SOLUTION OF SOME BOUNDARY VALUE PROBLEMS OF VEKUA SHELL THEORY WITH SYMMETRY AND ANTI-SYMMETRY CONDITIONS AT THE BOUNDARIES

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Abstract

Based on I.N.Vekua's shell theory (approximation $N = 1$) for rectangular plate and shallow shells a number of boundary value problems are effectively solved when conditions of free support (antisymmetry conditions) and sliding jam (symmetry conditions) are defined on the boundary of the domain or when on one part of the boundary outline symmetry conditions are defined, while on the other antisymmetry conditions are given. Both the classic case of Vekua shell theory and the case based on elastic mix theory are considered. Using the method of separation of variables the mentioned boundary value problems are reduced to the solution of an infinite system of linear algebraic equations with a block diagonal matrix.

Key words and phrases: Plate, spherical and cylindrical shells, binary mix theory, symmetry and anti-symmetry conditions.
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Introduction

The importance of analytical (exact) solutions in plate and shell theory is well known. A number of publications are devoted to this problem [1], [2], [3], [4], [5].

In the present paper we construct analytical solutions of elastic equilibrium problems for plates and shells according to I.N.Vekua [6]. Both the classic case of Vekua shell theory and the case based on elastic mix theory are considered.

1 Problem Statement

Firstly, we shall deal with the stress and strain state of a plate considered in the Cartesian system of coordinates $x_1x_2x_3$ occupying the domain $\Omega = \{0 < x_1 < a_1,\ 0 < x_2 < a_2,\ -h < x_3 < h\}$ where $a_1,\ a_2,\ h$ are constant.

Due to the fact that the main equations of zero order approximation using Vekua technique for the classic case actually coincide with the equations of the flat deformable state, we are not going to consider them in this paper.

In the case of the $N = 1$ order approximation the components of the displacement vector have the following form

$$u_j = \overset{(0)}{u}_j + \frac{x_3}{h}\overset{(1)}{u}_j,$$

where

$$\overset{(0)}{u}_j = \frac{1}{2h}\int\limits_{-h}^{h} u_j dx_j, \qquad \overset{(1)}{u}_j = \frac{3}{2h^2}\int\limits_{-h}^{h} x_3 u_j dx_3, \qquad j = 1, 2, 3.$$

On the boundary of the median surface of the plate the following conditions are considered.

Symmetry conditions [7]

$$\overset{(k)}{u}_\alpha = 0, \quad \partial_\alpha \overset{(k)}{u}_3 = 0, \quad \partial_\alpha \overset{(k)}{u}_{3-\alpha} = 0, \qquad x_\alpha = 0 \quad x_\alpha = a_\alpha. \tag{1.1}$$

Anti-symmetry conditions [7]

$$\overset{(k)}{u}_{3-\alpha} = 0, \quad \overset{(k)}{u}_3 = 0, \quad \partial_\alpha \overset{(k)}{u}_\alpha = 0, \quad x_\alpha = 0 \quad x_\alpha = a_\alpha, \quad \alpha = 1, 2 \;\; k = 0, 1. \tag{1.2}$$

$$\partial_\alpha = \frac{\partial}{\partial x_\alpha}.$$

Various types of mixed boundary conditions are considered when symmetry conditions are defined on some of the sides of the rectangle, while anti-symmetry conditions are given for the other ones.

It should be noted that symmetry conditions (based on the classic three-dimensional elasticity theory) imply that a normal component of the displacement vector and tangential components of the stress are defined at the boundary of the domain, while anti-symmetry conditions imply the opposite, i.e. the normal component of the stress and the tangential components of the displacement are defined at the boundary. It should be mentioned that symmetry and anti-symmetry conditions allow a continuous extension of the solution onto the domain specular with respect to the given one.

In our case the equations break up into two independent systems of stretching- compression and bend, with the desired values $\overset{(0)}{u}_1$, $\overset{(0)}{u}_2$, $\overset{(1)}{u}_3$ $\overset{(1)}{u}_1$, $\overset{(1)}{u}_2$, $\overset{(0)}{u}_3$, respectively. The given boundary conditions also break up in a similar way.

The elastic equilibrium of the plate with the corresponding boundary conditions is described by the following system of differential equations.

Stretching-compression system

$$\begin{cases} \mu\Delta \overset{(0)}{u}_1 + (\lambda+\mu)\partial_1 \overset{(0)}{\theta} + \dfrac{\lambda}{h}\partial_1 \overset{(1)}{u}_3 + \overset{(0)}{F}_1 = 0, \\ \mu\Delta \overset{(0)}{u}_2 + (\lambda+\mu)\partial_2 \overset{(0)}{\theta} + \dfrac{\lambda}{h}\partial_2 \overset{(1)}{u}_3 + \overset{(0)}{F}_2 = 0, \\ \mu\Delta \overset{(1)}{u}_3 - \dfrac{3\lambda}{h} \overset{(0)}{\theta} - \dfrac{3(\lambda+2\mu)}{h^2} \overset{(1)}{u}_3 + \overset{(1)}{F}_3 = 0. \end{cases} \tag{1.3}$$

Bend system

$$\begin{cases} \mu\Delta \overset{(1)}{u}_1 + (\lambda+\mu)\partial_1 \overset{(1)}{\theta} - \dfrac{3\mu}{h}\partial_1 \overset{(0)}{u}_3 - \dfrac{3\mu}{h^2} \overset{(1)}{u}_1 + \overset{(1)}{F}_1 = 0, \\ \mu\Delta \overset{(1)}{u}_2 + (\lambda+\mu)\partial_2 \overset{(1)}{\theta} - \dfrac{3\mu}{h}\partial_2 \overset{(0)}{u}_3 - \dfrac{3\mu}{h^2} \overset{(1)}{u}_2 + \overset{(1)}{F}_2 = 0, \\ \mu\Delta \overset{(0)}{u}_3 + \dfrac{\mu}{h} \overset{(1)}{\theta} + \overset{(0)}{F}_3 = 0, \end{cases} \tag{1.4}$$

where $\Delta = \partial_{11} + \partial_{22}$ is a flat Laplacian, $\overset{(k)}{\theta} = \partial_1 \overset{(k)}{u}_1 + \partial_2 \overset{(k)}{u}_2,\ k = 0, 1;$

$$\overset{(k)}{F}_j = \frac{2k+1}{2\pi} \int\limits_{-h}^{h} \Phi_j P_k\left(\frac{x_3}{h}\right) dx_3 + \frac{2k+1}{2h}\left(\overset{(+)}{\sigma}_{1j} - (-1)^k \overset{(-)}{\sigma}_{3j}\right).$$

Φ_j are components of volumetric forces, $P_k\left(\dfrac{x_3}{h}\right)$ is Legendre's polynomial of the order k, $\overset{(\pm)}{\sigma}_{3j} = \sigma_{3j}\ (x_1, x_2, \pm h)$ are stresses defined at the front surfaces of the plate; λ and μ are Lame's constants.

It can be easily seen that a certain class of boundary value problems for a plane is considered. The solution of all these boundary value problems will be considered for the case of boundary value problem (1.2), (1.3), (1.4), i.e. when anti-symmetry conditions are defined on the outline of the plate.

2 Solution of the problem for the plate

Arrange the functions $\overset{(k)}{F}_j$, $j = 1,2,3;\ k = 0,1$ into the corresponding trigonometric functions

$$
\begin{aligned}
\overset{(k)}{F}_1 &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \overset{(k)}{F}{}^{1}_{mn} \cos\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2},\\
\overset{(k)}{F}_2 &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \overset{(k)}{F}{}^{2}_{mn} \sin\frac{\pi m x_1}{a_1} \cos\frac{\pi n x_2}{a_2},\\
\overset{(k)}{F}_3 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \overset{(k)}{F}{}^{3}_{mn} \sin\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2}, \quad k = 0,1,
\end{aligned}
\tag{2.1}
$$

where $\overset{(k)}{F}{}^{j}_{mn}$ are Fourier coefficients of the functions $\overset{(k)}{F}_1$, $\overset{(k)}{F}_2$, $\overset{(k)}{F}_3$.

The desired values $\overset{(k)}{u}_1$, $\overset{(k)}{u}_2$, $\overset{(k)}{u}_3$ are represented as the following series

$$
\begin{aligned}
\overset{(k)}{u}_1 &= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \overset{(k)}{a}_{mn} \cos\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2},\\
\overset{(k)}{u}_2 &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \overset{(k)}{b}_{mn} \sin\frac{\pi m x_1}{a_1} \cos\frac{\pi n x_2}{a_2},\\
\overset{(k)}{u}_3 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \overset{(k)}{c}_{mn} \sin\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2}, \quad k = 0,1,
\end{aligned}
\tag{2.2}
$$

where $\overset{(k)}{a}_{mn}$, $\overset{(k)}{b}_{mn}$, $\overset{(k)}{c}_{mn}$ are the desired coefficients.

Substituting expressions (2.1), (2.2) into equation systems (1.3), (1.4) and comparing the coefficients for the same trigonometric functions, we obtain the following values for the coefficients $\overset{(k)}{a}_{01}$, $\overset{(k)}{b}_{10}$, $k = 1,2$

$$
\overset{(0)}{a}_{01} = \frac{a_2^2}{\pi^2}\frac{1}{\mu}\overset{(0)}{F}_{01}, \quad \overset{(0)}{b}_{10} = \frac{a_1^2}{\pi^2}\frac{1}{\mu}\overset{(0)}{F}_{10},
$$

$$
\overset{(1)}{a}_{01} = \frac{a_2^2 h^2}{\pi^2 h^2 + 3a_2^2}\frac{1}{\mu}\overset{(1)}{F}_{01}, \quad \overset{(1)}{b}_{10} = \frac{a_1^2 h^2}{\pi^2 h^2 + 3a_1^2}\frac{1}{\mu}\overset{(1)}{F}_{10}.
$$

For any fixed natural values of m and n we have the following systems of equations.

In the stretching-compression case

$$\begin{pmatrix} (\lambda+2\mu)\frac{\pi^2 m^2}{a_1^2}+\mu\frac{\pi^2 n^2}{a_2^2} & (\lambda+\mu)\frac{\pi^2 mn}{a_1 a_2} & -\frac{\lambda}{h}\frac{\pi m}{a_1} \\ (\lambda+\mu)\frac{\pi^2 mn}{a_1 a_2} & (\lambda+2\mu)\frac{\pi^2 n^2}{a_2^2}+\mu\frac{\pi^2 m^2}{a_1^2} & -\frac{\lambda}{h}\frac{\pi n}{a_2} \\ -\frac{3\lambda\pi}{h}\frac{m}{a_1} & -\frac{3\lambda\pi}{h}\frac{n}{a_2} & \mu\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)+\frac{3(\lambda+2\mu)}{h^2} \end{pmatrix}$$

$$\times \begin{pmatrix} \overset{(0)}{a}_{mn} \\ \overset{(0)}{b}_{mn} \\ \overset{(1)}{c}_{mn} \end{pmatrix} = \begin{pmatrix} \overset{(0)}{F}{}^1_{mn} \\ \overset{(0)}{F}{}^2_{mn} \\ \overset{(1)}{F}{}^3_{mn} \end{pmatrix}, \tag{2.3}$$

In the bend case

$$\begin{pmatrix} (\lambda+2\mu)\frac{\pi^2 m^2}{a_1^2}+\mu\frac{\pi^2 n^2}{a_2^2}+\frac{3\mu}{h^2} & (\lambda+\mu)\frac{\pi^2 mn}{a_1 a_2} & \frac{3\mu}{h}\frac{\pi m}{a_1} \\ (\lambda+\mu)\frac{\pi^2 mn}{a_1 a_2} & (\lambda+2\mu)\frac{\pi^2 n^2}{a_2^2}+\mu\frac{\pi^2 m^2}{a_1^2}+\frac{3\mu}{h^2} & \frac{3\mu}{h}\frac{\pi n}{a_2} \\ \frac{\mu}{h}\frac{\pi m}{a_1} & \frac{\mu}{h}\frac{\pi n}{a_2} & \mu\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right) \end{pmatrix}$$

$$\times \begin{pmatrix} \overset{(1)}{a}_{mn} \\ \overset{(1)}{b}_{mn} \\ \overset{(0)}{c}_{mn} \end{pmatrix} = \begin{pmatrix} \overset{(1)}{F}{}^1_{mn} \\ \overset{(1)}{F}{}^2_{mn} \\ \overset{(0)}{F}{}^3_{mn} \end{pmatrix}. \tag{2.4}$$

Denote the block matrix with arbitrary m and n by D_{mn} in the stretching-compression case and by Q_{mn} in the bend case.

$$\det D_{mn} = \mu^2\pi^4\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)^2\left[(\lambda+2\mu)\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)+\frac{12}{h^2}(\lambda+\mu)\right], \tag{2.5}$$

$$\det Q_{mn} = \mu^2(\lambda+2\mu)\pi^4\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)\left[\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)+\frac{3}{h^2})\right]. \tag{2.6}$$

As one can see from equalities (2.5) and (2.6), we have $\det D_{mn} > 0$, $\det Q_{mn} > 0$.

Hence the desired coefficients can be unambiguously defined for any m and n.

Quite similarly problems can be solved in the case when the conditions of symmetry and anti-symmetry are arbitrarily defined on the contour of the plate.

3 Solution of the problem for spherical and cylindrical shells

Now consider the case of zero approximation of Vekua's method for a strongly sloping spherical shell and sloping cylindrical shell. The shells are assumed to be rectangular in the plan. The solutions of the corresponding boundary value problems are constructed quite similarly to those of boundary value problems in the case of $N = 1$ approximation. Without dwelling on the statement of boundary value problems for the mentioned shells with anti-symmetry conditions on the contour we will just give expressions for the mn-th block of infinite block-diagonal matrices. Naturally, in this case the solution of the mentioned problems is also reduced to the infinite block-diagonal matrix.

In the case of strongly sloping spherical shell we have

$$S_{mn} = \begin{pmatrix} (\lambda+2\mu)\pi^2\frac{m^2}{a_1^2}+\mu\pi^2\frac{n^2}{a_2^2} & (\lambda+\mu)\pi^2\frac{m}{a_1}\frac{n}{a_2} & -\frac{2\lambda+3\mu}{R}\pi\frac{m}{a_1} \\ (\lambda+\mu)\pi^2\frac{m^2}{a_1^2}\frac{n^2}{a_2^2} & (\lambda+2\mu)\pi^2\frac{n^2}{a_2^2}+\mu\pi^2\frac{m^2}{a_1^2} & -\frac{2\lambda+3\mu}{R}\pi\frac{n}{a_2} \\ -\frac{2(\lambda+3\mu)}{R}\pi\frac{m^2}{a_1^2} & -\frac{2(\lambda+3\mu)}{R}\pi\frac{n}{a_2} & \mu\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right) \end{pmatrix}; \tag{3.1}$$

In the case of sloping cylindrical shell we have

$$C_{mn} = \begin{pmatrix} (\lambda+2\mu)\pi^2\frac{m^2}{a_1^2}+\mu\pi^2\frac{n^2}{a_2^2} & (\lambda+\mu)\pi^2\frac{m}{a_1}\frac{n}{a_2} & -\frac{\lambda}{\rho}\pi\frac{m}{a_1} \\ (\lambda+\mu)\pi^2\frac{m}{a_1}\frac{n}{a_2} & (\lambda+2\mu)\pi^2\frac{m^2}{a_1^2}+\mu\pi^2\frac{n^2}{a_2^2}+\frac{\mu}{\rho^2} & -\frac{\lambda+3\mu}{\rho}\pi\frac{n}{a_2} \\ -\frac{\lambda}{\rho}\pi\frac{m}{a_1} & -\frac{\lambda+3\mu}{\rho}\pi\frac{n}{a_2} & \mu\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)+\frac{\lambda+2\mu}{\rho^2} \end{pmatrix}, \tag{3.2}$$

where R and ρ denote radii of median surfaces of the spherical and cylindrical shells, respectively.

$$\det S_{mn} = \mu^2(\lambda+2\mu)\pi^4\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)^2\left[\mu\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)-\frac{2(\lambda+3\mu)(2\lambda+3\mu)}{\pi R^2(\lambda+2\mu)}\right].$$

Due to the expression $\det C_{mn}$ ($\det C_{mn} \neq 0$) being somewhat cumbersome we are not going to give it. On the basis of the strong sloping of the shell and estimating some expressions containing λ and μ we can show that $\det S_{mn}$ differs from zero.

4 Solution of boundary value problems for plates on the basis of binary mix theory

4.1. Problem statement. Let the above-mentioned plate occupy the same domain Ω and consist of a mix of two isotropic hard materials. Consider the corresponding boundary value problems for this case. As initial equations, we will take a three-dimensional system of equations of a binary mix version given in Green and Naghdi's and Steel's publication [8], [9].

In contrast to the classical theory, equilibrium equations and boundary conditions will also be given here for the case of zero approximation of Vekua's method, which coincide with the equations of flat deformable state of an infinitely long cylindrical body consisting of a binary mix. In the two-component theory at every point occupied by the body two displacement vectors and two tensors of deformations and stresses are considered, corresponding to two components of the mix. The components of the displacement vector u_j', u_j'', $j = 1,2,3$ will, for short, be united in the column matrix $u_j = (u_j', u_j'')^T$. In the zero approximation the desired functions $\overset{(0)}{u}_j$ denote averaged with respect to the width values

$$\overset{(0)}{u}_j = \frac{1}{2h}\int_{-1}^{1} u_j dx_3 = (\overset{(0)}{u'}_j, \overset{(0)}{u''}_j)^T.$$

On the borders of the median surface of the plate the following conditions are considered.

Symmetry conditions

$$\overset{(0)}{u}_\alpha = 0, \quad \partial_\alpha \overset{(0)}{u}_{3-\alpha} = 0 \qquad x_\alpha = 0 \qquad x_\alpha = a_\alpha; \tag{4.1}$$

Anti-symmetry conditions

$$\overset{(0)}{u}_{3-\alpha} = 0, \quad \partial_\alpha \overset{(0)}{u}_\alpha = 0 \qquad x_\alpha = 0 \qquad x_\alpha = a_\alpha, \quad \alpha = 1,2. \tag{4.2}$$

Problems are also considered when on some of the sides of the rectangle symmetry conditions are defined while on the others - those of anti-symmetry.

In the case of the $N = 1$ -th order approximation the statement of the problem will have form (1.1)¡ (1.2) if the value $\overset{(k)}{u}_j$, $j = 1, 2, 3$, $k = 0, 1$ implies a column matrix

$$\overset{(k)}{u}_j = (\overset{(k)}{u'}_j, \overset{(k)}{u''}_j)^T, \quad j = 1, 2, 3; \quad k = 0, 1.$$

The system of equilibrium equations in the case of the $N = 0$ -th order approximation will have the form

$$\begin{cases} A\Delta \overset{(0)}{u}_1 + B\partial_1 \left(\partial_1 \overset{(0)}{u}_1 + \partial_2 \overset{(0)}{u}_2 \right) + \overset{(0)}{F}_1 = 0, \\ \\ A\Delta \overset{(0)}{u}_2 + B\partial_2 \left(\partial_1 \overset{(0)}{u}_1 + \partial_2 \overset{(0)}{u}_2 \right) + \overset{(0)}{F}_2 = 0, \end{cases} \tag{4.3}$$

where

$$A = \begin{pmatrix} a_1 & c \\ c & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & d \\ d & b_2 \end{pmatrix}; \quad \overset{(0)}{F}_\alpha = (F'_\alpha, F''_\alpha)^T, \quad \alpha = 1, 2;$$

$$a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_5 + \lambda_1 - \tfrac{\alpha\rho_2}{\rho},$$
$$a_2 = \mu_2 - \lambda_5, \quad b_2 = \mu_2 + \lambda_5 + \lambda_2 + \tfrac{\alpha_2\rho_1}{\rho},$$

$$c = \mu_3 + \lambda_5, \quad d = \mu_3 + \lambda_3 - \lambda_5 - \tfrac{\alpha\rho_1}{\rho} = \mu_3 + \lambda_4 - \lambda_5 + \tfrac{\alpha\rho_2}{\rho}, \quad \alpha = \lambda_3 - \lambda_4,$$

$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \mu_1,\ \mu_2,\ \mu_3$ are elastic constants; $\rho_1,\ \rho_2$ are densities of the mix components, $\rho = \rho_1 + \rho_2$;

$$\overset{(0)}{F}_\alpha = \frac{1}{2h} \int_{-h}^{h} \Phi_\alpha dx_3 + \frac{1}{2h} \left(\overset{(+)}{\sigma}_{3\alpha} - \overset{(-)}{\sigma}_{3\alpha} \right), \quad \alpha = 1, 2.$$

$\Phi_\alpha = (\Phi'_\alpha, \Phi''_\alpha)^T$ is a column matrix consisting of the components of volumetric forces of two mix components, $\overset{(+)}{\sigma}_{3\alpha}$, $\overset{(-)}{\sigma}_{3\alpha}$ are stresses defined on two front surfaces and column matrices as well.

In the case under consideration, same as in the classical theory, equilibrium equations of the $N = 1$-th order approximation are divided into two independent systems of stretching-compression and bend. They have the following form:

The system of stretching-compression equations [10]

$$\begin{cases} A\Delta \overset{(0)}{u}_1 + B\partial_1 \overset{(0)}{\theta} + \dfrac{1}{h}\Lambda\partial_1 \overset{(1)}{u}_3 + \overset{(0)}{F}_1 = 0, \\ A\Delta \overset{(0)}{u}_2 + B\partial_2 \overset{(0)}{\theta} + \dfrac{1}{h}\Lambda\partial_2 \overset{(1)}{u}_3 + \overset{(0)}{F}_2 = 0, \\ A\Delta \overset{(1)}{u}_3 - \dfrac{3}{h^2}(A+B)\overset{(1)}{u}_3 - \dfrac{3}{h}\Lambda \overset{(0)}{\theta} + \overset{(1)}{F}_3 = 0; \end{cases} \tag{4.4}$$

the system of bend equations [10]

$$\begin{cases} A\Delta \overset{(1)}{u}_1 - \dfrac{3}{h^2}A\overset{(1)}{u}_1 + B\partial_1 \overset{(1)}{\theta} - \dfrac{3}{h}(B-\Lambda)\partial_1 \overset{(0)}{u}_3 + \overset{(1)}{F}_1 = 0, \\ A\Delta \overset{(1)}{u}_2 - \dfrac{3}{h^2}A\overset{(1)}{u}_2 + B\partial_2 \overset{(1)}{\theta} - \dfrac{3}{h}(B-\Lambda)\partial_2 \overset{(0)}{u}_3 + \overset{(1)}{F}_2 = 0, \\ A\Delta \overset{(0)}{u}_3 + \dfrac{1}{h}(B-\Lambda)\overset{(1)}{\theta} + \overset{(0)}{F}_3 = 0, \end{cases} \tag{4.5}$$

where

$$\overset{(k)}{\theta} = \partial_1 \overset{(k)}{u}_1 + \partial_2 \overset{(k)}{u}_2 = \left(\overset{(k)}{\theta'}, \overset{(k)}{\theta''}\right)^T, \quad k = 1, 2;$$

$$\Lambda = \begin{pmatrix} \lambda_1 - \frac{\alpha\rho_2}{\rho} & \lambda_3 - \frac{\alpha\rho_1}{\rho} \\ \lambda_4 + \frac{\alpha\rho_2}{\rho} & \lambda_2 + \frac{\alpha\rho_1}{\rho} \end{pmatrix},$$

$$\overset{(k)}{F}_j = \tfrac{2k+1}{2h}\int\limits_{-h}^{h} \Phi_j P_k\left(\tfrac{x_3}{h}\right)dx_3 + \tfrac{2k+1}{2h}\left(\overset{(+)}{\sigma}_{3j} - (-1)^k \overset{(-)}{\sigma}_{3j}\right).$$

Each of the systems (4.4) and (4.5) consists of six equations with six unknowns.

4.2. Solution of the stated problems. The solution of the boundary value problems stated in 4.1 will be considered here taking as an example the boundary value problem with anti-symmetry conditions on the sides of a rectangle.

In the case of zero approximation of the function $\overset{(0)}{F}_\alpha$, $\alpha = 1, 2$ is arranged with respect to the corresponding trigonometric functions, in particular,

$$\overset{(0)}{F}_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F^1_{mn} \cos\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2},$$
$$\overset{(0)}{F}_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} F^2_{mn} \sin\frac{\pi m x_1}{a_1} \cos\frac{\pi n x_2}{a_2}, \tag{4.6}$$

where

$$F^1_{mn} = \left(F^{1'}_{mn}, F^{1''}_{mn}\right)^T, \quad F^2_{mn} = \left(F^{2'}_{mn}, F^{2''}_{mn}\right)^T,$$

and the desired values $\overset{(0)}{u}_1$, $\overset{(0)}{u}_2$ are represented as the following series

$$\overset{(0)}{u}_1 = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} a_{mn} \cos\frac{\pi m x_1}{a_1} \sin\frac{\pi n x_2}{a_2},$$
$$\overset{(0)}{u}_2 = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} b_{mn} \sin\frac{\pi m x_1}{a_1} \cos\frac{\pi n x_2}{a_2}, \tag{4.7}$$

where a_{mn} and b_{mn} are column matrices consisting of the desired Fourier coefficients:

$$a_{mn} = \left(a'_{mn}, a''_{mn}\right)^T, \quad b_{mn} = \left(b'_{mn}, b''_{mn}\right)^T.$$

In the case of the $N = 1$-th order approximation we will have expressions absolutely similar to those of (2.1) and (2.2) if the arranged functions and coefficients imply column matrices consisting of the corresponding values for two mix components.

Substituting expressions (4.6), (4.7) in equation system (4.3) and comparing the coefficients for similar trigonometric functions we have

$$a_{01} = b_{10} = 0;$$

for any fixed natural values of m and n we obtain the following system of four equations with four unknowns

$$\begin{cases} \left[\dfrac{\pi^2 m^2}{a_1^2}(A+B) + \dfrac{\pi^2 n^2}{a_2^2}A\right] a_{mn} + \dfrac{\pi^2 mn}{a_1 a_2} B b_{mn} = F^1_{mn}, \\ \dfrac{\pi^2 mn}{a_1 a_2} B a_{mn} + \left[\dfrac{\pi^2 n^2}{a_2^2}(A+B) + \dfrac{\pi^2 m^2}{a_1^2}A\right] b_{mn} = F^2_{mn}. \end{cases} \tag{4.8}$$

In the case of the $N=1$-th order approximation we have

$$\overset{(0)}{a}_{01}=\frac{a_2^2}{\pi^2}A^{-1}\overset{(0)}{F}_{01},\quad \overset{(0)}{b}_{10}=\frac{a_1^2}{\pi^2}A^{-1}\overset{(0)}{F}_{10},$$

$$\overset{(1)}{a}_{01}=\frac{a_2^2h^2}{\pi^2h^2+3a_2^2}A^{-1}\overset{(1)}{F}_{01},\quad \overset{(1)}{b}_{10}=\frac{a_1^2h^2}{\pi^2h^2+3a_1^2}A^{-1}\overset{(1)}{F}_{10}.$$

For any fixed natural values of m and n we have the following systems of equations.

In the case of stretching and compression we have

$$\begin{cases}\left[\frac{\pi^2m^2}{a_1^2}(A+B)+\frac{\pi^2n^2}{a_2^2}A\right]\overset{(0)}{a}_{mn}+\frac{\pi^2mn}{a_1a_2}B\overset{(0)}{b}_{mn}-\frac{1}{h}\frac{\pi m}{a_1}\Lambda\overset{(1)}{c}_{mn}=\overset{(0)}{F}{}^1_{mn},\\[2ex] \frac{\pi^2mn}{a_1a_2}B\overset{(0)}{a}_{mn}+\left[\frac{\pi^2n^2}{a_2^2}(A+B)+\frac{\pi^2m^2}{a_1^2}A\right]\overset{(0)}{b}_{mn}-\frac{1}{h}\frac{\pi n}{a_2}\Lambda\overset{(1)}{c}_{mn}=\overset{(0)}{F}{}^2_{mn},\\[2ex] -\frac{3}{h}\frac{\pi m}{a_1}\Lambda\overset{(0)}{a}_{mn}-\frac{3}{h}\frac{\pi n}{a_2}\Lambda\overset{(0)}{b}_{mn}\\ +\left[\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)A+\frac{3}{h^2}(A+B)\right]\overset{(1)}{c}_{mn}=\overset{(1)}{F}{}^3_{mn}.\end{cases} \tag{4.9}$$

In the case of bend we have

$$\begin{cases}\left[\left(\frac{\pi^2m^2}{a_1^2}+\frac{\pi^2n^2}{a_2^2}+\frac{3}{h^2}\right)A+\frac{\pi^2m^2}{a_1^2}B\right]\overset{(1)}{a}_{mn}+\frac{\pi^2mn}{a_1a_2}B\overset{(1)}{b}_{mn}\\ +\frac{3}{h}\frac{\pi m}{a_1}(B-\Lambda)\overset{(0)}{c}_{mn}=\overset{(1)}{F}{}^1_{mn},\\[2ex] \frac{\pi^2mn}{a_1a_2}B\overset{(1)}{a}_{mn}+\left[\left(\frac{\pi^2m^2}{a_1^2}+\frac{\pi^2n^2}{a_2^2}+\frac{3}{h^2}\right)A+\frac{\pi^2n^2}{a_2^2}B\right]\overset{(1)}{b}_{mn}\\ +\frac{3}{h}\frac{\pi n}{a_2}(B-\Lambda)\overset{(0)}{c}_{mn}=\overset{(1)}{F}{}^2_{mn},\\[2ex] \frac{1}{h}\frac{\pi m}{a_1}(B-\Lambda)\overset{(1)}{a}_{mn}+\frac{1}{h}\frac{\pi n}{a_2}(B-\Lambda)\overset{(1)}{b}_{mn}+\pi^2\left(\frac{m^2}{a_1^2}+\frac{n^2}{a_2^2}\right)A\overset{(0)}{c}_{mn}=\overset{(0)}{F}{}^3_{mn}.\end{cases} \tag{4.10}$$

Let the matrices of the equation systems (4.8), (4.9) and (4.10) be denoted by L^1_{mn}, L^2_{mn}, L^3_{mn}, respectively. The fact that $\det L^j_{mn}$, $j=1,2,3$ is non-zero follows from the corresponding uniqueness theorems.

As for the strongly sloping spherical and cylindrical shells, similar to (3.1), (3.2), we can take the corresponding block matrices and study them. We cannot afford it here due to the brevity requirements.

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