HIGH ORDER OF ACCURACY DECOMPOSITION SCHEME FOR MULTIDIMENSIONAL HYPERBOLIC EQUATION

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(Received: 25.04.07; accepted: 28.09.07)

Abstract

In the present work, on the basis of rational splitting of cosine operator-function, there is constructed fourth order accuracy decomposition scheme for multidimensional hyperbolic equation, when the main operator is self-adjoint and positively. Stability of the constructed scheme is shown and the error of approximate solution is estimated

Key words and phrases: Decomposition method, Operator split, Abstract hyperbolic equation.

AMS subject classification: 65M12, 65M15, 65M55

1 Introduction

As it is known, the solution of Cauchy problem for an abstract hyperbolic equation can be given by means of sine and cosine operator functions, where square root from the main operator is included in the argument. Using this formula, for the equally distanced values of the time variable, the precise three-layer semi-discrete scheme can be constructed, whose transition operator is a cosine operator function. Main purpose of the work is to construct decomposition scheme for abstract hyperbolic equation by means of the above-mentioned scheme basing on splitting of cosine-operator function. Splitting of cosine operator-function can be carried out using cosine-operator functions, as well as using rational operator-functions. Schemes of rational splitting have important practical value, as, using them, one can carry out numerical calculations.

D. Gordeziani and A. Samarskii in the works [1] - [3] constructed and investigated first and second order precision decomposition schemes for hyperbolic equation. Qin Sheng, Voss David A., Khaliq Abdul Q. M. in the work [4] constructed second order precision decomposition scheme for sin-Gordon equation. It has to be pointed out that these authors constructed the scheme using exponential splitting and then obtained the corresponding rational splitting using Pade approximation.

In the present work, on the basis of rational splitting of cosine operator-function, there is constructed fourth order accuracy decomposition scheme for hyperbolic equation, when the main operator is self-adjoint positively defined and is represented as a sum of two addends. Stability of the constructed scheme is shown and the error of approximate solution is estimated.

2 Statement of the Problem and Rational Decomposition Scheme

Let us consider the Cauchy problem for abstract hyperbolic equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + Au\,(t) = 0, \quad t \in [0, T]\,, \tag{2.1}$$

$$u\,(0) = \varphi_0, \quad \frac{du\,(0)}{dt} = \varphi_1. \tag{2.2}$$

where A is a self-adjoint (A does not depend on t), positively defined (generally unbounded) operator with the definition domain $D\,(A)$, which is everywhere dense in H, i.e $\overline{D\,(A)} = H, \quad A = A^*$ and

$$(Au, u) \geq a\,\|u\|^2\,, \quad \forall u \in D\,(A)\,, \quad a = const > 0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined correspondingly the norm and scalar product in H; φ_0 and φ_1 are given vectors from H; $u\,(t)$ is a continuous, twice continuously differentiable, searched function with values in H.

It is known that if $\varphi_0 \in D\,(A)\,,\ \varphi_1 \in D\left(A^{1/2}\right)$, then there exists such twice continuously differentiable function $u\,(t)$, which satisfies equation (2.1) and initial conditions (2.2) (see [5], Chapter III, ♮1). In this case the solution is given by the following formula:

$$u(t) = \cos\left(tA^{1/2}\right)\varphi_0 + A^{-1/2}\sin\left(tA^{1/2}\right)\varphi_1, \tag{2.3}$$

where operator functions $\cos\left(tA^{1/2}\right)$ and $\sin\left(tA^{1/2}\right)$ are defined by Euler generalization formulas:

$$\begin{aligned}\cos\left(tA^{1/2}\right) &= \frac{1}{2}\left(e^{-it\sqrt{A}} + e^{it\sqrt{A}}\right),\\ \sin\left(tA^{1/2}\right) &= \frac{1}{2i}\left(e^{it\sqrt{A}} - e^{-it\sqrt{A}}\right),\end{aligned}$$

where $\left\{e^{\pm it\sqrt{A}}\right\}$ is a unitary group of operators generated by operators $\left(\pm iA^{1/2}\right)$.

It is proved, that there exists a limit $\lim\limits_{n\to\infty}\left(I\pm\frac{t}{n}iA^{1/2}\right)^{-n}\varphi$ (I is a unit operator), for any $\varphi\in H-$ and this limit is defined as $e^{\pm it\sqrt{A}}\varphi$ (see [6], Chapter IX).

Let $A=A_1+A_2+...+A_m$, where A_i $(i=1,2,...,m)$ are self-adjoint, positively defined operators.

Let us introduce a grid set:

$$\omega_\tau=\left\{t_k=k\tau,\quad k=0,1,...n,\quad n>1,\quad \tau=\frac{T}{n}\right\}.$$

From formula (2.3) it can be easily obtained the following three-point recurrent relation:

$$u(t_{k+1})=2\cos\left(\tau A^{1/2}\right)u(t_k)-u\left(t_{k-1}\right). \tag{2.4}$$

Let us construct decomposition scheme using the formula (2.4):

$$u_{k+1} = V\left(\tau\right)u_k-u_{k-1},\quad k=1,...,n-1, \tag{2.5}$$

$$u_0 = \varphi_0,\quad u_1=\frac{1}{2}\left(V\left(\tau\right)\varphi_0+\tau V\left(\frac{\tau}{\sqrt{3}}\right)\varphi_1\right), \tag{2.6}$$

where

$$\begin{aligned} V^{(m)}\left(\tau\right) &= \frac{2}{m+2}\Bigg[V_0^{(m)}\left(\tau;A_1,...A_m\right)+V_0^{(m)}\left(\tau;A_m,...A_1\right) \\ &\quad +\sum_{j=1}^{m}\left(I+\lambda\tau^2A_j\right)^{-1}\Bigg], \qquad (2.7)\\ V_0^{(m)}\left(\tau;A_1,...A_m\right) &= \left(I+\alpha\tau^2A_1\right)^{-1}...\left(I+\alpha\tau^2A_m\right)^{-1} \\ &\quad \times\left(I+\overline{\alpha}\tau^2A_m\right)^{-1}...\left(I+\overline{\alpha}\tau^2A_1\right)^{-1}, \end{aligned}$$

where $\lambda=\frac{m+2}{2}-\frac{\sqrt{m+2}}{\sqrt{6}}$, $\quad \alpha=\frac{\sqrt{m+2}}{4\sqrt{6}}\pm i\sqrt{\frac{m+2}{24}+4\lambda^2}$, $\overline{\alpha}$ is a conjugate of α.

We declare function u_k as an approximation of $u\left(t\right)$ in $t=t_k$ node.

In order to conduct numerical calculations of the scheme (2.5)-(2.6), it is necessary to inverse $I+\gamma\tau^2A_j$ $(j=1,2...,m,\quad \gamma=\lambda,\alpha,\overline{\alpha})$ operator, which is equivalent to solving of the following equation:

$$\varphi+\gamma\tau^2A_j\varphi=f,$$

where φ is unknown function and f is a given function.

3 Stability of the Rational Decomposition Scheme

To investigate stability of the scheme (2.5)-(2.6) we need the following lemma (see [7]).

Lemma 3.1 *Let the Recurrent relation*

$$u_{k+1} = Lu_k - Su_{k-1} + f_k$$

be given, where L *and* S *are the commutative operators acting in the linear space* X; u_0, u_1 *and* f_k *are the given vectors from this space. Then the following formula is valid:*

$$u_{k+1} = U_k(L,S)\,u_1 - SU_{k-1}(L,S)\,u_0 + \sum_{i=1}^{k} U_{k-i}(L,S)\,f_i, \tag{3.8}$$

where the operator polynomials $U_k(L,S)$ *are satisfy the following relation*

$$\begin{aligned} U_{k+1}(L,S) &= LU_k(L,S) - SU_{k-1}(L,S), \quad k = 1,2,..., \\ U_0(L,S) &= I, \quad U_1(L,S) = L. \end{aligned} \tag{3.9}$$

Note that (3.8) can be easily proved using method of induction.

In previous works, using formula (3.8), we have investigated three-layer semi-discrete schemes for abstract parabolic and hyperbolic equations (see [7], [8]).

Let us continue investigation of stability of the scheme (2.5)-(2.6). The following theorem takes place.

Theorem 3.1 *Suppose* A_1 *and* A_2 *are self-adjoint positively defined operators. Then for the scheme (2.5)-(2.6) the following estimate is valid:*

$$\|u_k\| \le \|\varphi_0\| + \nu\,\|\varphi_1\|, \quad k = 1,...,n,$$

where $\nu = \left(1 + \tau^2\nu_0\right)/\sqrt{2\nu_0}$, ν_0 *is minimal of lower boundaries of operators* A_1 *and* A_2.

proof.

According to formula (3.8), we have

$$u_{k+1} = U_k(L,I)\,u_1 - U_{k-1}(L,I)\,u_0, \tag{3.10}$$

where $L = V(\tau)$. Substituting the value of u_1 into (3.10), we obtain:

$$u_{k+1} = \left(\frac{1}{2}LU_k(L,I) - U_{k-1}(L,I)\right)\varphi_0 + \frac{1}{2}\tau U_k(L,I)\,V\left(\frac{\tau}{\sqrt{3}}\right)\varphi_1. \tag{3.11}$$

Let us consider scalar polynomial $U_k(x,1)$ corresponding to operator polynomial $U_k(L,I)$. It is important that the polynomials $U_k(2x,1)$ are the second kind Chebyshev polynomials, for which the following representation is valid (see e.g. [9], Chapter II)

$$U_k(2x,1) = \frac{\sin((k+1)\arccos x)}{\sqrt{1-x^2}}, \quad x \in]-1,1[.$$

Hence it follows that

$$U_k(x,1) = \frac{2\sin\left((k+1)\arccos \frac{x}{2}\right)}{\sqrt{4-x^2}}, \quad x \in]-2,2[. \tag{3.12}$$

Therefore we obtain the following well-known estimate:

$$|U_k(x,1)| \leq \frac{2}{\sqrt{4-x^2}}, \quad x \in]-2,2[. \tag{3.13}$$

Let us estimate the norm of the operator $\left(I+\alpha\tau^2A_1\right)^{-1}$. As, due to conditions of the theorem, A_1 is self-adjoint and positively defined operator, we have:

$$\begin{aligned}
\left\|\left(I+\alpha\tau^2A_i\right)^{-1}\right\| &= \sup_{x\in[\nu_0,+\infty)} \frac{1}{|1+\alpha\tau^2x|} \\
&= \sup_{x\in[\nu_0,+\infty)} \frac{1}{\sqrt{1+(\alpha+\overline{\alpha})\tau^2x+\alpha\overline{\alpha}\tau^4x^2}} \\
&= \sup_{x\in[\nu_0,+\infty)} \frac{1}{\sqrt{1+\frac{\sqrt{m+2}}{4\sqrt{6}}\tau^2x+\left(\frac{m+2}{48}+\lambda^2\right)\tau^4x^2}} \\
&\leq \frac{1}{1+\frac{1}{2\sqrt{6}}\tau^2\nu_0}.
\end{aligned} \tag{3.14}$$

Analogously we obtain:

$$\left\|\left(I+\overline{\alpha}\tau^2A_i\right)^{-1}\right\| \leq \frac{1}{1+\frac{1}{2\sqrt{6}}\tau^2\nu_0}, \tag{3.15}$$

$$\left\|\left(I+\lambda\tau^2A_i\right)^{-1}\right\| \leq \frac{1}{1+\tau^2\nu_0}. \tag{3.16}$$

From the estimates (3.14) and (3.15) it follows:

$$\|V_0(\tau;A_1,...,A_m)\| \leq \frac{1}{\left(1+\frac{1}{4}\tau^2\nu_0\right)^{2m}} \leq \frac{1}{1+\tau^2\nu_0}. \tag{3.17}$$

Analogously we obtain:

$$\|V_0(\tau;A_m,...,A_1)\| \leq \frac{1}{1+\tau^2\nu_0}. \tag{3.18}$$

From (2.7), taking into account (3.16), (3.17) and (3.18), we obtain:

$$\|V(\tau)\| \leq \frac{2}{1+\tau^2\nu_0}. \tag{3.19}$$

As $V(\tau)$ is self-adjoint operator, from (3.19) it follows:

$$Sp(V(\tau)) \subset [-\nu_1, \nu_1], \tag{3.20}$$

where $\nu_1 = 2/\left(1+\tau^2\nu_0\right)$.

Let us estimate the norm of the operator $\tau U_k(L,I)$. As is known, when the argument represents a self-adjoint bounded operator, the norm of the operator polynomial is equal to the C-norm of the corresponding scalar polynomial on the spectrum (see, e.g., [10] Chapter VII). Due to this fact, from (3.13) with account of (3.20) we obtain

$$\tau\|U_k(L,I)\| = \tau \max_{x\in Sp(L)} |U_k(x,1)| \leq \tau \max_{x\in[-\nu_1,\nu_1]} \frac{2}{\sqrt{4-x^2}} = \frac{2\tau}{\sqrt{4-\nu_1^2}} \leq \nu. \tag{3.21}$$

Now let us estimate the norm of the operator $\frac{1}{2}LU_k(L,I) - U_{k-1}(L,I)$. The scalar polynomial $U_k(x,1)$ satisfies the following recurrent relation:

$$\begin{aligned} U_{k+1}(x,1) &= xU_k(x,1) - U_{k-1}(x,1), \quad k=1,2,..., \\ U_0(x,1) &= 1, \quad U_1(x,1) = x. \end{aligned} \tag{3.22}$$

Due to recurrent relation (3.22) and formula (3.12), we have:

$$\begin{aligned} \frac{1}{2}xU_k(x,1) - U_{k-1}(x,1) &= \frac{1}{2}\left[(xU_k(x,1) - U_{k-1}(x,1)) - U_{k-1}(x,1)\right] \\ &= \frac{1}{2}\left[U_{k+1}(x,1) - U_{k-1}(x,1)\right] \\ &= \frac{\sin\left((k+2)\arccos\frac{x}{2}\right) - \sin\left(k\arccos\frac{x}{2}\right)}{\sqrt{4-x^2}} \\ &= \frac{2\cos\left((k+1)\arccos\frac{x}{2}\right)\sin\left(\arccos\frac{x}{2}\right)}{\sqrt{4-x^2}} \\ &= \frac{2\cos\left((k+1)\arccos\frac{x}{2}\right)\sqrt{1-\frac{x^2}{4}}}{\sqrt{4-x^2}} \\ &= \cos\left((k+1)\arccos\frac{x}{2}\right), \quad x\in[-2,2]. \end{aligned}$$

Hence we obtain

$$\left|\frac{1}{2}xU_k(x,1) - U_{k-1}(x,1)\right| \leq 1, \quad x\in[-2,2]. \tag{3.23}$$

Analogously to (3.21), according to the inequality (3.23) we have:

$$\left\| \frac{1}{2} L U_k\left(L,1\right) - U_{k-1}\left(L,1\right) \right\| \leq 1. \tag{3.24}$$

From (2.7) the following estimate follows:

$$\left\| V\left(\frac{\tau}{\sqrt{3}}\right) \right\| \leq 2. \tag{3.25}$$

From (3.11), taking into account (3.21), (3.24) and (3.25), we obtain the proving inequality. ■

Now let us show that the scheme (2.5)-(2.6) remains stable after small perturbation of the operator $V\left(\tau\right)$. With this purpose, along with the scheme (2.5)-(2.6), we consider the following scheme:

$$\widetilde{u}_{k+1} = \widetilde{V}\left(\tau\right)\widetilde{u}_k - \widetilde{u}_{k-1}, \quad k = 1, ..., n-1, \tag{3.26}$$

$$\widetilde{u}_0 = \widetilde{\varphi}_0, \quad \widetilde{u}_1 = \frac{1}{2}\left(\widetilde{V}\left(\tau\right)\widetilde{\varphi}_0 + \tau\widetilde{V}\left(\frac{\tau}{\sqrt{3}}\right)\widetilde{\varphi}_1\right), \tag{3.27}$$

where $\widetilde{V}\left(\tau\right)$ is a bounded operator in H, $\widetilde{\varphi}_0$ and $\widetilde{\varphi}_1$ are the given vectors from H.

The following theorem takes place.

Theorem 3.2 *If* $\left\| V\left(\tau\right) - \widetilde{V}\left(\tau\right) \right\| \leq \varepsilon\tau^2$, $\varepsilon = const > 0$, *then the estimate is valid:*

$$\left\| u_{k+1} - \widetilde{u}_{k+1} \right\| \leq \varepsilon\tau\nu \sum_{i=1}^{k} \exp\left(\varepsilon\nu t_{k-i}\right)\delta_{i-1} + \delta_k, \quad k = 1, ..., n-1,$$

where

$$\delta_k = \left\|\varphi_0 - \widetilde{\varphi}_0\right\| + \nu\left\|\varphi_1 - \widetilde{\varphi}_1\right\| + \frac{1}{2}\varepsilon\nu\tau\left(\left\|\widetilde{\varphi}_0\right\| + \frac{1}{3}\tau\left\|\widetilde{\varphi}_1\right\|\right) + \varepsilon\nu t_k\left(\left\|\varphi_0\right\| + \nu\left\|\varphi_1\right\|\right),$$

u_k *and* $\widetilde{u}_k$ *are solutions of the systems (2.5)-(2.6) and (3.26)-(3.27), respectively.*

proof.

From (2.5) and (3.26) we have:

$$u_{k+1} - \widetilde{u}_{k+1} = V\left(\tau\right)\left(u_k - \widetilde{u}_k\right) - \left(u_{k-1} - \widetilde{u}_{k-1}\right) + \left(V\left(\tau\right) - \widetilde{V}\left(\tau\right)\right)\widetilde{u}_k.$$

Hence, using the formula (3.8), we obtain:

$$\begin{aligned} u_{k+1} - \widetilde{u}_{k+1} &= U_k(L,I)(u_1 - \widetilde{u}_1) - U_{k-1}(L,I)(u_0 - \widetilde{u}_0) \\ &+ \sum_{i=1}^{k} U_{k-i}(L,I)\left(V(\tau) - \widetilde{V}(\tau)\right)\widetilde{u}_i. \end{aligned}$$

Due to formulas (2.6) and (3.27), we obtain:

$$\begin{aligned} u_{k+1} - \widetilde{u}_{k+1} &= \left(\frac{1}{2}LU_k(L,I) - U_{k-1}(L,I)\right)(\varphi_0 - \widetilde{\varphi}_0) \\ &+ \frac{1}{2}\tau U_k(L,I)V\left(\frac{\tau}{\sqrt{3}}\right)(\varphi_1 - \widetilde{\varphi}_1) \\ &+ \frac{1}{2}U_k(L,I)\left[\left(V(\tau) - \widetilde{V}(\tau)\right)\widetilde{\varphi}_0\right. \\ &\left.+ \tau\left(V\left(\frac{\tau}{\sqrt{3}}\right) - \widetilde{V}\left(\frac{\tau}{\sqrt{3}}\right)\right)\widetilde{\varphi}_1\right] \\ &+ \sum_{i=1}^{k} U_{k-i}(L,I)\left(V(\tau) - \widetilde{V}(\tau)\right)\widetilde{u}_i. \end{aligned} \tag{3.28}$$

From (3.28), according to inequalities (3.21), (3.24) and (3.25) and conditions of the theorem, we have:

$$\|u_{k+1} - \widetilde{u}_{k+1}\| \le \delta + c\sum_{i=1}^{k}\|\widetilde{u}_i\| \le \delta + c\sum_{i=1}^{k}\|u_i\| + c\sum_{i=1}^{k}\|u_i - \widetilde{u}_i\|, \tag{3.29}$$

where $c = \varepsilon\tau\nu$ and

$$\delta = \|\varphi_0 - \widetilde{\varphi}_0\| + \nu\|\varphi_1 - \widetilde{\varphi}_1\| + \varepsilon\nu\tau\left(\frac{1}{2}\|\widetilde{\varphi}_0\| + \frac{1}{6}\tau\|\widetilde{\varphi}_1\|\right).$$

From (3.29), with account of the estimate obtained in theorem 3.2, we have:

$$\varepsilon_{k+1} \le c\sum_{i=1}^{k}\varepsilon_i + \delta_k, \tag{3.30}$$

where $\varepsilon_i = \|u_i - \widetilde{u}_i\|$ and

$$\delta_k = \delta + \varepsilon\nu t_k\left(\|\varphi_0\| + \nu\|\varphi_1\|\right).$$

Using induction method, from (3.30) we obtain (discrete analog of Gronwall's lemma):

$$\varepsilon_{k+1} \le c(1+c)^{k-1}\varepsilon_1 + c\sum_{i=1}^{k-1}(1+c)^{k-i-1}\delta_i + \delta_k.$$

Hence, taking into account that $\varepsilon_1 \leq \delta_0$ and $(1+c)^k \leq \exp(\varepsilon\nu t_k)$, we obtain the inequality under proof. ■

Result: If $\|\varphi_0 - \widetilde{\varphi}_0\| \to 0$, $\|\varphi_1 - \widetilde{\varphi}_1\| \to 0$ and $\varepsilon \to 0$, then $\|u_k - \widetilde{u}_k\| \to 0$, $k = 1, ..., n$.

4 Estimate of Error of the Approximated Solution

We will need natural degrees of the operator $A = A_1 + A_2 + ... + A_m \quad (A^s, \quad s = 2, 3)$. In case of two addends ($m = 2$) they are defined as follows:

$$A^2 = \left(A_1^2 + A_2^2\right) + (A_1A_2 + A_2A_1),$$

$$A^3 = \left(A_1^3 + A_2^3\right) + \left(A_1^2A_2 + ... + A_2^2A_1\right) + (A_1A_2A_1 + A_2A_1A_2),$$

Analogously is defined A^s ($s = 2, 3$) when $m > 2$.

Obviously, the domain $D(A^s)$ of the operator A^s is the intersection of the domains of its addends.

Let us introduce the following definitions:

$$\|\varphi\|_A = \|A_1\varphi\| + ... + \|A_m\varphi\|, \quad \varphi \in D(A),$$

$$\|\varphi\|_{A^2} = \sum_{i,j=1}^{m} \|A_iA_j\varphi\|, \quad \varphi \in D\left(A^2\right),$$

where $\|\cdot\|$ is a norm in H, similarly is defined $\|\varphi\|_{A^3}$.

The following theorem takes place:

Theorem 4.1 *Let the following conditions be fulfilled:*

(a) $\lambda = \frac{m+2}{2} - \frac{\sqrt{m+2}}{\sqrt{6}}, \quad \alpha = \frac{\sqrt{m+2}}{4\sqrt{6}} \pm i\sqrt{\frac{m+2}{4\cdot 24} + \frac{\lambda^2}{2}}$;

(b) $A, A_1, A_2, ..., A_m$ *are self-adjoint, positively defined (generally unbounded) operators;*

(c) $\varphi_0 \in D\left(A^3\right)$, $\varphi_1 \in D\left(A^{2+1/2}\right)$.

Then for error of approximate solution obtained by scheme (2.5)-(2.6), the following estimate holds:

$$\|u(t_k) - u_k\| \quad \leq \quad c\tau^4 \left(\|\varphi_1\|_{A^2} + \tau\|\varphi_0\|_{A^3} + t_k \max_{1\leq i\leq k} \|u(t_i)\|_{A^3}\right).$$

proof.

Let us note that if $\varphi_0 \in D\left(A^3\right)$ and $\varphi_1 \in D\left(A^{2+1/2}\right)$, then from formula (2.3) automatically follows that $u(t) \in D\left(A^3\right)$ for every $t \geq 0$.

According to the following formula (see. [6], p. 603):

$$A\int_r^t e^{-sA}ds = e^{-rA} - e^{-tA}, \quad 0 \le r \le t,$$

we can obtain the following expansion:

$$e^{(-tA)} = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + (-A)^k \int_0^t \int_0^{s_1} ... \int_0^{s_{k-1}} e^{(-sA)} ds ds_{k-1} ... ds_1.$$

Using this formula we obtain the following expansion:

$$\cos\left(\tau A^{1/2}\right) = \sum_{i=0}^{k} (-1)^i \frac{\tau^{2i}}{(2i)!} A^i + R_k(\tau, A), \tag{4.31}$$

where $R_k(\tau, A)$ is a residual member, for which the following estimation is true:

$$\|R_k(\tau, A)\varphi\| \le \frac{1}{(2k+2)!}\tau^{2k+2}\|\varphi\|_{A^{k+1}}, \quad \varphi \in D\left(A^{k+1}\right). \tag{4.32}$$

We denote an error of the approximate solution at $t = t_k$ by z_k, $z_k = u(t_k) - u_k$. Due to formulas (2.4) and (2.5), we have:

$$z_{k+1} = V(\tau) z_k - z_{k-1} + R(\tau) u(t_k), \tag{4.33}$$

where

$$R(\tau) = 2\cos\left(\tau A^{1/2}\right) - V(\tau). \tag{4.34}$$

Using induction method, the following expansion can be obtained:

$$\left(I + \tau^2 A\right)^{-1} = \sum_{i=0}^{k} (-1)^i \tau^{2i} A^i + \widetilde{R}_k(\tau, A), \tag{4.35}$$

where

$$\widetilde{R}_k(\tau, A) = (-1)^k \tau^{2k+2}\left(I + \tau^2 A\right)^{-1} A^{k+1}. \tag{4.36}$$

It is obvious that for residual member of $\widetilde{R}_k(\tau, A)$ the following estimate is valid:

$$\left\|\widetilde{R}_k(\tau, A)\varphi\right\| \le \tau^{2k+2}\|\varphi\|_{A^{k+1}}, \quad \varphi \in D\left(A^{k+1}\right). \tag{4.37}$$

Let us estimate the operator $R(\tau)$. We decompose the operator $V(\tau)$ from right to left using the formula (4.35) in the way that each residual member be of sixth degree respect to τ. Therefore we obtain.

$$\begin{aligned} V(\tau) &= 2I - \tau^2 \frac{2(2\alpha + 2\overline{\alpha} + \lambda)}{m+2} \sum_{j=1}^{m} A_j \\ &\quad + \tau^4 \left(\frac{4\left(\alpha^2 + \alpha\overline{\alpha} + \overline{\alpha}^2\right) + 2\lambda^2}{m+2} \sum_{j=1}^{m} A_j^2 + \right. \\ &\quad \left. \frac{2(\alpha + \overline{\alpha})^2}{m+2} \sum_{j,k=1\ (j<>k)}^{m} A_k A_j \right) + R_V(\tau), \end{aligned} \tag{4.38}$$

where for residual member we have:

$$\|R_V(\tau)\varphi\| \le c\tau^6 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{4.39}$$

Due to theorem condition (a), parameters α and λ satisfy the following equalities:

$$\begin{aligned} 2(\alpha + \overline{\alpha}) + \lambda &= \frac{m+2}{2}, \\ (\alpha + \overline{\alpha})^2 - \alpha\overline{\alpha} + \frac{\lambda^2}{2} &= \frac{m+2}{48}, \\ \alpha + \overline{\alpha} &= \frac{\sqrt{m+2}}{2\sqrt{6}}. \end{aligned}$$

from here we have

$$\begin{aligned} \lambda &= \frac{m+2}{2} - \frac{\sqrt{m+2}}{\sqrt{6}}, \\ \alpha\overline{\alpha} &= \frac{m+2}{48} + \frac{\lambda^2}{2}, \\ \alpha + \overline{\alpha} &= \sqrt{\frac{m+2}{24}}. \end{aligned}$$

With account of these equalities, from (4.38) we obtain:

$$V(\tau) = 2I - \tau^2 A + \frac{\tau^4}{12} A^2 + R_V(\tau). \tag{4.40}$$

Due to (4.31), we have:

$$2\cos\left(\tau A^{1/2}\right) = 2I - \tau^2 A + \frac{\tau^4}{12} A^2 + 2R_2(\tau, A). \tag{4.41}$$

From (4.34), taking into account equalities (4.40), (4.41) and inequalities (4.39), (4.32), we obtain:

$$\|R(\tau)\varphi\| \leq 2\|R_2(\tau, A)\varphi\| + \|R_V(\tau)\varphi\| \leq c\tau^6 \|\varphi\|_{A^3}, \quad \varphi \in D\left(A^3\right). \tag{4.42}$$

According to formula (3.8), from (4.33), we obtain:

$$\begin{aligned} z_{k+1} &= U_k(L, I) z_1 - U_{k-1}(L, I) z_0 + \sum_{i=1}^{k} U_{k-i}(L, I) R(\tau) u(t_i) \\ &= U_k(L, I) z_1 + \sum_{i=1}^{k} U_{k-i}(L, I) R(\tau) u(t_i). \end{aligned} \tag{4.43}$$

For z_1 we have:

$$z_1 = u(t_1) - u_1 = \frac{1}{2} R(\tau)\varphi_0 + \left(A^{-1/2}\sin\left(\tau A^{1/2}\right) - \frac{1}{2}\tau V\left(\frac{\tau}{\sqrt{3}}\right)\right)\varphi_1. \tag{4.44}$$

Analogously to estimate (4.42), we obtain:

$$\left\|\left(\frac{1}{2}\tau V\left(\frac{\tau}{\sqrt{3}}\right) - A^{-1/2}\sin\left(\tau A^{1/2}\right)\right)\varphi_1\right\| \leq c\tau^5 \|\varphi_1\|_{A^2}, \quad \varphi \in D\left(A^2\right). \tag{4.45}$$

From (4.44), with account of (4.42) and (4.45), the following estimate can be obtained:

$$\|z_1\| \leq c\tau^5 \left(\|\varphi_1\|_{A^2} + \tau \|\varphi_0\|_{A^3}\right). \tag{4.46}$$

From the formula (4.43), taking into account inequalities (4.42), (3.21) and (4.46), we obtain the following estimates:

$$\begin{aligned} \|z_{k+1}\| &\leq c\nu\tau^4 \left(\|\varphi_1\|_{A^2} + \tau\|\varphi_0\|_{A^3} + \tau\sum_{i=1}^{k}\|u(t_i)\|_{A^3}\right) \\ &\leq c\nu\tau^4 \left(\|\varphi_1\|_{A^2} + \tau\|\varphi_0\|_{A^3} + t_k \max_{1\leq i\leq k}\|u(t_i)\|_{A^3}\right), \end{aligned}$$

where $\varphi_0 \in D\left(A^3\right)$, $\varphi_1 \in D\left(A^{2+1/2}\right)$ ■

Acknowledgment: This work is supported by Georgian National Science Foundation (Project GNSF/ST06/3-035).

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