NEW MATHEMATICAL MODELS FOR THIN-WALLED SOLID STRUCTURES AND PROJECTIVE METHODS FOR THEIR SOLUTION

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(Received: 27.04.07; accepted: 15.09.07)

Abstract

The main objective of this article is construction and justification of the new mathematical models for anisotropic nonhomogeneous visco-poro-elastic, piezo-electric and electrically conductive binary mixture and their application in case of thin-walled structures based on works [12,13] with variable thickness in thermodynamic and stationary nonlinear problems of definition of stress-strain states for same ones. This investigation could have interesting applications in the areas of pseudo-xsantoma, medical tomography and land mine detection and possible could have an impact in the fields of geophysics, energy exploration, composite manufacturing, earthquake engineering, biomechanics, and many other areas. For the relevant applications it would be necessary to develop and justify new projective numerical-analytical methods. These new methods will be compared with existing methods for problems of that kind and used for recomputating of Basic Elements of Aircrafts. Above proposed models in abstract settings may be presented by the operator equation

$$a\frac{d^2}{dt^2}A_1u+b\frac{d}{dt}A_2u=A_3\,u+A_4\,(t,\,u)\,+f,\quad t\in[0;\;T]\,.$$

Here A_1, A_2, A_3- are linear strongly positive operators, A_4- nonlinear operator of Monge-Ampere type acting in some Banach space; a, b - matrices with constant coefficients, $T \leq \infty$, u - unknown vector. For different choices of parameters one can obtain equations of various types. We are concern with abstract linear parabolic, hyperbolic with damping and nonlinear variant of such equations. Numerical methods for these problems will be developed and studied (they will be closed to and based on works [6,7,–9,12]). Recently constructed methods by Makarov (see f.e. [4,8.10]), provide an exponentially convergence or convergence without accuracy saturation. These methods lead to sequence of stationary problems which will be solved by FEM and FVEM developed for example in [6,7].

Key words and phrases: Mathematical models, thin-walled structures, an effect of boundary layer, exponentially convergent algorithms, inhomogeneous problems, error estimate.

AMS subject classification: 74Kxx, 74Hxx, 65jxx.

1 New Mathematical Models for Thin-walled Solid Structures

1.1. Spatial nonlinear governing equations for poro-elastic media

Let us consider the problem of constructing $3D$ model with respect to spatial variable for poro-elastic media. We denote the domain in the three-dimensional Euclidean space by $\boldsymbol{R}^3$. In the Cartesian coordinates the point is denoted by $\boldsymbol{\Omega}$, time interval - $(0,T)$. Thus, in each point of the mixture (macro-point) we consider the following average quantities of tensors of stresses and strains and displacement vectors, respectively:

$$\begin{aligned}
\boldsymbol{\sigma} &= (\sigma_{11},\sigma_{22},\sigma_{33},\sigma_{32},\sigma_{31},\sigma_{12})^T, \quad \boldsymbol{\varepsilon} = (\varepsilon_{11},\varepsilon_{22},\varepsilon_{33},\varepsilon_{32},\varepsilon_{31},\varepsilon_{12})^T,\\
\boldsymbol{u} &= (u_1,u_2,u_3)^T, \boldsymbol{p} = (p_{11},p_{22},p_{33},p_{32},p_{31},p_{12})^T,\\
\boldsymbol{\zeta} &= (\zeta_{11},\zeta_{22},\zeta_{33},\zeta_{32},\zeta_{31},\zeta_{12})^T, \quad \boldsymbol{w} = (v_1,w_2,w_3)^T,\\
\varepsilon_{ij} &= \frac{1}{2}(u_{i,j}+u_{j,i}+u_{k,i}u_{k,j}), \quad \zeta_{ij} = \frac{1}{2}(w_{i,j}+w_{j,i}+w_{k,i}w_{k,j}). \qquad (1)
\end{aligned}$$

The equilibrium equations for the mixture are of the following form

$$\begin{aligned}
&\partial_j(\sigma_{ij}+\sigma_{kj}\cdot u_{i,k}) = \partial_{tt}(\rho_1 u_i+\rho_2 w_i)+f_i\,,\\
&\partial_j(p_{ij}+p_{kj}\cdot w_{i,k}) = \partial_{tt}(\rho_2 u_i+\rho_3 w_i)+\frac{\eta}{\mu}\,\partial_t w_i+\varphi_i\,, \quad (x,t)\in Q_T.
\end{aligned} \qquad (2)$$

Here $f=(f_1,f_2,f_3)^T$, $\varphi=(\varphi_1,\varphi_2,\varphi_3)^T$ are the vectors of volume forces, ρ_i - densities, and the quantity η/μ is defined analogously as in [2].

Instead of law (5.1) from [3] we define the Hooke type low as follows:

$$\boldsymbol{\sigma} = \boldsymbol{B}\varepsilon+\boldsymbol{C}\zeta, \qquad (3)$$

$$\boldsymbol{p} = \boldsymbol{C}\varepsilon+\boldsymbol{M}\zeta, \qquad (4)$$

where $B, A=B^{-1}$ - are rigidity and compliance 6×6 symmetric matrices, respectively, $\boldsymbol{C}=\{c_{ij}\}_{6\times 6}$, $\boldsymbol{M}=\{m_{ij}\}_{6\times 6}$ are also symmetric matrices.

As we have mentioned above, we assume, that in each point of the body passes at least one plane of flexible symmetry, which is parallel to Ox_1x_2

plane, i.e. in matrices C and M at most 13 constants do not equal to zero and

$$b_{i4} = b_{i5} = b_{16} = b_{56} = C_{i4} = C_{i5} = C_{46} = C_{56} = m_{i4} = m_{i5} = m_{46} = b_{56} = 0. \tag{5}$$

From (3) and (4) we have

$$\sigma_{ii} = B_i\varepsilon + C_i\zeta, \quad p_{ii} = C_i\varepsilon + M_i\zeta, \quad \sigma_{ij} = B_{9-(i+j)}\varepsilon + C_{9-(i+j)}\zeta,$$
$$\sigma_{ij} = B_{9-(i+j)}\varepsilon + C_{9-(i+j)}\zeta, \quad p_{ij} = C_{9-(i+j)}\varepsilon + M_{9-(i+j)}\zeta,$$

where B_i, C_i, M_i are i-th rows of the corresponding matrices.

Let us introduce the following denotations

$$\tau_{ij} = (\sigma_{ij}, p_{ij})^T, \quad \varepsilon_{ij} = (\varepsilon_{ij}, \zeta_{ij})^T, \quad U_i = (u_{ij}, w_{ij})^T.$$

On the basis of introduced denotations and assumptions (5) the equilibrium equations (2) and relations (3), (4) are written in the following form (see [13]):

$$\begin{aligned}
&\partial_j\big(\tau_{ij} + \tau_{kj} \oplus U_{i,k}\big) = \rho\partial_{tt}U_i + \rho_0\partial_t U_i + F_i,\\
&\tau_{ii} = A_{i1}\varepsilon_{11} + A_{i2}\varepsilon_{22} + A_{i3}\varepsilon_{33} + A_{16}\varepsilon_{12},\\
&\tau_{\alpha 3} = A_{6-\alpha\,4}\varepsilon_{32} + A_{6-\alpha\,5}\varepsilon_{33} + A_{66}\varepsilon_{12},\\
&\tau_{12} = A_{61}\varepsilon_{11} + A_{62}\varepsilon_{22} + A_{63}\varepsilon_{33} + A_{66}\varepsilon_{12},
\end{aligned} \tag{6}$$

where

$$A_{mn} = \begin{pmatrix} b_{mn} & C_{mn} \\ C_{mn} & m_{mn} \end{pmatrix}, \quad \rho = \begin{pmatrix} \rho_1 & \rho_3 \\ \rho_3 & \rho_2 \end{pmatrix}, \quad \rho_0 = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{\eta}{\mu} \end{pmatrix}, \quad F_i = (f_i, \varphi_i)^T.$$

Here a symbol $\otimes$ denotes the following operation: $(a_1, a_2)^T \otimes (b_1, b_2)^T = (a_1b_1, a_2b_2)^T$.

Analogous three-dimensional nonlinear model for anisotropic binary mixtures are presented in the work [10], which generalizes previously known model for poro-elastic and binary mixtures. The constructed models together with certain independent scientific interest represent such form of spatial models, which allow not only to construct, but also to justify von Karman-Mindlin-Reissner (KMR) type systems of differential equations as in the stationary, as well in nonstationary cases. Under justification we mean assumption of "Physical Soundness" to these models in view of Truesdell-Ciarlet (see for example details in [1], [5, ch.5]). As is known, even in case of isotropic elastic plate with constant thickness the subject of justification constituted an unsolved problem. The point is that von Karman, Love, Timoshenko, Landau & Lifshits and others considered one of the compatibility conditions of Saint-Venant-Beltrami as one of the equations of the corresponding system of differential equations. This fact was verified also by Podio-Guidugli [11] recently. In the presented model is constructed a correct equation that is especially important for dynamic problems. The corresponding system in this case contains wave processes not only in the vertical, but also in the horizontal direction. The equation has the following form:

$$\left(\Delta^2 - \frac{1-v^2}{E}\,\rho\Delta\partial_{tt}\right)\Phi = -\frac{E}{2}\,[w,w] + \frac{\nu}{2}\left(\Delta - \frac{2\rho}{E}\,\partial_{tt}\right)\left(g_3^+ + g_3^-\right) + \frac{1+\nu}{2h}\,f. \tag{7}$$

The precision of the presented mathematical model is also conditioned by a new quantity, introduced by Vashakmadze, which describes an effect of boundary layer. Existence of this member not only explains a set of paradoxes in the two-dimensional elasticity theory (Babushka, Lukasievicz), but also is very important for example for process of generating cracks and holes (details see in [12], ch.1, par. 3.3). Further, let us note that in works [13] equations of (1.7) type are constructed with respect to certain components of stress tensor by differentiation and summation of two differential equations. Also other equations of KMR type, which differ from (1.7) type equation, are equivalent to the system, where the order of each equation is not higher than two. Above proposed models in abstract settings may be presented as

$$a\,\frac{d^2}{dt^2}\,A_1u + b\,\frac{d}{dt}\,A_2u = A_3u + A_4(t,u) + f, \quad t \in [0;T], \tag{8}$$

where A_1, A_2, A_3 are linear strong positive operators, A_4 - nonlinear operator of Monge-Ampere type acting in some Banach space; a, b - matrices, $T \leq +\infty$, u - unknown vector. For different choices of parameters one can obtain equations of various types. We concern on abstract linear parabolic, hyperbolic with damping and nonlinear variation of such equations. Numerical methods for these problems will be based on works of Lazarov [6,7], Makarov [4,8–10], Vashakmadze [12].

Recently constructed methods by Makarov with co-authors, provide an exponentially convergence or convergence without accuracy saturation. These methods lead to sequence of stationary problems which will be solved by FEM and FVEM developed by Lazarov with co-authors.

1.2. Conclusions

Thus, we intend to obtain the following new results:

1. New nonlinear mathematical models for poro-elastic and elastic (with piezo-electric and electrically conductive viscous processes) binary mixtures will be created and justified;

2. Questions of solvability of stationary and thermo-dynamical models (spatial case) will be investigated both in the linear and nonlinear anisotropic cases(it should be noted that even in the linear case in the elasticity theory when at least one plane of elastic symmetry exists at each point of body, a question of strong elliptic of system of differential equations(DE) of spatial elasticity theory is yet an unsolved problem);

3. New two-dimensional with respect to spatial coordinates mathematical models of KMR type will be created and justified for poro-elastic binary mixtures when it represents a thin-walled structure; These models even in isotropic elastic case contain and justify (in sense of physical soundness) the well-known von Karman system of DE for elastic plates;

4. Optimal models especially for nonhomogeneous systems of KMR type will be created and chosen without contracting a class of admissible solutions even in classical (only elastic) case;

5. Effective numerical methods will be constructed and justified; questions of convergence and error estimate will be studied for problems for thermo-poro elastic structures;

6. Questions of influence of new terms (which firstly were introduced by Vashakmadze in poroelastic case) in the equation of form (1.7) will be investigated. Presence of these terms are very important, especially for seismic problems: in nonstationary problems these terms are of type $\partial_{tt}\Delta\Phi$, in stationary problems there are of type $\Delta(g_3^+ + g_3^-)$;

7. In case when the problem (1.8) is splitting into two such equations, that one of them is stationary and nonlinear, effective numerical method for compute eigenvalues would be proposed.

8. Problem of influence of new boundary value effect (firstly introduced by Vashakmadze) in problems of thermo poro-elasticity will be investigated and analyzed;

Using above mentioned we formulate the Basic problems and Innovations for “Re-computation of principal elements of aircraft’s”.

1.3. Re-computation of principal elements of aircraft’s

I. Objectives:

For realization of these purposes the foundation is the possibility of investigation and decision of the following problems:

1. a. Conducting of perfect comparatively analysis of well-known methods designing of principal elements of aircrafts.

b. Creation of new mathematical theory, corresponding numerical schemes and software with a view to refinement and justification (recomputation) of designing databases.

c. Discover and experimentally confirmation of new effects of statical and thermo-dynamical behavior of principal elements of constructions of aircrafts by means of mathematical theory and technology basing on exact nonlocal representations of main characteristics of a stress-strain state of thin-walled structures.

1. Main goals of the Project are:

1. 1.1. Study and analysis of the main prerequisites for constructions of models;

   1.2. The main computation elements - decomposition of construction of aircrafts;

   1.3. Geometry of basic surfaces of elements;

   1.4. Physical - mechanical properties (including f.e.a possibility to obtain datatables for anisotropic materials);

   1.5. Distribution fields of forces (power and temperature loads)

2. Mathematical statement of boundary-value and variational problems of thermomechanics for elements of aircraft constructions in different cases.

1. 2.1. Isotropy and anisotropy,

   2.2. Piecewise nonhomogeneity of geometry and physical-mechanical properties, homogeneous and heterogeneous structures.

   2.3. Constant and variable thickness.

3. Numerical schemes by finite and boundary element methods for elements of an aircraft constructions. Analysis and development of explicit methods of solution of corresponding linear and nonlinear systems of algebraic equations.

1. 3.1. Construction and analysis of systems of differential and finite-difference equations of computation of principal elements of aircraft constructions by varialtional setting of problems of theory of plates and shells, using exact nonlocal representations (cases of statics and dynamics, finite and semifinite time intervals).

   3.2. Creation of oriented computing center with modern hardware and software.

4. Carrying out numerical experiments and comparative analysis.

1. 4.1. Calculation of elements of constructions and junctions of aircrafts according to different refined theories of thin-walled structures in limits of equal approach.
   4.2. Analysis of obtained results and their comparison with the experimental data.

II. Innovation

Application of mathematical theory of plates and shells, constructed on the basis of exact nonlocal representations (free of simplifying hypotheses of geometric and physical character) for calculation and comparative analysis of the existing theories as applied to investigations with thermodynamic behavior of the principal elements of aircrafts.

Mathematical theory of plates and shells by means of precise nonlocal representations show that the equations of existing refined theories lack members which can have very important influence on behavior of thin-walled structures in certain conditions:

1. In the nonlinear dynamic equations of von-Karman type there is absent a member describing a wave propagation in the longitudinal direction. The influence of this member can appear very important at the description of behavior of wings and tail parts of aircraft construction. Analogous event holds in the statistical problems too. Introduction of the corresponding summands explains the essence of the well-known problem of Trusdell-Ciarlet.

2. The theory allows to refine the description of thermo-poro-pieso-electric and electrical-conductive processes in composites and binary mixtures.

3. The equations of the theory point to presence of the two-dimensional soliton waves of sound frequences, which can cause significant changes in the calsulation of the stress-strain state of the principal elements of aircraft, especially in junctions of wings with fuselage.

4. The possibility of applications of this theory must be especially emphasized at presence of inhomogeneities , anisotropy,piecewise heterogeneities in the thin-walled structures.

5. The correction, introduced according to this theory, in the average boundary conditions, is a refinement of the influence of boundary layer. It can cause significant changes in the neighborhood of cuts (porthole, doors and etc.).

Introduction of this member also explains a set of paradoxes, usually characteristic to existing refined theories (f.e. Kirchhof-von Karman-Mindlin-Reissner and all other one).

2 Exponentially Convergent Algorithms for the Operator Exponential with Applications to Inhomogeneous Problems in Banach Spaces

2.1. Introduction

We consider the problem

$$\frac{du(t)}{dt} + Au(t) = f(t), \quad u(0) = u_0, \tag{9}$$

where A is a strongly positive operator in a Banach space X, $u_0 \in X$ is a given vector, $f(t)$ is a given and $u(t)$ is the unknown vector valued function. A simple example of a partial differential equation covered by the abstract setting (9) is the classical inhomogeneous heat equation

$$\frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = f(t,x)$$

with corresponding boundary and initial conditions, where the operator A is defined by

$$D(A) = \{v \in H^2(0,1) : \ v(0) = 0, v(1) = 0\},$$
$$Av = -\frac{d^2 v}{dx^2} \quad \text{for all} \quad v \in D(A).$$

The homogeneous equation

$$\frac{dT(t)}{dt} + AT(t) = 0, \quad T(0) = I, \tag{10}$$

where I is the identity operator and $T(t)$ is an operator valued function defines the semi-group of bounded operators $T(t) = e^{-At}$ generated by A (called also the operator exponential or the solution operator of the homogeneous equation (9)). Given the solution operator, the initial vector u_0 and the right-hand side $f(t)$, the solution of the homogeneous initial value problem (9) can be represented by

$$u(t) = u_o(t) = T(t)u_0 = e^{-At}u_0 \tag{11}$$

and the solution of the inhomogeneous problem by

$$u(t) = e^{-At}u_0 + u_p(t) \tag{12}$$

with

$$u_p(t) = \int_0^t e^{-A(t-\xi)} f(\xi) d\xi. \tag{13}$$

We can see that an efficient approximation of the operator exponential is needed in order to get an efficient discretization of both (11) and (12). Further, having in mind a discretization of the second summand in (12) by a quadrature sum we need an efficient approximation of the operator exponential for all $t \geq 0$ including the point $t = 0$.

A convenient representation of the operator exponential is the one provided by the improper Dunford-Cauchy integral

$$e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-tz} (zI - A)^{-1} dz, \tag{14}$$

where Γ_I is an integration path enveloping the spectrum of A. After parametrizing Γ we get an improper integral of the type

$$e^{-At} = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-tz} (zI - A)^{-1} dz = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t, \xi) d\xi. \tag{15}$$

The last integral can be discretized by a quadrature rule (desirable exponentially convergent) involving a short sum of resolvents. Such an algorithm inherits a two-level parallelism with respect to both the computation of resolvents and the treatment of different time values.

In contrast to various other approximation methods with a polynomial convergence rate for the problem (9) using finite differences or the Padé fractions (both discrete in time), the Cayley transform (continuous in time) and other ideas, we present and analyzes new efficient *exponentially convergent* algorithms for the operator exponential including $t = 0$ which are also applied to inhomogeneous problems with certain holomorphic right-hand sides. The algorithms under consideration are parallelizable in an evident way.

2.2. New algorithm for the operator exponential with an exponential convergence estimate including $t = 0$

We consider the following representation of the operator exponential

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-zt} (zI - A)^{-1} u_0 dz. \tag{16}$$

Our aim is to approximate this integral by a quadrature with exponential convergence rate including $t = 0$. It is of great importance having in mind the representation of the solution of the non-homogeneous initial value problem (9) by

$$u(t) = e^{-At}u_0 + \int_0^t e^{-A(t-\xi)} f(\xi) d\xi, \tag{17}$$

where the argument of the operator exponential under the integral becomes zero for $\xi = t$. We can represent

$$u(t) = \frac{1}{2\pi i} \int_{\Gamma_I} e^{-zt} \left[(zI - A)^{-1} - \frac{1}{z} I \right] u_0 dz \tag{18}$$

instead of (16) (for $t > 0$ the integral from the second summand is equal to zero due to the analyticity of the integrand inside of the integration path) and this integral represents the solution of the problem (9) for $u_0 \in D(A^\alpha),\ \alpha > 0$. We call the hyperbola

$$\Gamma_0 = \{z(\xi) = a_0 \cosh \xi - ib_0 \sinh \xi :\ \xi \in (-\infty, \infty),\ b_0 = a_0 \tan \varphi\} \tag{19}$$

the spectral hyperbola, which pathes through the vertex $(a_0, 0)$ of the spectral angle and possesses asymptotes which are parallel to the rays of the spectral angle Σ. We choose the following hyperbola as an integration path

$$\Gamma_I = \{z(\xi) = a_I \cosh \xi - ib_I \sinh \xi :\ \xi \in (-\infty, \infty)\}. \tag{20}$$

$$\begin{aligned} a_I &= a_0 \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) - b_0 \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \\ &= \sqrt{a_0^2 + b_0^2} \cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = a_0 \frac{\cos\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos \varphi}, \\ b_I &= a_0 \sin\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) + b_0 \cos\left(\frac{\pi}{4} - \frac{\varphi}{2}\right) \\ &= \sqrt{a_0^2 + b_0^2} \sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right) = a_0 \frac{\sin\left(\frac{\pi}{4} + \frac{\varphi}{2}\right)}{\cos \varphi}, \end{aligned} \tag{21}$$

After parametrizing of the integral (18) by (20) we get

$$u(t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \mathcal{F}(t, \xi) d\xi \tag{22}$$

with

$$\begin{aligned} &\mathcal{F}(t, \xi) = F_A(t, \xi) u_0, \\ &F_A(t, \xi) = e^{-z(\xi)t} (a_I \sinh \xi - ib_I \cosh \xi) \left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)} I \right]. \end{aligned} \tag{23}$$

We approximate integral (22) by the following Sinc-quadrature

$$u_N(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} \mathcal{F}(t, z(kh)) \tag{24}$$

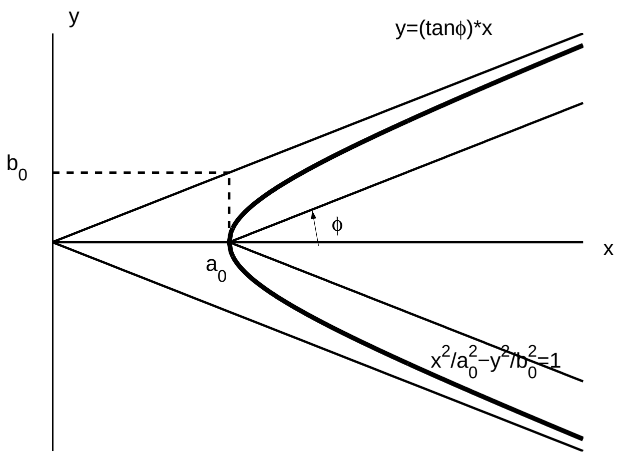

Figure 0.1: Spectral characteristics of the operator A.

Theorem 2.1 *Let A be a densely defined strongly positive operator and $u_0 \in D(A^\alpha)$, $\alpha \in (0,1)$, then Sinc-quadrature* (24) *represents an approximate solution of the homogeneous initial value problem* (9) *(i.e. $u(t) = e^{-At}u_0$) and possesses a uniform with respect to $t \geq 0$ exponential convergence rate with estimate which is of the order $\mathcal{O}(e^{-c\sqrt{N}})$ uniformly in $t \geq 0$ provided that $h = \mathcal{O}(1/\sqrt{N})$ and of the order $\mathcal{O}\left(\max\left\{e^{-\pi dN/(c_1 \ln N)},\ e^{-c_1 a_I t N/2 - c_1 \alpha \ln N}\right\}\right)$ for each fixed $t \geq 0$ provided that $h = c_1 \ln N/N$.*

Remark 2.2 *Note that taking $(zI - A)^{-1}$ instead of $(zI - A)^{-1} - \frac{1}{z}I$ in* (18) *we remain with a difference given by*

$$D_I(t) = -\frac{1}{2\pi i}\int_{\Gamma_I} e^{-zt}\frac{1}{z}u_0 dz. \tag{25}$$

For the integration path Γ_I and $t = 0$ this difference can be calculated analytically. Actually, taking into account that the real part is an odd function and the integral of it in the sense of Cauchy is equal to zero we further get for the integral of the imaginary part

$$\begin{aligned} D_I(0) &= -\frac{1}{2\pi i}\int_{\Gamma_I}\frac{1}{z}u_0 dz = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{a_I b_I d\xi}{a_I^2\cosh^2\xi + b_I\sinh^2\xi}u_0 \\ &= \frac{a_I b_I}{2\pi}\int_{-\infty}^{\infty}\frac{d(\tanh\xi)}{a_I^2 + b_I^2\tanh^2\xi}u_0 = \frac{1}{\pi}\arctan\frac{b_I}{a_I}u_0 = \frac{1}{\pi}\left(\frac{\pi}{4}+\frac{\varphi}{2}\right)u_0, \end{aligned} \tag{26}$$

where the factor in the front of u_0 is less then $1/2$. It means that one can expect a large error for t small enough when using $(zI - A)^{-1}$ instead *of* $(zI - A)^{-1} - \frac{1}{z}I$ *in* (18). *This phenomena can be observed in the next example. Note that for* $t > 0$ *integral* (25) *is equal to* 0 *due to the analyticity of the integrand inside of the integration path.*

Example 2.3 *Let us choose* $a_0 = \pi^2$, $\varphi = 0.8\pi/2$, *then the next Table 2.3 gives the values of* $\|D_I(t)\|/\|u_0\|$ *for various* t.

t	$\|D_I(t)\|/\|u_0\|$
0	*0.45*
$0.1 \cdot 10^{-}8$	*0.404552*
$0.1 \cdot 10^{-}7$	*0.081008*
$0.1 \cdot 10^{-}6$	*0.000257*
$0.1 \cdot 10^{-}5$	$0.147153 \cdot 10^{-}6$

Table 2.1: The unremovable error when using the resolvent instead of $(zI - A)^{-1} - \frac{1}{z}I$.

2.3. Inhomogeneous differential equation

In this section we consider the inhomogeneous problem (9) with the solution

$$u(t) = u_o(t) + u_p(t), \tag{27}$$

where

$$u_o(t) = e^{-At}u_0, \quad u_p(t) = \int_0^t e^{-A(t-s)}f(s)ds. \tag{28}$$

Note that there exist algorithms for convolution integrals of the type like the ones from previous sections and also based on Sinc quadratures (F.Stenger). Since these algorithms use the inverse Laplace transformation combined with Tikhonov's regularization their justification is rather complicated and the convergence order is $\mathcal{O}(\sqrt{N}e^{-c\sqrt{N}})$. In order to shake off the factor $\sqrt{N}$ in the front of the exponential we propose in this section a discretization different from F.Stenger.

Using representation (18) of the operator exponential we get

$$\begin{aligned} u_p(t) &= \int_0^t \frac{1}{2\pi i}\int_{\Gamma_I} e^{-z(t-s)}[(zI - A)^{-1} - \frac{1}{z}I]f(s)dzds \\ &= \frac{1}{2\pi i}\int_{\Gamma_I}\left[(z(\xi)I - A)^{-1} - \frac{1}{z(\xi)}I\right]\int_0^t e^{-z(\xi)(t-s)}f(s)dsz'(\xi)d\xi, \\ z(\xi) &= a_I\cosh\xi - ib_I\sinh\xi. \end{aligned} \tag{29}$$

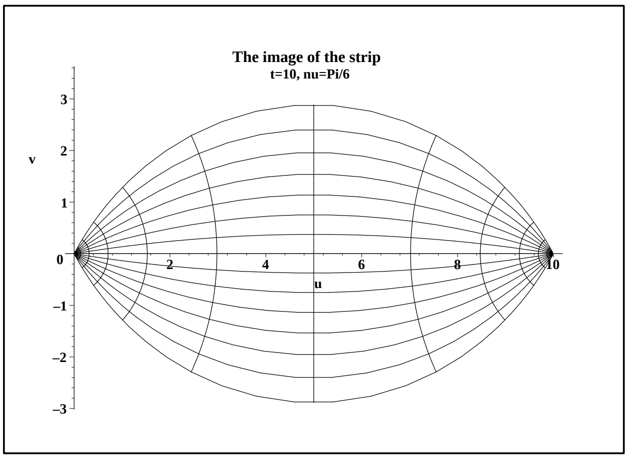

Figure 0.2: The image of the strip for $t = 10,\ \nu = \pi/6$.

Replacing here the first integral by quadrature (24) we get

$$u_p(t) \approx u_{ap}(t) = \frac{h}{2\pi i} \sum_{k=-N}^{N} z'(kh) \left[(z(kh)I - A)^{-1} - \frac{1}{z(kh)} I \right] f_k(t) \quad (30)$$

with

$$f_k(t) = \int_0^t e^{-z(kh)(t-s)} f(s) ds,\ k = -N, \dots, N. \quad (31)$$

In order to construct an exponentially convergent quadrature for these integrals we change the variables by

$$\frac{t}{2} - s = \frac{t}{2} \tanh \xi \quad (32)$$

and get instead of (31)

$$f_k(t) = \int_{-\infty}^{\infty} \mathcal{F}_k(t, \xi) d\xi, \quad (33)$$

where

$$\mathcal{F}_k(t, \xi) = \frac{t}{2\cosh^2 \xi} \exp[-z(kh)t(1 + \tanh \xi)/2] f(t(1 - \tanh \xi)/2). \quad (34)$$

Note that with the complex variables $z = \xi + i\nu$ and $w = u + iv$ equation (32) represents the conformal mapping $w = \psi(z) = t[1 - \tanh z]/2,\ z = \phi(w) = \frac{1}{2} \ln \frac{t-w}{w}$ of the strip D_ν onto the domain $\mathcal{A}_\nu$. The integrand can be estimated on the real axis by

$$\begin{aligned}\| \mathcal{F}_k(t,\xi)\| &\le \frac{t}{2\cosh^2\xi}\exp[-a_I\cosh(kh)t(1+\tanh\xi)/2]\|f(t(1-\tanh\xi)/2)\| \\ &\le 2te^{-2|\xi|}\|f(t(1-\tanh\xi)/2)\|.\end{aligned} \tag{35}$$

Lemma 2.4 *Let the right hand side $f(t)$ in (9) for $t\in[0,\infty]$ can be analytically extended into the sector $\Sigma_f=\{\rho e^{i\theta_1}:\ \rho\in[0,\infty],\ |\theta_1|<\varphi\}$ and for all complex $w\in\Sigma_f$ we have*

$$\|f(w)\|\le ce^{-\delta|\Re w|} \tag{36}$$

with $\delta\in(0,\sqrt{2}a_0]$, then the integrand $\mathcal{F}_k(t,\xi)$ can be analytically extended into the strip D_{d_1}, $0<d_1<\varphi/2$ and belongs to the class $H^1(D_{d_1})$ with respect to ξ, where a_0, φ are the spectral characterizations of A.

The assumptions of Lemma 2.4 can be weakened if we consider problem (9) on some finite interval $(0,T]$.

Lemma 2.5 *Let the right hand side $f(t)$ in (9) for $t\in[0,T]$ can be analytically extended into the domain $\mathcal{A}(T)$, then the integrand $\mathcal{F}_k(t,\xi)$ can be analytically extended into the strip D_{d_1}, $0<d_1<\varphi/2$ and belongs to the class $H^1(D_{d_1})$ with respect to ξ.*

Let the assumptions of Lemma 2.4 hold, then we can use the following quadrature rule to compute the integrals (33)

$$f_k(t)\approx f_{k,N}(t)=h\sum_{p=-N}^{N}\mu_{k,p}(t)f(\omega_p(t)), \tag{37}$$

where

$$\begin{aligned}\mu_{k,p}(t)&=\frac{t}{2}\exp\{-\frac{t}{2}z(kh)[1+\tanh(ph)]\}/\cosh^2(ph),\\ \omega_p(t)&=\frac{t}{2}[1-\tanh(ph)],\ h=\mathcal{O}(1/\sqrt{N}),\\ z(\xi)&=a_I\cosh\xi-ib_I\sinh\xi.\end{aligned} \tag{38}$$

Substituting (37) into (30) we get the following algorithm to compute an approach $u_{ap,N}(t)$ to $u_{ap}(t)$

$$u_{ap,N}(t)=\frac{h}{2\pi i}\sum_{k=-N}^{N}z'(kh)[(z(kh)I-A)^{-1}-\frac{1}{z(kh)}I]h\sum_{p=-N}^{N}\mu_{k,p}(t)f(\omega_p(t)). \tag{39}$$

The next theorem characterizes the error of this algorithm.

Theorem 2.6 *Let A be a densely defined strongly positive operator with the spectral characterization a_0, φ and the right hand side $f(t) \in D(A^\alpha)$, $\alpha > 0$ for $t \in [0, \infty]$ can be analytically extended into the sector $\Sigma_f = \{\rho e^{i\theta_1} : \rho \in [0, \infty], |\theta_1| < \varphi\}$ where the estimate*

$$\|A^\alpha f(w)\| \le c_\alpha e^{-\delta_\alpha |\Re w|},\ w \in \Sigma_f \tag{40}$$

with $\delta_\alpha \in (0, \sqrt{2}a_0]$ holds, then algorithm (39) converges with the error estimate

$$\|\mathcal{E}_N(t)\| = \|u_p(t) - u_{ap,N}(t)\| \le ce^{-c_1\sqrt{N}} \tag{41}$$

uniformly in t with positive constants c, c_1 depending on α, φ, a_0 and independent of N.

Example 2.7 *We consider the inhomogeneous problem (9) with the operator A defined by*

$$\begin{aligned} &D(A) = \{u(x) \in H^2(0,1) :\ u(0) = u(1) = 0\}, \\ &Au = -u''(x)\ \forall u \in D(A). \end{aligned} \tag{42}$$

The initial function is $u_0 = u(0, x) = 0$ and the right hand side $f(t)$ is given by

$$f(t,x) = x^3(1-x)^3 \frac{1-t^2}{(1+t^2)^2} - \frac{6t}{1+t^2} x(1-x)(5x^2 - 5x + 1). \tag{43}$$

it is easy to see that the exact solution is $u(t,x) = x^3(1-x)^3\frac{t}{1+t^2}$. The algorithm (39) was implemented for $t = 1$, $x = 1/2$ in Maple 8 with Digits = 16. The next table shows an exponential decay of the error $\varepsilon_N = |u(1, 1/2) - u_{ap,N}(1)|$ with growing N.

N	ε_N
8	*0.485604499*
16	*0.184497471*
32	*0.332658314 e-1*
64	*0.196729786 e-2*
128	*0.236757688 e-4*
256	*0.298766899 e-7*

Table 2.2: The error of algorithm (39) for $t = 0$, $x = 1/2$.

3 Exponentially Convergent Algorithm for Nonlinear Equations

We consider the problem

$$\begin{aligned} &\frac{\partial u(t)}{\partial t} + Au(t) = f(t, u(t)), \quad t \in (0, 1], \\ &u(0) = u_0, \end{aligned} \tag{44}$$

where $u(t)$ is an unknown vector valued function with values in a Banach space X, $u_0 \in X$ is a given vector, $f(t, u) : \ (\mathbb{R}_+ \times X) \to X$ is a given function (nonlinear operator) and A is a linear densely defined closed operator with the domain $D(A)$ acting in X. The abstract setting (44) covers many applied problems such as nonlinear heat conduction or diffusion in porous media, the flow of electrons and holes in semiconductors, nerve axon equations, chemically reacting systems, equations of the population genetics theory, dynamics of nuclear reactors, Navier-Stokes equations of the viscous flow etc. This fact together with theoretical interest are important reasons to study efficient discrete approximations of problem (44).

Given a discretization parameter N we are interesting in approximations possessing an exponential convergence rate with respect to $N \to \infty$ which for a given tolerance ε provide algorithms of optimal or low complexity.

Problem (44) is equivalent to the nonlinear Volterra integral equation

$$u(t) = u_h(t) + u_{nl}(t), \tag{45}$$

where

$$u_h(t) = T(t)u_0, \tag{46}$$

$T(t) = e^{-At}$ is the operator exponential (the semi-group) generated by A and the nonlinear term is given by

$$u_{nl}(t) = \int_0^t e^{-A(t-s)} f(s, u(s)) ds. \tag{47}$$

3.1. A discretization scheme of Chebyshev type

Changing in (45) the variables by

$$t = \frac{x+1}{2} \tag{48}$$

we transform problem (45) to the following problem on the interval $[-1, 1]$

$$u(\frac{x+1}{2}) = g_h(x) + g_{nl}(x, u) \tag{49}$$

with

$$g_h(x) = e^{-A\frac{x+1}{2}}u_0, \\ g_{nl}(x,u) = \frac{1}{2}\int_{-1}^{x} e^{-A\frac{x-\xi}{2}} f(\frac{\xi+1}{2}, u(\frac{\xi+1}{2}))d\xi. \tag{50}$$

Using the representation of the operator exponential by the Dunford-Cauchy integral along the integration path Γ_I and enveloping the spectral curve Γ_0 we obtain

$$\begin{aligned} g_h(x) &= e^{-A\frac{x+1}{2}}u_0 = \frac{1}{2\pi i}\int_{\Gamma_I} e^{-z\frac{x+1}{2}}[(zI-A)^{-1} - \frac{1}{z}I]u_0 dz, \\ g_{nl}(x,u) &= \frac{1}{2}\int_{-1}^{x} e^{-A\frac{x-\eta}{2}} f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2}))d\eta \\ &= \frac{1}{4\pi i}\int_{-1}^{x}\int_{\Gamma_I} e^{-z\frac{x-\eta}{2}}[(zI-A)^{-1} - \frac{1}{z}I] f(\frac{\eta+1}{2}, u(\frac{\eta+1}{2}))dz d\eta \end{aligned} \tag{51}$$

(note, that $P.V. \int_{\Gamma_I} z^{-1}dz = 0$ but this term in the resolvent provides the numerical stability of the algorithm below when $t \to 0$). After parametrizing the integrals in (51) we have

$$g_h(x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathcal{F}_h(x,\xi)d\xi \tag{52}$$

with

$$\mathcal{F}_h(x,\xi) = F_A((x+1)/2, \xi)u_0 \tag{53}$$

(in the case $A = 0$ we define $F_A(t,\xi) = 0$).

We approximate integral (52) by the following Sinc-quadrature

$$g_{h,N_1}(x) = \frac{h}{2\pi i}\sum_{k=-N_1}^{N_1} \mathcal{F}_h(x,kh), \quad h = \sqrt{\frac{2\pi d}{\alpha(N_1+1)}} \tag{54}$$

with the error

$$\|\eta_{N_1}(\mathcal{F}_h,h)\| = \|\mathcal{E}((x+1)/2)u_0\| \le \frac{c}{\alpha} exp\left(-\sqrt{\frac{\pi d\alpha}{2}(N_1+1)}\right)\|A^\alpha u_0\|, \tag{55}$$

where

$$\mathcal{E}((x-\eta)/2) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} F_A((x-\eta)/2,\xi)d\xi - \frac{1}{2\pi i}\sum_{k=-N_1}^{N_1} F_A((x-\eta)/2, kh) \tag{56}$$

and the constant c independent of x, N_1. Analogously we transform the second integral in (51)to

$$g_{nl}(x,u) = \frac{1}{4\pi i}\int_{-1}^{x}\int_{\Gamma_I} e^{-z\frac{x-\eta}{2}}[(zI-A)^{-1}-\frac{1}{z}I]f(\frac{\eta+1}{2},u(\frac{\eta+1}{2}))dzd\eta$$
$$= \frac{1}{4\pi i}\int_{-1}^{x}\int_{-\infty}^{\infty}\mathcal{F}_{nl}(x,\xi,\eta)d\xi d\eta, \tag{57}$$

where

$$\mathcal{F}_{nl}(x,\xi,\eta) = F_A((x-\eta)/2,\xi)f(\frac{\eta+1}{2},u(\frac{\eta+1}{2})). \tag{58}$$

Replacing the infinite integral by quadrature rule (54) we arrive at the approximation

$$g_{nl,N_1}(x,u) = \frac{h}{4\pi i}\int_{-1}^{x}\sum_{k=-N_1}^{N_1}\mathcal{F}_{nl}(x,kh,\eta)d\eta. \tag{59}$$

In order to approximate the nonlinear operator $g_{nl,N_1}(x,u)$ we choose the mesh $\omega_N = \{x_{k,N} = \cos\frac{(2k-1)\pi}{2N},\ k=1,\dots,N\}$ on $[-1,1]$, where $x_{k,N}$ are zeros of Chebyshev orthogonal polynomial of first kind $T_N(x) = \cos(N\arccos x)$. For the step-sizes $\tau_{k,N} = x_{k,N} - x_{k-1,N}$ it is well known that

$$\tau_{k,N} = x_{k+1,N} - x_{k,N} < \tfrac{\pi}{N}, k=1,\dots,N,$$
$$\tau_{max} = \max_{1\le k\le N}\tau_{k,N} < \tfrac{\pi}{N}. \tag{60}$$

Let

$$P_{N-1}(x;f(\cdot,u)) = \sum_{p=1}^{N} f((x_{p,N}+1)/2, u((x_{p,N}+1)/2))L_{p,N-1}(x) \tag{61}$$

be the interpolation polynomial for the function $f(x,u(x))$ on the mesh ω_N, i.e. $P_{N-1}(x_{k,N};f(\cdot,u)) = f((x_{k,N}+1)/2, u((x_{k,N}+1)/2)), k=1,2,\dots,N$, where $L_{p,N-1} = \frac{T_N(x)}{T'_N(x_{p,N})(x-x_{p,N})}, p=1,\dots,N$ are the Lagrange fundamental polynomials. Given a vector $y=(y_1,\dots,y_N), y_i \in X$ let

$$P_{N-1}(x;f(\cdot,y)) = \sum_{p=1}^{N} f((x_{p,N}+1)/2, y_p))L_{p,N-1}(x) \tag{62}$$

be the polynomial which interpolates $f(x,y)$, i.e. $P_{N-1}(x_{k,N};f(\cdot,y)) = f((x_{k,N}+1)/2, y_k)), k=1,2,\dots,N$. Substituting $P_{N-1}(t;f(\cdot,y))$ instead of $f(t,u)$ into (58), (59) we get the approximation

$$g_{nl,N,N_1}(x,y) = \frac{h}{4\pi i}\int_{-1}^{x}\sum_{k=-N_1}^{N_1} F_A((x-\eta)/2,kh)P_{N-1}(\eta; f(\cdot,y))d\eta. \quad (63)$$

Substituting approximations (54) and (63) into (54) and collocating the resulting equation on the grid ω_N we arrive at the following

Algorithm A1 for solving problem (49): find $y = (y_1,\dots,y_N),\ y_i \in X$ such that

$$y_j = g_{h,N_1}(x_{j,N}) + g_{nl,N,N_1}(x_{j,N},y), j = 1,\dots,N \quad (64)$$

or

$$\begin{aligned} y_j &= \frac{h}{2\pi i}\sum_{k=-N_1}^{N_1}\mathcal{F}_h(x_{j,N},kh) \\ &+ \frac{h}{4\pi i}\sum_{k=-N_1}^{N_1}\int_{-1}^{x_{j,N}} F_A((x_{j,N}-\eta)/2,kh)P_{N-1}(\eta; f(\cdot,y))d\eta, \\ j &= 1,\dots,N. \end{aligned} \quad (65)$$

Equations (64) or (65) define a nonlinear operator $\mathcal{A}$ so that

$$y = \mathcal{A}(y) + \phi, \quad (66)$$

where

$$\begin{aligned} &y = (y_1,y_2,\dots,y_N), \quad y_i \in X, \\ &[\mathcal{A}(y)]_j = \frac{h}{4\pi i}\sum_{k=-N_1}^{N_1}\int_{-1}^{x_{j,N}} F_A((x_{j,N}-\eta)/2,kh)P_{N-1}(\eta; f(\cdot,y))d\eta, \\ &(\phi)_j = \frac{h}{2\pi i}\sum_{k=-N_1}^{N_1}\mathcal{F}_h(x_{j,N},kh) = \frac{h}{2\pi i}\sum_{k=-N_1}^{N_1} F_A((x_{j,N}+1)/2,kh)u_0, \\ &j = 1,\dots,N. \end{aligned} \quad (67)$$

This is a system of nonlinear equations which can be solved by an iteration method. Since the integrands in

$$I_{j,k} = \int_{-1}^{x_{j,N}} F_A((x_{j,N}-\eta)/2,kh)P_{N-1}(\eta; f(\cdot,y))d\eta, \quad (68)$$

$j = 1,\dots,N,\ k = -N_1,\dots,N_1$ are products of the exponential function and polynomials, these integrals can be calculated analytically, for example, by computer algebra tools.

Given the vector $y = (y_1,\dots,y_N)$ the interpolation polynomial $\tilde{u}(x) = P_{N-1}(x;y)$ represents an approximation for $u((x+1)/2) = u(t)$, i.e. $u((x+1)/2) = u(t) \approx P_{N-1}(x;y)$.

3.2. The error analysis for a small Lipschitz constant

We assume that

(i)

$$f(t,u(t)) \in D(A^\alpha)\ \forall\, t \in [0,1] \quad \text{and} \quad \int_0^1 \|A^\alpha f(t,u(t))\|dt < \infty. \tag{69}$$

(ii) the vector valued function $A^\alpha f(\frac{1+\xi}{2}, u(\frac{1+\xi}{2}))$ of ξ can be analytically extended from the interval $B = [-1,1]$ into the domain $\mathcal{D}_\rho$ enveloped by the so called Bernstein's regularity ellipse $\mathcal{E}_\rho = \mathcal{E}_\rho(B)$ (with the foci at $z = \pm 1$ and the sum of semi-axes equal to $\rho > 1$):

$$\mathcal{E}_\rho = \{z \in \mathbb{C} : z = \frac{1}{2}\left(\rho e^{i\varphi} + \frac{1}{\rho}e^{-i\phi}\right)\}$$

$$= \{(x,y) : \frac{x^2}{a^2} + \frac{x^2}{a^2} = 1,\ a = \frac{1}{2}\left(\rho + \frac{1}{\rho}\right),\ b = \frac{1}{2}\left(\rho - \frac{1}{\rho}\right)\}$$

.

(iii) The function $f(t,y) = f(t,y;N)$ in the domain $G = \{(t,y,N) : 0 \le t \le 1,\ |||y-u||| < \gamma, N \ge N_0\}$ in addition to **(i)**, **(ii)** satisfies

$$|||A^\alpha[f(t,y_1) - f(t,y_2)]||| \le L|||y_1 - y_2|||\ \forall y_1,\ y_2 \in G, \tag{70}$$

for all $(t, y_i, N) \in G,\ i = 1,2$, where
$|||Z||| = |||y-u||| = \max_{j=1,\dots,N} \|y_j - u(t_j)\|$, γ is a positive real constant and N_0 is a fixed natural number large enough.

$$\begin{aligned} |||Z||| = |||u-y||| &\le \frac{c}{\alpha - c^* L} \ln N_1 e^{-c_1\sqrt{N_1}} \\ &\times \left(\|A^\alpha u_0\| + \int_0^1 \|A^\alpha f(t,u(t))\|dt + \sup_{z\in D_\rho}\|A^\alpha f(z,u(z))\|\right) \end{aligned} \tag{71}$$

in a more strong norm

$$\begin{aligned} |||A^\alpha Z||| = |||A^\alpha(u-y)||| &\le \frac{c}{\beta - \alpha - c^* L} \ln N_1 e^{-c_1\sqrt{N_1}} \\ &\times \left(\|A^\beta u_0\| + \int_0^1 \|A^\beta f(t,u(t))\|dt + \sup_{z\in D_\rho}\|A^\beta f(z,u(z))\|\right), \\ &\forall\, \beta > \alpha > 0,\ \beta - \alpha > c^* L. \end{aligned} \tag{72}$$

Theorem 3.1 *Let A be a densely defined, closed, strongly positive linear operator with the domain $D(A)$ in a Banach space X and the assumptions* **(i)**, **(ii)**, **(iii)** *hold, then algorithm* **A1** *defined by* (65) *for the numerical solution of the nonlinear problem* (44) *possesses an uniform with respect to t exponential convergence rate with estimates* (71), (72) *provided that $N \asymp \sqrt{N_1}$ and the Lipschitz constant L is sufficiently small.*

Remark 3.2 *The same result can be obtained if one uses the interpolation polynomial on the Chebyshev-Gauss-Lobatto grid*

$$\omega_N^{CGL} = \{x_{k,N} = x_{k,N}^{CGL} = \cos\frac{(N-j)\pi}{N}, k = 0, 1, \dots, N, \} \tag{73}$$

where the nodes are zeros of the polynomial $(1-x^2)T_N'(x)$.

3.3. Modified algorithm for arbitrary Lipschitz constant

In this section we show, how the algorithm above can be modified for a nonlinearity with an arbitrary Lipschitz constant. To this end we suppose that $u(t) \in D(A^\sigma)$, $\sigma > c^*L/2$. We cover the interval $[0,1]$ by the grid $\omega_G = \{t_i = i \cdot \tau : i = 0, 1, \dots, K, \tau = 1/K\}$ and consider problem (44) on each subinterval $[t_{k-1}, t_k], k = 1, \dots, K$. The substitution $t = t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2, v(\xi) = u(t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2)$ translates the original equation into the differential equation

$$v'(\xi) + \tilde{A}v = \tilde{f}(\xi, v) \tag{74}$$

on the reference interval $[-1,1]$ with $\tilde{A} = \frac{\tau}{2}A$ and with the function $\tilde{f}(\xi, v) = \frac{\tau}{2}f(t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2, u(t_{k-1}(1-\xi)/2 + t_k(1+\xi)/2))$ satisfying the Lipschitz condition with the Lipschitz constant $\tilde{L} = \tau L/2$ which can be made arbitrarily small by the appropriate choice of τ. We cover each subinterval $[t_{k-1}, t_k]$ by the Chebyshev-Gauss-Lobatto grid

$$\begin{aligned}&\omega_{k,N}^{CGL} = \{t_{k,j} : t_{k,j} = t_{k-1}(1-x_{j,N})/2 + t_k(1+x_{j,N})/2, j = 0, 1, \dots, N\},\\ &x_{j,N} = \cos\left(\pi(N-j)/N\right)\end{aligned} \tag{75}$$

and denote $v_k(x_{j,N}) = v_{k,j} = u(t_{k,j}), v_{k,0} = v_k, u(t_{k,0}) = u(t_k) = u_k, \vec{v}_k = [v_{k,j}]_{j=1,\dots,N}, \vec{u}_k = [u(t_{k,j})]_{j=1,\dots,N}$. Then, algorithm (65) with the corresponding Chebyshev-Gauss-Lobatto interpolation polynomial can be applied which provides an exponential accuracy on the subinterval $[t_{k-1}, t_k]$ under the assumption that the initial vector u_{k-1} is known. This is exactly the case for $k = 1$ and by algorithm (65) we obtain a value $v_{1,N} = v_1$ as an approximation for $u(t_1)$. Starting on the subinterval $[t_1, t_2]$ with the approximate initial value v_1 we obtain an approximate solution for this subinterval and so on.

In order to write down this idea as an algorithm we derive from (74) the relation

$$v_{k,j} = e^{-\tilde{A}(1+x_{j,N})} u_{k-1} + \int_{-1}^{x_{j,N}} e^{-\tilde{A}(x_{j,N}-\eta)} \tilde{f}(\eta, v_k(\eta)) d\eta. \tag{76}$$

Denoting by $y_{k,j}$ approximations to $v_{k,j}$, approximating the operator exponential with N_1 nodes and the nonlinearity by the Chebyshev-Gauss-Lobatto interpolation polynomial

$$\begin{aligned} P_N(\eta, \vec{\tilde{f}}) &= \sum_{l=0}^{N} \tilde{f}(x_{l,N}, y_{k,l}) L_{l,N}(\eta), \\ L_{l,N}(\eta) &= \frac{(1-\eta^2)T'_N(\eta)}{(\eta - x_{l,N}) \frac{d}{d\eta}[(1-\eta^2)T'_N(\eta)]_{\eta = x_{l,N}}}, \\ \vec{\tilde{f}} &= [\tilde{f}(x_{j,N}, y_{k,j})]_{j=0}^{N} \end{aligned} \tag{77}$$

we arrive at the following system of nonlinear equations (analogous to (65))

$$y_{k,j} = e_{N_1}^{-\tilde{A}(1+x_{j,N})} y_{k-1} + \int_{-1}^{x_{j,N}} e_{N_1}^{-\tilde{A}(x_{j,N}-\eta)} P_N(\eta, \vec{\tilde{f}}) d\eta, \tag{78}$$

which expresses $y_{k,j}, j = 1, 2, \ldots, N$ (in particular $y_{k,N} = y_{k+1}$) through y_{k-1}.

Now, we can formulate the following algorithm.

Algorithm A2.

Given K satisfying (79), and N_1 computes the approximate solution of nonlinear problem (44) with an arbitrary Lipschitz constant by solving of the nonlinear discrete system (78) on each subinterval

1. Choose K satisfying (79) and N_1 and set $\tau = 1/K,\ t_0 = 0,\ y_0 = u_0$.

2. For $i := 1$ step 1 to K do

2.1. Set $t_i = t_{i-1} + \tau$ and find the approximate solution $y_{i,j}, j = 1, 2, \ldots, N$ of problem (9) on the Chebyshev-Gauss-Lobatto grid (75) covering the interval $[t_{i-1}, t_i]$ by algorithm (78) using y_{i-1} as the initial value.

2.2. Set $y_i = y_{i,N}$.

$$\frac{c^*}{\alpha_k} \frac{L\tau}{2} < 1 \tag{79}$$

$$\max_{1\le k\le K} |||A^{\alpha_2}\vec{z}_k||| \le \max\{q^K, q\} \ln N_1 e^{-c_1\sqrt{N_1}}$$
$$\sum_{k=1}^{K}\left[\int_{t_{k-1}}^{t_k} \|A^{\alpha_k} f(t,u(t))\|dt + \|A^{\alpha_k}u(t_{k-1})\| + \frac{\tau}{2}\sup_{z\in D_{\rho_k}} \|A^{\alpha_k} f(t_k(z), u(t_k(z)))\|\right]. \tag{80}$$

Theorem 3.3 *Let A be a densely defined closed strongly positive linear operator with the domain $D(A)$ in a Banach space X and the assumptions* **(i)**, **(ii)**, **(iii)** *hold. If the solution of the nonlinear problem* (44) *belongs to the domain $D(A^\sigma)$ with $\sigma > c^*L/2$ then algorithm* **A2** *possesses an uniform with respect to t exponential convergence rate with estimate* (80), *provided that $N \asymp \sqrt{N_1}$ and the chosen number of subintervals K satisfies* (79).

Example 3.4 *Let us consider the problem*

$$\begin{aligned}&\frac{\partial u}{\partial t} + Au = f(t,u(t)),\\ &u(-1) = u_0\end{aligned} \tag{81}$$

with the linear operator A given by

$$\begin{aligned}&D(A) = \{w(x) \in H^2(0,1):\ w'(0) = 0, w'(1) = 0\},\\ &Av = -w''\ \forall w \in D(A),\end{aligned} \tag{82}$$

with the nonlinear operator f given by

$$f(t,u) = -2tu^2 \tag{83}$$

and with the initial condition given by

$$u_0 = u(-1,x) = 1/2. \tag{84}$$

Since the numerical algorithm above supposes that the operator coefficient is strongly positive we shift its spectrum by the variables transform $u(t,x) = e^{d^2t}v(t,x)$ with a real number d. Then we obtain the problem

$$\begin{aligned}&\frac{\partial v}{\partial t} + A_d v = f_d(t,v(t)),\\ &v(-1) = v_0\end{aligned} \tag{85}$$

with the linear operator A_d given by

$$\begin{aligned}&D(A_d) = D(A),\\ &A_d w = Aw + d^2w\ \forall w \in D(A_d),\end{aligned} \tag{86}$$

with the nonlinear operator f_d given by

$$f_d(t,v) = -2te^{d^2t}v^2 \tag{87}$$

and with the initial condition

$$v_0 = v(-1,x) = e^{d^2}/2. \tag{88}$$

It is easy to check that the exact solution of this problem is

$$v(t,x) = e^{-d^2t}/(1+t^2). \tag{89}$$

The equivalent Volterra integral equation for v has the form

$$v(t,x) = \frac{1}{2}e^{-A_d(t+1)}e^{d^2} - 2\int_{-1}^{t} e^{-A_d(t-s)}se^{d^2s}[v(s,\cdot)]^2ds. \tag{90}$$

Returning to the unknown function u the integral equation takes the form

$$u(t,x) = \frac{1}{2}e^{-A_d(t+1)}e^{d^2(t+1)} - 2\int_{-1}^{t} e^{-A_d(t-s)}se^{-d^2s}[u(s,\cdot)]^2ds. \tag{91}$$

Our algorithm was implemented in Maple with numerical results given by Table 3 where $\varepsilon_N = \max_{1\le j\le N}\varepsilon_{j,N}$, $\varepsilon_{j,k,N} = |u(x_{j,N},kh) - y_{j,k}|$, $j = 1,\dots,N$, $k = -N_1,\dots,N_1$. The numerical results are in a good agreement with Theorem 3.1.

N	ε_N	It
4	0.8 e-1	12
8	0.7 e-3	10
16	0.5 e-6	11
32	0.3 e-12	12

Table 3.3: The error of algorithm (65).

Example 3.5 *This example deals with the two-dimensional nonlinear problem*

$$\begin{aligned}&\frac{\partial u}{\partial t} + Au = f(t,u(t)),\\ &u(0) = u_0\end{aligned} \tag{92}$$

where

$$D(A) = \{w(x,y) \in H^2(\Omega) : \ w|_{\partial\Omega} = 0\},$$
$$Av = -\Delta v \ \forall v \in D(A), \tag{93}$$
$$\Omega = [0,1] \times [0,1]$$

with the nonlinear operator f *given by*

$$f(t,u) = -u^3 + e^{-6\pi^2 t} \sin^3 \pi x \sin^3 \pi y \tag{94}$$

and with the initial condition given by

$$u_0 = u(0,x,y) = \sin \pi x \sin \pi y. \tag{95}$$

The exact solution is given by $u = e^{-2\pi^2 t} \sin \pi x \sin \pi y$. *Algorithm* (65) *with* $N = \sqrt{N_1}$ *Chebyshev-Gauss-Lobatto nodes combined with the fixed point iteration provides the error which is presented in Table* 4.

N	ε_N	*It*
4	*.3413e-6*	*12*
8	*.1761e-6*	*10*
16	*.8846e-7*	*14*
32	*.5441e-8*	*14*

Table 3.4: The error ε_N of algorithm (65) for problem (92)- (95).

Example 3.6 *Let us consider again the nonlinear initial value problem and apply the algorithm* **A2** *for various values of the Lipschitz constant* 2μ. *Numerical experiments indicate the convergence for* $\mu > 0.4596747673$ *but beginning with* $\mu \approx 1$ *the process becomes divergent and algorithm* **A2** *should be applied. The corresponding results for various* μ *are presented in Table* 3.6.

Here the degree of the interpolation polynomial is $N = 16$, K *is the number of subintervals of the whole interval* $[-1;1]$, *It denotes the number of the iterations in order to arrive at the accuracy* $exp(-N) * 0.01$.

Conclusions

The present results are generalized to the cases when the operator coefficient is nonconstant and posses a variable domain.

μ	K	It
0.9	1	22
1	2	20
10	32	20
20	50	25
50	128	25
100	256	24

Table 3.5: The results of algorithm **A2** with various values of the Lipschitz constant μ.

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