatrix} + \frac{C_2}{R} m J_m(\omega_2 R)\begin{pmatrix} B_{m2} \\ -A_{m2}\end{pmatrix} = \begin{pmatrix} \alpha_{m1} \\ \beta_{m1}\end{pmatrix},$$

$$C_1\Big[\frac{2\mu m\omega_1}{R}J_m'(\omega_1 R) - \frac{\mu m}{R^2}J_m(\omega_1 R)\Big]B_{m1} -$$

$$-C_2\Big[\mu\omega_2^2 J_m''(\omega_2 R) + \frac{\mu m^2}{R^2}J_m(\omega_2 R)\Big]A_{m2} = \alpha_{m2},$$

$$C_1\Big[-\frac{2\mu m\omega_1}{R}J_m'(\omega_1 R) + \frac{\mu m}{R^2}J_m(\omega_1 R)\Big]A_{m1} -$$

$$-C_2\Big[\mu\omega_2^2 J_m''(\omega_2 R) + \frac{\mu m^2}{R^2}J_m(\omega_2 R)\Big]B_{m2} = \beta_{m2}, \; m = 1,2,\ldots,m_0, \tag{20}$$

where $J_m'(\zeta)$ is a derivative of $J_m(\zeta)$ by ζ argument:

$$J_m'(\omega_k r) = \frac{1}{\omega_k}\partial_r J_m(\omega_k r), \quad J_m''(\omega_k r) = \frac{1}{\omega_k^2}\partial_r^2 J_m(\omega_k r).$$

from the (19) we obtain:

$$A_{01} = \frac{\alpha_{01}}{2C_1\omega_1 J_0'(\omega_1 R)}, \quad A_{02} = -\frac{\alpha_{02}}{2C_2\mu\omega_2^2 J_0''(\omega_2 R)}. \tag{21}$$

From the (20) we can see that it is possible to separate a system towards the unknowns A_{m1} and B_{m2}. The main determinant of which is:

$$\Delta_m' = -C_1C_2\mu\Big\{\omega_1 J_m'(\omega_1 R)\Big[\omega_2^2 J_m''(\omega_2 R) + \frac{m^2}{R^2}J_m(\omega_2 R)\Big] +$$

$$+\frac{m^2}{R^2}\Big[-2\omega_1 J_m'(\omega_1 R) + \frac{1}{R}J_m(\omega_1 R)\Big]J_m(\omega_2 R)\Big\}.$$

We have a system towards the unknowns A_{m2} and B_{m1}, main determinant of which is:

$$\Delta_m'' = -C_1C_2\mu\Big\{\omega_1 J_m'(\omega_1 R)\Big[\omega_2^2 J_m''(\omega_2 R) + \frac{m^2}{R^2}J_m(\omega_2 R)\Big] -$$

$$-\frac{m^2}{R^2}\Big[-2\omega_1 J_m'(\omega_1 R) + \frac{1}{R}J_m(\omega_1 R)\Big]J_m(\omega_2 R)\Big\}$$

Δ'_m and Δ''_m determinants differ from zero because of the only solution of the problem.

Let us insert the found A_{m1}, B_{m1}, A_{m2}, B_{m2} values of the systems (19) and (20) in the (13), then in the (12). We will obtain problem $(A)_\tau$ solution. Reasoning analogously we will obtain problem $(B)_\tau$ solution.

Let us require the agreement conditions to be completed.

$$\partial_t^k f^{(i)}(z,0) = 0, \quad k = 0,1,2,3, \quad i = 1,2 \tag{22}$$

and simultaneously for the big t the conditions are satisfied:

$$|\partial_t^k u(x,t)| < Me^{\xi_0 t}, \quad |\partial_t^q f^{(i)}(z,t)| < Me^{\xi_0 t}, \tag{23}$$

where $\xi_0 \geq 0$, $M > 0$-are constants, $k = 0,1,2$; $q = 0,1,2,3,4$. In these conditions by analyzing the A_{mj}, B_{mj} solutions of system (20), on the basis of asymptotical conditions of Bessel's functions (e.g. [5]),

$$J_m(\zeta) \sim \sqrt{\frac{2}{\pi\zeta}} \cos\left(\zeta - \frac{m\pi}{2} - \frac{\pi}{4}\right)$$

we conclude that from the conditions $f^{(i)} \in C^4$ $t \geq 0$ and $f^{(i)} \in C^2(L)$ occurs the following estimation for $\widetilde{u}(x,t)$ vector.

$$|\widetilde{u}(x,t)| \leq \frac{C}{|\tau|^4}, \quad C = \text{const} \tag{24}$$

for every t equally toward each x.

In these conditions we may use the Laplace inverted transform. The original is given with the following formula:

$$u(x,t) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{\tau t} \widetilde{u}(x,\tau) \partial\tau,$$

where $\operatorname{Re}\tau = \xi > \xi_0 \geq 0$, ξ_0-is an original growth indicator. The condition (24) is also sufficient for the existence of $\partial_t u(x,t)$ and $\partial_t^2 u(x,t)$ originals.

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