FRACTIONAL-STEP DIFFERENCE SCHEMES FOR MULTIDIMENSIONAL PARABOLIC TYPE EQUATION

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Abstract

An algorithm is given for the first time for constructing simple, absolutely approximating and absolutely stable schemes for the multidimensional parabolic equation. The stability is investigated by using the method of harmonic analysis.

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Introduction

For multidimensional parabolic type equation the construction of absolutely stable economical schemes having complete approximation is a problematical question in the case $p > 2$. First such type scheme for $p = 2$, $a = 0$ have been written by American scientific Peaceman, Douglas and Rachford in 1955 (see [1,2]). In the present paper it is given the generalization of these results for the equation (1) if $p > 2$ and $a \neq 0$.

We note that the fractional-step schemes not having complete approximation introduce additional complications relatively to the boundary conditions. This reflects on the exactness of schemes. Here represented schemes don't have such deficiency.

1 Statement of the problem

Let us consider the first initial-boundary problem in the cylinder Q_T for the equation

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = Lu + f, \quad Lu = \sigma^2 \sum_{i=1}^{p} \frac{\partial^2 u}{\partial x_i^2} - au + f, \\
&(x,t) \in Q_T = G \times (0 \le t \le T), \quad u(x,0) = u_0(x), \quad u(x,t)\big|_{\Gamma} = 0, \qquad (1.1)\\
&x = (x_1, x_2, \ldots, x_p) \in \overline{G}, \quad \overline{G} = G + \Gamma, \\
&\qquad p \text{ - dimensional cube}, \quad 0 \le x_i \le 1, \quad i = \overline{1,p}.
\end{aligned}
$$

2 Fractional-step schemes for the problem (1.1)

Let us divide integrable domain $Q_T = \overline{G} \times [0,T]$ in the elementary cells by planes $x_i = kh_i$, $k = 0, 1, \ldots, N_1$ and $t_j = j\tau$, $\tau > 0$, $j = 0, 1, \ldots, N_0$, $\overline{\omega}_h = \{x_i = (i_1 h_1, \ldots, i_p h_p) \in G\}$ is the cubic net with the step h_i, $\overline{\omega}_\tau = \{t_j = j\tau,\ j = 0, 1, \ldots, N_0\}$ is the cete on the segment $0 \le t \le T$ with the step $\tau = \frac{T}{N_0}$.

In difference schemes relative to a right hand f and a solution u we remain identical notations. Simply the corresponding discrete values are taken in the base knot $(x_{1i}, x_{2i}, \ldots, x_{pi}, t_j)$.

The symmetric scheme for the problem (1.1) has the following form

$$
\begin{aligned}
\frac{u^{n+k} - u^{n+k-2}}{2\tau} &= 0,5\, p\, \Delta_{kk}(u^{n+k} - u^{n+k-2}) + \sum_{i=1}^{p} \Delta_{ii}\, u^{n+k-2} \\
&\quad -0,5\, a(u^{n+k} - u^{n+k-2}) + f^{n+k-1}, \qquad (2.1)\\
&(x_i, t_j) \in \overline{\omega}_h \times \overline{\omega}_\tau, \quad u^{n-1}(x_i, 0) = u_0(x_i), \\
&u^{n+k}(x_i, t_j)\big|_{\Gamma_h} = 0, \qquad k = \overline{1,p},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta_{ii} u = \sigma^2 u_{x_i \overline{x}_i} &= \frac{\sigma^2}{h_i^2}\big[u(x_1, \ldots, x_i + h_i, \ldots, x_p, t_j) \\
&- 2u(x_1, \ldots, x_p, t_j) + u(x_1, \ldots, x_i - h_i, \ldots, x_p, t_j)\big] \approx \sigma^2 \frac{\partial^2 u}{\partial x_i^2}\,.
\end{aligned}
$$

Let us write out the fractional-step scheme which corresponds to the symmetric scheme (2.1)

$$\frac{u^{n+\frac{k}{p}} - u^{n+\frac{k-1}{p}}}{\frac{1}{p}\,\tau} = 0,5\,p\,\Delta_{kk}\big(u^{n+\frac{k}{p}} - u^{n+\frac{k-1}{p}}\big) + \sum_{i=1}^{p} \Delta_{ii}\, u^{n+\frac{k-1}{p}} - 0,5\,a\big(u^{n+\frac{k}{p}} - u^{n+\frac{k-1}{p}}\big) + f^{n+\frac{2k-1}{p}}, \quad k=\overline{1,p}. \tag{2.2}$$

3 Investigation of the scheme (2.2)

Let us introduce the space H of net functions which are defined on the net $\overline{\omega}_h$ and vanished on the Γ_h, with the inner product

$$(u,v) = \sum_{x\in\omega_h} u(x)\,v(x)\,h_1\cdots h_p = \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2-1}\cdots\sum_{i_p=1}^{N_p-1} u(i_1h_1,\ldots,i_ph_p)\,v(i_1h_1,\ldots,i_ph_p)\,h_1\cdots h_p$$

and norm $\|u\| = \sqrt{(u,u)}$.

Let us define $A = -\sum\limits_{i=1}^{p}\Delta_{ii}$. The operator A is selfadjoint and positive in H. Square of norm in the energetic space H_A has the form

$$\|u\|_A^2 = \sum_{i_1=1}^{N_1}\sum_{i_2=1}^{N_2-1}\cdots\sum_{i_p=1}^{N_p-1}\big(u_{\overline{x}_1}(i_1h_1,\ldots,i_ph_p)\big)^2 h_1\cdots h_p + \cdots + \sum_{i_1=1}^{N_1-1}\sum_{i_2=1}^{N_2-1}\cdots\sum_{i_{p-1}=1}^{N_p-1}\sum_{i_p=1}^{N_p}\big(u_{\overline{x}_p}(i_1h_1,\ldots,i_ph_p)\big)^2 h_1\cdots h_p$$

or

$$\|u\|_A^2 = \|u_{\overline{x}_1}\|_1^2 + \cdots + \|u_{\overline{x}_p}\|_p^2.$$

Consider $u=u(t)$, $t\in\omega_\tau$ as an abstract function with the values in the space H. For the investigation of the scheme (2.2) we assume that $f=0$ and exclude the intermediate values $u^{n+\frac{k}{p}}$, $k=1,2,\ldots,p-1$. Initially we rewrite the scheme (2.2) in the form

$$(A_k + B_1)\,u^{n+\frac{k}{p}} = (A_k - B_1 + C_1)\,u^{n+\frac{k-1}{p}}, \tag{3.1}$$

where

$$A_k = E - 0,5\,\tau\,\Delta_{kk}, \quad B_1 = 0,5\,\frac{\tau}{p}\,a\,E, \quad C_1 = \frac{\tau}{p}\sum_{i=1}^{p}\Delta_{ii}, \quad k=\overline{1,p}.$$

Equivalent to the (3.1) homogeneous scheme has the form

$$\begin{aligned}(A_1+B_1)(A_2+B_1)\cdots(A_p+B_1)\,u^{n+1}\\ =(A_1-B_1+C_1)(A_2-B_1+C_1)\cdots(A_p-B_1+C_1)\,u^n.\end{aligned} \tag{3.2}$$

If we introduce the notations (3.1) in (3.2) and expend to the powers τ, obtain

$$\begin{aligned}\frac{u^{n+1}-u^n}{\tau}&=(\Delta_{11}+\cdots+\Delta_{pp})\,\frac{u^{n+1}+u^n}{2}-a\,\frac{u^{n+1}+u^n}{2}\\ &-\frac{\tau^2}{4}\,(\Delta_{11}\Delta_{22}+\cdots+\Delta_{p-1,p-1}\Delta_{pp})\,\frac{u^{n+1}-u^n}{\tau}+o(\tau^2).\end{aligned} \tag{3.3}$$

Let us write (3.3) in the canonical form

$$B\,u_t+\widetilde{A}\,u=f,\quad t_j\in\omega_\tau, \tag{3.4}$$

where

$$B=E+0,5\,\tau\,A+0,5\,a\tau\,E,\quad \widetilde{A}=A+aE\quad(\widetilde{A}=-L_h).$$

From (3.4) follows that

$$B\geq 0,5\,\tau\,\widetilde{A}. \tag{3.5}$$

This means that the scheme (3.4) is stable in the space H_A.

Let us verify the inequality (3.5)

$$B-0,5\,\tau\,\widetilde{A}=E+0,5\,\tau\,A+0,5\,\tau\,aE-0,5\,\tau(A+aE)=E>0$$

The scheme (2.2) has the exactness $o(\tau^2+|h|^2)$. This follows immediately from (3.3).

Theorem. *Let us assume that the condition* (3.5) *holds. Then the fractional-step scheme* (2.2) *is stable relative to the right-hand relative to the solution of the problem* (1.1) *is valid the following a priori estimate*

$$\|u_{n+1}\|_A\leq\|u_0\|_A+\sum_{k=0}^{n}\tau\left\|B^{-1}f_k\right\|_A$$

and since B is selfadjoiut positive operator, we have

$$\|u_{n+1}\|_B\leq\|u_0\|_B+\sum_{k=0}^{n}\tau\,\|f_k\|_{B^{-1}}\,.$$

The proof is similar to the Theorem 4 (see [3], Ch. VI, §2).

Remark. If we apply the method of harmonic analysis by investigation of the fractional step scheme (2.2), the equation of dispersion obtains the following form

$$\Big(1+\frac{p}{2}\,a_i+\frac{1}{2}\,a\Big)\rho_i^2+\Big(\sum_{i=1}^{p} a_i-\frac{p}{2}\,a_i-1+\frac{1}{2}\,a\Big)=0, \quad i=\overline{1,p},$$

where

$$u^n=\rho^n\cdot e^{i(k_1x_1+\cdots+k_px_p)}, \quad \rho^n=e^{\omega\, n\,\tau}, \quad i=\sqrt{-1},$$
$$a_i=8\,\frac{\sigma^2\tau}{h_i^2}\,\sin^2\frac{k_ih_i}{2}$$

(see [4]). The following condition of stability holds

$$|\rho|=|\rho_1\cdots\rho_p|\le 1,$$

where

$$\rho_i=\sqrt{\frac{1+\frac{p}{2}\,a_i-\sum\limits_{i=1}^{p}a_i-\frac{a}{2}}{1+\frac{p}{2}\,a_i+\frac{a}{2}}}\,, \quad i=\overline{1,p}.$$

Really, if $\frac{\tau}{h_i^2}\to\infty$ ($a_i=\text{const}=c$, $i=\overline{1,p}$), we obtain for any i, that

$$|\rho|=\sqrt{\left|\frac{1-\frac{1}{2}(pc+a)}{1+\frac{1}{2}(pc+a)}\right|^p}\le 1.$$

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