ON THE APPROXIMATE SOLUTION OF THE KIRCHHOFF - BERNSTEIN NONLINEAR WAVE EQUATION

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Abstract

The initial boundary value problem is considered for a hyperbolic integro-differential equation. The numerical method of its solution and the results of a method error are presented.

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Let us consider the nonlinear equation

$$w_{tt}(x,t) = \varphi\left(\int_0^\pi w_x^2(x,t)\,dx\right) w_{xx}(x,t),\ 0 < x < \pi, \quad 0 < t \le T, \tag{1}$$

with the initial boundary conditions

$$w(x,0) = w^0(x), \quad w_t(x,0) = w^1(x), \tag{2}$$

$$\begin{gathered} w(0,t) = w(\pi,t) = 0, \\ 0 \le x \le \pi, \quad 0 \le t \le T. \end{gathered} \tag{3}$$

Here $\varphi(z)$, $w^i(x)$ are the known functions, $i = 0, 1$, where

$$\varphi(z) \ge \alpha > 0, \quad 0 \le z < \infty, \tag{4}$$

and T, α are some constants.

In 1876, when studying vibration G. Kirchhoff [1] obtained the equation

$$w_{tt}(x,t) - \left(\alpha_0 + \alpha_1 \int_0^L w_x^2(x,t)\,dx\right) w_{xx}(x,t) = 0, \tag{5}$$

where α_0, α_1 and L are some positive integers. (5) is a particular case of equation (1), when $\varphi(z)$ is some linear function.

The physical sense of equation (1) consists in modelling vibration for material nonlinearities, i.e. for nonlinear stress-strain relations, while equation (5) itself obeys Hooke's linear law. Equation (1) was for the first time mathematically investigated by S. Bernstein [2] in 1940.

Equation (1) is usually called the Kirchhoff's equation though there exists its other name – the Kirchhoff-Bernstein equation. Equation (1) together with (5) and their natural generalizations

$$w_{tt} - \left(1 + \int_{\Omega} |\nabla w|^2 dx\right) \Delta w = 0,\ x \in R^n, \quad n > 1,$$

and

$$w_{tt}(t) + a\left(\|A^{\frac{1}{2}} w(t)\|^2\right) Aw(t) = f(t),$$
$$a(s) \geq a_0 > 0, \quad A = A^* > 0,$$

had been the object of interest on the part of researchers - A. Arosio, J. Ball, M. Böhm, G. Carrier, P. D'Ancona, R. Dickey, L. Medeiros, K. Nishihara, S. Panizzi, S. Pohožaev, R. Rodriguez, S. Spagnolo and others. For the literature on this topic see, for example, [3].

The subject of investigation was as a rule the local or global solvability, the uniqueness of a solution and its continuous dependence on the initial data, but insufficient attention was given to such an important problem as the construction of an approximate solution and establishment of its accuracy. Besides the works of F.Attigui, S.Bilbao, L.Liu, M.Rincon and our, where this problem was to this extent or another touched upon, we hardly know of any other works published in this direction. In this paper we propose a numerical algorithm of the solution of problem (1)-(3) and estimate its accuracy.

The algorithm consists of three parts.

First part-space discretization. An approximate solution of problem (1)–(3) is written in the form

$$w_n(x,t) = \sum_{i=1}^{n} w_{ni}(t) \sin ix,$$
$$0 \leq x \leq \pi, \quad 0 \leq t \leq T,$$

where the coefficients $w_{ni}(t)$ are defined by Galerkin's method from the system of nonlinear differential equations

$$w''_{ni}(t) + \varphi\left(\frac{\pi}{2} \sum_{j=1}^{n} j^2 w_{nj}^2(t)\right) i^2 w_{ni}(t) = 0, \quad i = 1, 2, \ldots, n, \ \ 0 < t \leq T, \tag{6}$$

with the conditions

$$w_{ni}(0) = a_i^{(0)}, \quad w'_{ni}(0) = a_i^{(1)}, \quad i = 1, 2, \ldots, n, \tag{7}$$

where

$$a_i^{(p)} = \frac{2}{\pi}(w^p, \sin ix)_{L^2(0,\pi)}, \quad p = 0, 1.$$

We introduce the functions

$$u_{ni}(t) = w'_{ni}(t), \quad v_{ni}(t) = iw_{ni}(t), \quad i = 1, 2, \dots, n,$$

and replace system (6), (7) by an equivalent system of first order

$$\begin{aligned} &u'_{ni}(t) + \varphi\left(\frac{\pi}{2}\sum_{j=1}^{n} v_{nj}^2(t)\right) iv_{ni}(t) = 0, \\ &v'_{ni}(t) = iu_{ni}(t), \quad 0 < t \le T, \quad i = 1, 2, \dots, n, \end{aligned} \tag{8}$$

$$u_{ni}(0) = a_i^{(1)}, \quad v_{ni}(0) = ia_i^{(0)}, \quad i = 1, 2, \dots, n. \tag{9}$$

Second part-time discretization. Now we proceed to solving problem (8), (9) by means of the difference method. On the time interval $[0, T]$, let us introduce the grid $\{t_m \mid 0 = t_0 < t_1 < \cdots < t_M = T\}$ with a generally variable step $\tau_m = t_m - t_{m-1} > 0$, $m = 1, 2, \dots, M$.

Approximate values of $u_{ni}(t)$ and $v_{ni}(t)$ on the mth time layer, i.e. for $t = t_m$, $m = 1, 2, \dots, M$, denoted by u_{ni}^m and v_{ni}^m are defined by the implicit symmetric scheme

$$\begin{gathered} \frac{u_{ni}^m - u_{ni}^{m-1}}{\tau_m} + \left[\frac{\pi}{2}\sum_{j=1}^{n}\left((v_{nj}^m)^2 - (v_{nj}^{m-1})^2\right)\right]^{-1} \\ \times\left(\Phi\left(\frac{\pi}{2}\sum_{j=1}^{n}(v_{nj}^m)^2\right) - \Phi\left(\frac{\pi}{2}\sum_{j=1}^{n}(v_{nj}^{m-1})^2\right)\right) i\frac{v_{ni}^m + v_{ni}^{m-1}}{2}, \quad \frac{v_{ni}^m - v_{ni}^{m-1}}{\tau_m} = i\frac{u_{ni}^m + u_{ni}^{m-1}}{2}, \\ m = 1, 2, \dots, M, \quad i = 1, 2, \dots, n, \end{gathered} \tag{10}$$

$$u_{ni}^0 = a_i^{(1)}, \quad v_{ni}^0 = ia_i^{(0)}, \quad i = 1, 2, \dots, n, \tag{11}$$

where $\Phi(z)$ is the primitive function of $\varphi(z)$

$$\Phi(z) = \int_0^z \varphi(\xi)d\xi.$$

Third part-an iteration process. The last part of the algorithm is aimed at solving the system of nonlinear equations (10), (11). It is assumed that the counting is performed layerwise by iteration. After getting a solution on the $(m-1)$th layer, we proceed to the mth layer. Denote by $u_{ni}^{m,k}$ and

$v_{ni}^{m,k}$ the kth iteration approximation of u_{ni}^m and v_{ni}^m, $k = 0, 1, \dots$. Let us use the iteration method

$$\begin{gathered} u_{ni}^{m,k+1} = u_{ni}^{m-1} - \tau_m \left[\frac{\pi}{2} \sum_{j=1}^{n} \left((v_{nj}^{m,k})^2 - (v_{nj}^{m-1})^2 \right) \right]^{-1} \\ \times \left(\Phi\left(\frac{\pi}{2} \sum_{j=1}^{n} (v_{nj}^{m,k})^2 \right) - \Phi\left(\frac{\pi}{2} \sum_{j=1}^{n} (v_{nj}^{m-1})^2 \right) \right) i \frac{v_{ni}^{m,k} + v_{ni}^{m-1}}{2}, \qquad (12) \\ v_{ni}^{m,k+1} = v_{ni}^{m-1} + \tau_m i \frac{u_{ni}^{m,k} + u_{ni}^{m-1}}{2}, \\ m = 1, 2, \dots, M, \quad k = 0, 1, \dots , \quad i = 1, 2, \dots, n. \end{gathered}$$

We calculate the components $u_{ni}^{m,k}$ and $v_{ni}^{m,k}$ by formulas (12). Then, for chosen n and for $t = t_m$, the series $\sum_{i=1}^{n} w_{ni}^{m,k} \sin ix$, where

$$w_{ni}^{m,k} = \frac{1}{i} v_{ni}^{m,k},$$

gives, at the kth iteration step, an approximate value of the exact solution $w(x, t_m)$ of the problem. Therefore we can characterize the error of the algorithm by the difference

$$w(x, t_m) - \sum_{i=1}^{n} w_{ni}^{m,k} \sin ix.$$

Now we formulate the result on the algorithm accuracy.

Theorem. *Suppose that the restrictions* (4) *and* $\varphi(z) \in C^p[0, \infty)$, *where the parameter* p *can be equal both to* 1 *and to* 2, *are fulfilled,* $w^0(x)$ *and* $w^1(x)$ *are* 2π*-periodic functions of the form*

$$w^l(x) = \sum_{i=1}^{\infty} a_i^{(l)} \sin ix, \quad l = 0, 1,$$

and there hold the inequalities

$$|a_i^{(0)}| \le \frac{\omega}{i^{p+s+2,5}}, \quad |a_i^{(1)}| \le \frac{\omega}{i^{p+s+1,5}}, \quad i = 1, 2 \dots ,$$

where ω *and* s *are some positive constants, thereby ensuring the existence of a local solution of problem* (1)-(3), *namely, for* T, *satisfying the requirement* [2]

$$0 < T < \rho. \qquad (13)$$

Choose a value σ such that the inequalities $0<\sigma<1$ are fulfilled. Assume that for each $m=1,2,\dots,m_0$, $1\le m_0\le M$, the step τ_m is such that for $0<q_m<1$ it satisfies the inequality

$$\tau_m \max\left[1,\sum_{l=0}^{1}\left(\frac{4}{3}\big(\max(z_*,z_n^{m-1}) + \max(z_*,z_n^{m,m-1})\big)\right)^l \max_z\left|\frac{d^l\varphi}{dz^l}(z)\right|\right] \le \frac{2q_m(1-\sigma)}{n}, \tag{14}$$

where the following definitions are used

$$0\le z\le \max(z_*,z_n^{m-1},z_n^{m,m-1}),\quad z_*=\tfrac{1}{\alpha}\Big(\left\|w^1(x)\right\|^2_{L^2(0,\pi)}+\Phi\Big(\left\|w^{0\prime}(x)\right\|^2_{L^2(0,\pi)}\Big)\Big),$$

$$z_n^{m-1}=\tfrac{1}{\alpha}\left(\tfrac{\pi}{2}\sum_{i=1}^n\big(u_{ni}^{m-1}\big)^2+\Phi\left(\tfrac{\pi}{2}\sum_{i=1}^n\big(v_{ni}^{m-1}\big)^2\right)\right),$$

$$z_n^{m,m-1}=\Bigg\{\max\Bigg[\left(\tfrac{\pi}{2}\sum_{i=1}^n\big((u_{ni}^{m,0})^2+(v_{ni}^{m,0})^2\big)\right)^{\frac12},\ \left(\tfrac{\pi}{2}\sum_{i=1}^n\big((u_{ni}^{m-1})^2 +(v_{ni}^{m-1})^2\big)\right)^{\frac12}+\tau_m n h_m\Bigg]+\left(\left(\tfrac{\pi}{2}\sum_{i=1}^n\left(\big(u_{ni}^{m,0}-u_{ni}^{m-1}\big)^2+\big(v_{ni}^{m,0}-v_{ni}^{m-1}\big)^2\right)\right)^{\frac12} +\tau_m n h_m\right)\tfrac{q_m}{1-q_m}\Bigg\}^2,\quad h_m=\tfrac12\Bigg[\left(\tfrac{\pi}{2}\sum_{i=1}^n\left(\big(u_{ni}^{m,0}+u_{ni}^{m-1}\big)^2+\big(v_{ni}^{m,0}+v_{ni}^{m-1}\big)^2\right)\right)^{\frac12} +\max_{\widetilde z}|\varphi(\widetilde z)|\left(\tfrac{\pi}{2}\sum_{i=1}^n\big(v_{ni}^{m,0}+v_{ni}^{m-1}\big)^2\right)^{\frac12}\Bigg],$$

$$0\le\widetilde z\le \max\frac{\pi}{2}\left(\sum_{i=1}^n\big(v_{ni}^{m-1}\big)^2,\ \sum_{i=1}^n\big(v_{ni}^{m,0}\big)^2\right).$$

Then, with chosen n and for $t=t_{m_0}$, the error of the algorithm at the kth iteration step, $k=1,2,\dots$, is estimated by the relation

$$\begin{aligned}&||w(x,t_{m_0})-\sum_{i=1}^n w_{ni}^{m_0,k}\sin ix||_{L^2(0,\pi)}\\ &\le\sqrt{\frac{\pi^3}{2}}\left(\sum_{l=0}^1 c_{2l+1}\left(\frac{1}{n^{p+s+1}}\right)^{2-l}+c_2e^{\lambda n}\left(\max_{1\le m\le m_0}\tau_m\right)^p+\frac{q_{m_0}^k}{1-q_{m_0}}\right.\\ &\left.\times\left(\frac{\pi}{2}\sum_{i=1}^n\left(\big(u_{ni}^{m_0,1}-u_{ni}^{m_0,0}\big)^2+\big(v_{ni}^{m_0,1}-v_{ni}^{m_0,0}\big)^2\right)\right)\right)^{\frac12},\end{aligned}$$

where c_l, $l=1,2,3$, λ and ρ from (13) are constants expressed through the initial data of the problem.

Note that the step τ_m which satisfies inequality (14), can be found prior to performing the iteration on the mth layer.

The case of a global solution for equation (5) is studied in [4].

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