ON THE FREDHOLM PROPERTY AND INDEX OF WIENER-HOPF PLUS/MINUS HANKEL OPERATORS WITH PIECEWISE ALMOST PERIODIC SYMBOLS [1]

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Abstract

It is obtained a Fredholm property characterization for matrix Wiener-Hopf plus/minus Hankel operators with piecewise almost periodic Fourier symbols. The conditions that ensure the Fredholm property are mainly based on factorizations of certain almost periodic matrix functions, and spectral properties of some others. In addition, Fredholm index formulas are also obtained based on an extension of the Cauchy index notion which is therefore applied to some new functions derived from the symbols of the operators.

Key words and phrases: Wiener-Hopf operator, Hankel operator, Fredholm property, Fredholm index, piecewise almost periodic function, semi-almost periodic function, almost periodic function, factorization.

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1 Introduction

Motivated by the needs of different kinds of applications, there is presently a growing interest in the study of Fredholm and invertibility properties of the so-called Wiener-Hopf plus/minus Hankel operators (cf., e.g., [2, 3, 4, 7, 8, 12]). In fact, these operators occur in a natural manner e.g. in the analysis of problems of wave diffraction by wedges (cf. [5, 6]). Therefore, an eventual additional knowledge about the Fredholm characteristics of these operators is welcome from both theoretical and practical reasons.

In the present paper we will consider matrix Wiener-Hopf plus/minus Hankel operators of the form

$$W_\Phi \pm H_\Phi : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \,, \tag{1.1}$$

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with W_φ and H_φ being matrix Wiener-Hopf and Hankel operators defined by

$$W_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F} : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N, \tag{1.2}$$

$$H_\Phi = r_+\mathcal{F}^{-1}\Phi \cdot \mathcal{F}J : [L^2_+(\mathbb{R})]^N \to [L^2(\mathbb{R}_+)]^N \ , \tag{1.3}$$

respectively. As usual, $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$ denote the Banach spaces of complex-valued Lebesgue measurable functions φ, for which $|\varphi|^2$ is integrable on $\mathbb{R}$ and $\mathbb{R}_+$, respectively. In (1.1)–(1.3), $L^2_+(\mathbb{R})$ denotes the subspace of $L^2(\mathbb{R})$ formed by all functions supported in the closure of $\mathbb{R}_+ = (0, +\infty)$, the operator r_+ performs the restriction from $L^2(\mathbb{R})$ into $L^2(\mathbb{R}_+)$, $\mathcal{F}$ denotes the Fourier transformation, J is the reflection operator given by the rule $J\Phi(x) = \widetilde{\Phi}(x) = \Phi(-x)$, $x \in \mathbb{R}$, and $\Phi \in [L^\infty(\mathbb{R})]^{N\times N}$ is the so-called Fourier symbol.

The main result in the present work (Theorem 7.4) provides a Fredholm characterization and an index formula for the following diagonal matrix operator:

$$\mathfrak{D}_\Phi = \mathrm{diag}\,[W_\Phi + H_\Phi, W_\Phi - H_\Phi] \ , \tag{1.4}$$

where Φ belongs to the piecewise almost periodic function class (which will be defined below in a detailed way). Therefore, the present paper extends the results of [3] where the Fredholm property and index of the operator $\mathfrak{D}_\Phi$ were described but only for Fourier symbols in the subclass of semi-almost periodic matrix functions. In addition, it complements some other recent results like the ones of [12].

In the next sections we will prepare the necessary material for the main theorem which will only appear in the last section. In this sense, in the previous sections to the last one, we will present some well-known results and generalize some others to a corresponding matrix version.

2 Preliminary notation and results

In this section we present some additional notation and recall certain known results that will be used throughout the paper.

For a Banach algebra B, $B^{N\times N}$ will denote the Banach algebra of all $N \times N$ matrices with entries in B. Moreover, we are going to denote by $\mathcal{G}B$ the group of all invertible elements in B.

Let $C(\dot{\mathbb{R}})$ (with $\dot{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$) represent the (bounded) continuous functions φ on the real line for which the two limits $\varphi(-\infty) := \lim_{x\to-\infty} \varphi(x)$ and $\varphi(+\infty) := \lim_{x\to+\infty} \varphi(x)$ exist and coincide. The common value of these two limits will be denoted by $\varphi(\infty)$. Further, $C_0(\dot{\mathbb{R}})$ will stand for the functions $\varphi \in C(\dot{\mathbb{R}})$ for which $\varphi(\infty) = 0$. In addition, $PC := PC(\dot{\mathbb{R}})$ denotes the C^*-algebra of all bounded piecewise continuous functions on $\dot{\mathbb{R}}$, and we also put $C(\bar{\mathbb{R}}) := C(\mathbb{R}) \cap PC$. Use will be also made of the C^*-algebra $PC_0 := \{\varphi \in PC : \varphi(\pm\infty) = 0\}$.

We recall here some of the essential facts from the theory of Wiener-Hopf and Hankel operators. The following equality is well-known [15]:

$$W_{\Phi\Psi} = W_\Phi W_\Psi + H_\Phi H_{\widetilde{\Psi}} \ , \tag{2.5}$$

for $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$. The next proposition is the matrix version of the classical scalar case, which is obviously also valid for the matrix case (one can derive the matrix case result by using the scalar one entrywise).

Proposition 2.1 *If $\Theta \in [C(\dot{\mathbb{R}})]^{N\times N}$, then the Hankel operators H_Θ and $H_{\widetilde{\Theta}}$ are compact.*

We can equivalently rewrite (2.5) as $W_{\Phi\Psi} - W_\Phi W_\Psi = H_\Phi H_{\widetilde{\Psi}}$, and therefore Proposition 2.1 directly yields the following known result.

Theorem 2.2 *If $\Phi, \Psi \in [L^\infty(\mathbb{R})]^{N\times N}$ and at least one of the functions Φ, Ψ belongs to $[C(\dot{\mathbb{R}})]^{N\times N}$, then $W_{\Phi\Psi} - W_\Phi W_\Psi$ is compact.*

Now, employing a continuous partition of the identity, one can sharpen Theorem 2.2 as follows.

Theorem 2.3 *If $\Phi, \Psi \in PC^{N\times N}$ and if at each point $x_0 \in \dot{\mathbb{R}}$ at least one of the functions Φ and Ψ is continuous, then $W_{\Phi\Psi} - W_\Phi W_\Psi$ is compact.*

proof. The result can be proved by following the same arguments as in the scalar case [11, Lemma 16.2], with corresponding changes for matrices in the places of functions. Namely, let $x_1, \ldots, x_\ell$ and $x_{\ell+1}, \ldots, x_r$ denote all the points of discontinuity of the matrix functions Φ and Ψ, respectively. Then, let Θ and Ξ be continuous matrix functions on $\dot{\mathbb{R}}$ with the following properties: $\Theta(x_k) = 0_{N\times N},\ k = 1, \ldots, \ell,\ \Xi(x_k) = 0_{N\times N},\ k = \ell+1, \ldots, r$, and $\Theta + \Xi \equiv I_{N\times N}$. This construction of Θ and Ξ turn clear that $\Phi\Theta$ and $\Xi\Psi$ are continuous on $\dot{\mathbb{R}}$. From Theorem 2.2 and $\Theta + \Xi = I_{N\times N}$, we have

$$\begin{aligned}
W_{\Phi\Psi} &= W_{\Phi(\Theta+\Xi)\Psi} = W_{\Phi\Theta\Psi} + W_{\Phi\Xi\Psi} \\
&= W_{\Phi\Theta} W_\Psi + K_1 + W_\Phi W_{\Xi\Psi} + K_2 \\
&= W_{\Phi\Theta} W_\Psi + W_\Phi W_{\Xi\Psi} + K_3 \\
&= (W_\Phi W_\Theta + K_4) W_\Psi + W_\Phi (W_\Xi W_\Psi + K_5) + K_3 \\
&= W_\Phi W_\Theta W_\Psi + K_6 + W_\Phi W_\Xi W_\Psi + K_7 + K_3 \\
&= W_\Phi (W_\Theta + W_\Xi) W_\Psi + K_8 \\
&= W_\Phi W_\Psi + K_8 \ ,
\end{aligned}$$

where K_i are compact operators ($i = 1, \ldots, 8$). From here we derive that $W_{\Phi\Psi} - W_\Phi W_\Psi$ is compact.

3 Matrix-valued PC symbols

For $\Phi \in PC^{N\times N}$, it is well-known the importance for the following auxiliary extension of Φ:

$$\Phi_2(x,\mu) := (1-\mu)\Phi(x-0) + \mu\Phi(x+0), \quad (x,\mu) \in \dot{\mathbb{R}} \times [0,1] \ ,$$

where $\Phi(x \pm 0)$ denotes the one-sided limits at the point x. This obviously yields $\det \Phi_2$ to have the form

$$\det \Phi_2(x,\mu) = \det[(1-\mu)\Phi(x-0) + \mu\Phi(x+0)], \quad (x,\mu) \in \dot{\mathbb{R}} \times [0,1] \, ,$$

and maps $\dot{\mathbb{R}} \times [0,1]$ into $\mathbb{C}$. One of the peculiarities of $\det \Phi_2$ is that it allows the consideration of

$$:= \{\det \Phi_2(x,\mu) \in \mathbb{C} : x \in \dot{\mathbb{R}} \ , \ \mu \in [0,1]\} \ ,$$

as a closed continuous curve formed by the union of the curve generated by the image of Φ and the curve that joins $\det \Phi_2(x-0)$ to $\det \Phi_2(x+0)$ through a line segment, at the discontinuity points of Φ. In the case of $0 \notin$, it is therefore possible to consider the winding number of , with respect to the origin, as the number of the counter-clockwise circuits around the origin performed by the image of $\det \Phi_2$. In such a case, this winding number will be denoted by $\mathrm{wind}[\det \Phi_2]$.

The next theorem is now considered a classical result in the Fredholm theory of Wiener-Hopf operators, and there the winding number plays a fundamental role.

Theorem 3.1 *Let $\Phi \in PC^{N\times N}$.*

(a) If $\det \Phi_2(x_0,\mu_0) = 0$ for some $(x_0,\mu_0) \in \dot{\mathbb{R}} \times [0,1]$, then W_Φ is not semi-Fredholm.

(b) If $\det \Phi_2(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$, then W_Φ is Fredholm and its Fredholm index is given by

$$\mathrm{Ind} W_\Phi = -\mathrm{wind}[\det \Phi_2] \ .$$

Suppose $\det \Phi_2(x,\mu) \neq 0$ for all $(x,\mu) \in \dot{\mathbb{R}} \times [0,1]$. Then $\Phi(x-0)$ and $\Phi(x+0)$ are invertible for all $x \in \dot{\mathbb{R}}$. Assume in addition that the set $\Delta_\Phi := \{x \in \dot{\mathbb{R}} : \ \Phi(x-0) \neq \Phi(x+0)\}$ is finite. For a connected component ℓ of $\dot{\mathbb{R}} \setminus \Delta_\Phi$, it is denoted by $\mathrm{ind}_\ell[\det \Phi]$ the increment of any continuous argument of $\det \Phi$ on ℓ, times 1 over 2π. Taking into account the possible jump at infinity, the winding number introduced above can be given in the following way (cf., e.g., [1, page 100]):

$$\mathrm{wind}[\det \Phi_2] = \mathrm{ind}[\det \Phi_2] + \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(\infty) \right\} \right) , \quad (3.1)$$

where

$$\mathrm{ind}[\det \Phi_2] = \sum_{\ell} \mathrm{ind}_{\ell}[\det \Phi] + \sum_{x \in \Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) ,(3.2)$$

and $\xi_1(x), \dots, \xi_N(x)$ are the eigenvalues of $\Phi^{-1}(x-0)\Phi(x+0)$ for $x \in \Delta_\Phi$, and where $\{c\}$ stands for the fractional part of the number c.

Thus, the last characterization of the Fredholm property can be reformulated in the following way.

Theorem 3.2 *(cf., e.g., [1, Theorem 5.10]) Let $\Phi \in \mathcal{G}PC^{N\times N}$. For W_Φ to be a Fredholm operator it is necessary and sufficient that*

$$\mathrm{sp}(\Phi^{-1}(x-0)\Phi(x+0)) \cap (-\infty, 0] = \emptyset \, ,$$

for all $x \in \dot{\mathbb{R}}$. Here $\mathrm{sp}(\Phi^{-1}(x-0)\Phi(x+0))$ stands for the set of eigenvalues of the matrix $\Phi^{-1}(x-0)\Phi(x+0)$.
If W_Φ is Fredholm and Φ has at most finitely many jumps then

$$\mathrm{Ind} W_\Phi = -\mathrm{wind}[\det \Phi_2] \, ,$$

where $\mathrm{wind}[\det \Phi_2]$ is given by (3.1)–(3.2).

4 Matrix-valued AP symbols

In this section we will consider the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions e_λ (with $\lambda \in \mathbb{R}$), where $e_\lambda(x) := e^{i\lambda x}$, $x \in \mathbb{R}$. This algebra is usually denoted by AP, and called the algebra of *almost periodic functions*. The following subalgebras will be also useful in our reasoning

$$AP_+ := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \geq 0\}, \quad AP_- := \mathrm{alg}_{L^\infty(\mathbb{R})}\{e_\lambda : \lambda \leq 0\}.$$

In fact, one of the reasons why the last two algebras are very useful is due to the fact that $AP^\pm = AP \cap H^\infty_\pm$ (where $H^\infty_\pm$ are the closed subalgebras of $L^\infty(\mathbb{R})$ of all elements which are non-tangential limits of functions in $H^\infty(\mathbb{C}_\pm)$, which are the well-known Hardy spaces).

Proposition 4.1 *[1, Proposition 2.22] Let $A \subset (0, \infty)$ be an unbounded set and consider $\{I_\alpha\}_{\alpha \in A} = \{(x_\alpha, y_\alpha)\}_{\alpha \in A}$ to be a family of intervals $I_\alpha \subset \mathbb{R}$ such that $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ as $\alpha \to \infty$. If $\varphi \in AP$, then the limit*

$$M(\varphi) := \lim_{\alpha \to \infty} \frac{1}{|I_\alpha|} \int_{I_\alpha} \varphi(x) dx$$

exists, is finite, and is independent of the particular choice of the family $\{I_\alpha\}$.

Definition 4.2 Let $\varphi \in AP$. The number $M(\varphi)$ given by Proposition 4.1 is called the *Bohr mean value* of φ (or simply the *mean value* of φ).

Remark 4.3 In the matrix case the *mean value* is defined entrywise.

Definition 4.4 ([9]) A matrix function $\Phi \in \mathcal{G}AP^{N \times N}$ is said to admit a *right AP factorization* if it can be represented in the form

$$\Phi(x) = \Phi_-(x) D(x) \Phi_+(x),$$

for all $x \in \mathbb{R}$, with

$$\Phi_- \in \mathcal{GAP}_-^{\mathcal{N}\times\mathcal{N}}, \quad \oplus_+ \in \mathcal{GAP}_+^{\mathcal{N}\times\mathcal{N}}, \tag{4.1}$$

$D(x) = \mathrm{diag}[e^{i\lambda_1 x}, \ldots, e^{i\lambda_N x}]$, and $\lambda_j \in \mathbb{R}$ are the so-called *right AP indices.* A right AP factorization with $D = I_{N \times N}$ is called a *canonical right AP factorization.*

In another way, it is said that a matrix function $\Phi \in \mathcal{G}AP^{N \times N}$ admits a *left AP factorization* if instead of (4.1) we have $\Phi(x) = \Phi_+(x)\, D(x)\, \Phi_-(x)$, for all $x \in \mathbb{R}$, and $\Phi_\pm$ and D having the same properties as above.

Remark 4.5 It is readily seen from the above definitions that if an invertible almost periodic matrix function Φ admits a right AP factorization, then $\widetilde{\Phi}$ admits a left AP factorization, and also Φ^{-1} admits a left AP factorization.

The vector containing the right AP indices will be denoted by $k(\Phi)$, i.e., in the above case $k(\Phi) := (\lambda_1, \ldots, \lambda_N)$. If we consider the case with equal *right AP indices* ($k(\Phi) = (\lambda, \ldots, \lambda)$), then the matrix $\mathbf{d}(\Phi) := M(\Phi_-)M(\Phi_+)$ is independent of the particular choice of the right AP factorization (cf., e.g., [1, Proposition 8.4]). In this case the matrix $\mathbf{d}(\Phi)$ is called the *geometric mean of* Φ*.*

5 Matrix-valued SAP symbols

Definition 5.1 The C^*-algebra SAP of all *semi-almost periodic functions* on $\mathbb{R}$ is defined as the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains AP and $C(\bar{\mathbb{R}})$.

The next theorem proved by D. Sarason in [13] reveals the structure of the SAP algebra.

Theorem 5.2 *Let $u \in C(\bar{\mathbb{R}})$ be any function for which $u(-\infty) = 0$ and $u(+\infty) = 1$. If $\varphi \in SAP$, then there exist $\varphi_\ell, \varphi_r \in AP$ and $\varphi_0 \in C_0(\dot{\mathbb{R}})$ such that*

$$\varphi = (1-u)\varphi_\ell + u\varphi_r + \varphi_0 \ .$$

The functions φ_ℓ, φ_r are uniquely determined by φ, and independent of the particular choice of u. In addition, the maps $\varphi \mapsto \varphi_\ell$ and $\varphi \mapsto \varphi_r$ are C^*-algebra homomorphisms of SAP onto AP.

Remark 5.3 The last theorem is also valid for the matrix case.

The following theorem gives a description of the Fredholm property for Wiener-Hopf operators with SAP symbols.

Theorem 5.4 *[1, Theorem 10.11] Let $\Phi \in SAP^{N\times N}$ and assume that the almost periodic representatives Φ_ℓ , Φ_r admit a right AP factorization. Then W_Φ is Fredholm if and only if*

(i) $\Phi \in \mathcal{G}SAP^{N\times N}$,

(ii) $k(\Phi_\ell) = k(\Phi_r) = (0, \dots, 0)$,

(iii) $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)) \cap (-\infty, 0] = \emptyset$,

where $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell))$ *stands for the set of the eigenvalues of the matrix*

$$\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell) := [\mathbf{d}(\Phi_r)]^{-1}\mathbf{d}(\Phi_\ell) \ .$$

Let $\mathcal{G}SAP_{0,0}$ denote the set of all functions $\varphi \in \mathcal{G}SAP$ for which $k(\varphi_\ell) = k(\varphi_r) = 0$. To define the Cauchy index of $\varphi \in \mathcal{G}SAP_{0,0}$ we need the next lemma.

Lemma 5.5 *[1, Lemma 3.12] Let $A \subset (0, \infty)$ be an unbounded set and let $\{I_\alpha\}_{\alpha\in A} = \{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ be a family of intervals such that $x_\alpha \geq 0$*

and $|I_\alpha| = y_\alpha - x_\alpha \to \infty$ *as* $\alpha \to \infty$. *If* $\varphi \in \mathcal{G}SAP_{0,0}$, *and* $\arg\varphi$ *is any continuous argument of* φ, *then the limit*

$$\mathrm{ind}\varphi := \frac{1}{2\pi}\lim_{\alpha\to\infty}\frac{1}{|I_\alpha|}\int_{I_\alpha}((\arg\varphi)(x) - (\arg\varphi)(-x))dx \tag{5.1}$$

exists, is finite, and is independent of the particular choices of $\{(x_\alpha, y_\alpha)\}_{\alpha\in A}$ *and* $\arg\varphi$.

The limit $\mathrm{ind}\varphi$ defined in (5.1) is called the *Cauchy index* of $\varphi \in \mathcal{G}SAP_{0,0}$.

Theorem 5.6 *[1, Theorem 10.12] If* $\Phi \in SAP^{N\times N}$, *the almost periodic representatives* Φ_ℓ, Φ_r *admit right AP factorizations, and* W_Φ *is Fredholm, then*

$$\mathrm{Ind}W_\Phi = -\mathrm{ind}[\det\Phi] - \sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\xi_k\right\}\right) \tag{5.2}$$

where $\xi_1, \ldots, \xi_N \in \mathbb{C}\backslash(-\infty, 0]$ *are the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$. *When choosing* $\arg\xi_k \in (-\pi, \pi)$, *we have*

$$\mathrm{Ind}W_\Phi = -\mathrm{ind}[\det\Phi] - \frac{1}{2\pi}\sum_{k=1}^{N}\arg\xi_k\ .$$

6 Matrix-valued PAP symbols

Let us consider the closed subalgebra of $L^\infty(\mathbb{R})$ formed by almost periodic and piecewise continuous functions. We will denote it by $PAP := \mathrm{alg}_{L^\infty(\mathbb{R})}\{AP, PC\}$. It is readily seen that $SAP \subset PAP$. In the scalar case it was proved that $PAP = SAP + PC_0$. The same situations is also valid in the matrix case considering the decomposition entrywise.

The next proposition is the matrix version of a well-known corresponding result for the scalar case (cf. e.g. [1, Proposition 3.15]).

Proposition 6.1 *(i) If* $\Phi \in [SAP + PC_0]^{N\times N}$, *then there are uniquely determined matrix-valued functions* $\Theta_\ell, \Theta_r \in AP^{N\times N}$ *and* $\Phi_0 \in PC_0^{N\times N}$ *such that*

$$\Phi = (1-u)\Theta_\ell + u\Theta_r + \Phi_0\ ,$$

where $u \in C(\mathbb{R})$, $0 \le u \le 1$, $u(-\infty) = 0$ *and* $u(+\infty) = 1$.

(ii) If $\Phi \in \mathcal{G}[SAP + PC_0]^{N\times N}$, then there exist matrix-valued functions $\Theta \in \mathcal{G}SAP^{N\times N}$ and $\Xi \in \mathcal{G}PC^{N\times N}$ such that $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$,

$$\Phi = \Theta\,\Xi\,, \tag{6.1}$$

and

$$W_\Phi = W_\Theta W_\Xi + K_1 = W_\Xi W_\Theta + K_2 \tag{6.2}$$

with compact operators K_1, K_2.

(iii) In addition, the Θ_ℓ and Θ_r elements used in (i) coincide with the local representatives of $\Theta \in \mathcal{G}SAP^{N\times N}$ used in (ii), and their unique existence is ensured by Theorem 5.2 and Remark 5.3.

proof. The proof of the part (i) can be given as the proof of the scalar case (cf. [1, Proposition 3.15]) upon reasoning entrywise, and therefore it is omitted in here.

The proof of part (ii) can also be done in a similar way to the scalar case but contains some additional small differences. Therefore, it will be performed here for the reader convenience. Suppose $\Phi \in \mathcal{G}[SAP + PC_0]^{N\times N}$, and put $\Upsilon := (1-u)\Theta_\ell + u\Theta_r$. Then $\Phi = \Upsilon + \Phi_0$. There is an $M \in (0,\infty)$ such that $|\det \Upsilon(x)|$ is bounded away from zero for $|x| > M$, and therefore we can find an element $\Upsilon_0 \in [C_0(\dot{\mathbb{R}})]^{N\times N}$ such that $\Theta := \Upsilon + \Upsilon_0 \in \mathcal{G}SAP^{N\times N}$. This allows us to look to Φ in the form

$$\begin{aligned} \Phi = \Theta + \Phi_0 - \Upsilon_0 &= \Theta[I + \Theta^{-1}(\Phi_0 - \Upsilon_0)] =: \Theta\Xi\,, \\ &(= [I + (\Phi_0 - \Upsilon_0)\Theta^{-1}]\Theta =: \Xi\Theta)\,, \end{aligned}$$

being clear that $\Xi = \Theta^{-1}\Phi \in \mathcal{G}PC^{N\times N}$ and $\Xi(-\infty) = \Xi(+\infty) = I_{N\times N}$. Since Θ is continuous on $\mathbb{R}$ and Ξ is continuous at ∞, we deduce from Theorem 2.3 that (6.2) holds for compact operators K_1 and K_2.

The part (iii) follows immediately from the construction made for (ii).

Remark 6.2 Due to the item (iii) of Proposition 6.1, we also call Θ_ℓ and Θ_r the local representatives of Φ at $-\infty$ and $+\infty$, respectively.

The matrix formulation presented in the next proposition is also an adaptation of the corresponding known scalar case (cf. e.g. [1, Theorem 3.16]).

Proposition 6.3 *Let $\Phi \in [SAP + PC_0]^{N\times N}$. If $\Phi \notin \mathcal{G}[SAP + PC_0]^{N\times N}$, then W_Φ is not semi-Fredholm. Assume now that $\Phi \in \mathcal{G}[SAP + PC_0]^{N\times N}$, and Φ_ℓ and Φ_r have a right AP factorization, then W_Φ is Fredholm if and only if* *(i)* $k(\Phi_\ell) = k(\Phi_r) = (0, \dots, 0)$,

(ii) $\mathrm{sp}(\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)) \cap (-\infty, 0] = \emptyset$,

(iii) $\mathrm{sp}(\Phi^{-1}(x-0)\Phi(x+0)) \cap (-\infty, 0] = \emptyset$,

for all $x \in \mathbb{R}$*.*

In the last case (under conditions (i)–(iii)), the Fredholm index of W_Φ *is provided by:*

$$\begin{aligned} \mathrm{Ind} W_\Phi = & - \sum_\ell \mathrm{ind}_\ell[\det \Xi] - \mathrm{ind}[\det \Theta] \\ & - \sum_{x\in\Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) && (6.3) \\ & - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) , && (6.4) \end{aligned}$$

where $\xi_k(x)$ *are the eigenvalues of the matrix function* $\Phi^{-1}(x-0)\Phi(x+0)$*, and* η_k *are the eigenvalues of the matrix* $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$*.*

proof. If $\Phi \notin \mathcal{G}[SAP+PC_0]^{N\times N}$, then $\Phi \notin \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$ and therefore W_Φ is not semi-Fredholm due to the corresponding I. B. Simonenko result [14].

Let us now consider $\Phi \in \mathcal{G}[SAP + PC_0]^{N\times N}$. Then we can write (cf. formula (6.1)) $\Phi = \Theta\Xi$ (with $\Theta \in \mathcal{G}SAP^{N\times N}$, $\Xi \in \mathcal{G}PC^{N\times N}$ and $\Xi(\pm\infty) = I_{N\times N}$) such that

$$W_\Phi = W_\Theta W_\Xi + K , \tag{6.5}$$

for a compact operator K. From here we infer that W_Φ is a Fredholm operator if and only if W_Θ and W_Ξ are also Fredholm operators. In the present context, these last two operators are Fredholm if and only if the conditions of the theorem are satisfied. More precisely, since W_Θ is a Wiener-Hopf operator with an invertible semi-almost periodic matrix symbol, and with lateral representatives $\Theta_\ell = \Phi_\ell$ and $\Theta_r = \Phi_r$ (cf. Proposition 6.1) which admit right AP factorizations, then W_Θ is Fredholm if and only if (cf. Theorem 5.4) $k(\Theta_\ell) = k(\Theta_r) = (0, \dots, 0)$, and $\mathrm{sp}(\mathbf{d}^{-1}(\Theta_r)\mathbf{d}(\Theta_\ell)) \cap (-\infty, 0] = \emptyset$.

We turn now to the operator W_Ξ. This operator has an invertible piecewise continuous matrix symbol. Therefore, applying Theorem 3.2, we obtain that W_Ξ is Fredholm if and only if

$$\mathrm{sp}(\Xi^{-1}(x-0)\Xi(x+0)) \cap (-\infty, 0] \neq \emptyset .$$

Now we simply have to observe that $\Xi^{-1}(x-0)\Xi(x+0) = \Phi^{-1}(x-0)\Phi(x+0)$, to reach the final conclusion.

To prove the index formula (6.3), assume that W_Φ (with $\Phi \in PAP^{N\times N}$) is a Fredholm operator. It is clear that from the equality (6.5) we can derive the index formula:

$$\mathrm{Ind} W_\Phi = \mathrm{Ind} W_\Theta + \mathrm{Ind} W_\Xi \ . \tag{6.6}$$

Using formulas (3.1), (3.2) and (5.2), from (6.6) it follows that

$$\begin{aligned}\mathrm{Ind} W_\Phi = - & \sum_\ell \mathrm{ind}_\ell [\det \Xi] \\ & - \sum_{x\in\Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) \\ - & \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(\infty) \right\} \right) \\ & - \mathrm{ind}[\det \Theta] - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \eta_k \right\} \right) ,\end{aligned} \tag{6.7}$$

where $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi^{-1}(x-0)\Phi(x+0)$, η_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r)\mathbf{d}(\Phi_\ell)$. Therefore, (6.3) follows from (6.7) by just taking into account that Ξ does not have a jump at infinity.

7 Wiener-Hopf plus/minus Hankel operators with matrix-valued PAP symbols

To give the corresponding result as the Proposition 6.3 for the operator $\mathfrak{D}_\Phi$ (cf. (1.4)) we need the notion of equivalence after extension relation, which is defined as follows.

Definition 7.1 We will say that a linear bounded operator $S : X_1 \to X_2$ (acting between Banach spaces) is equivalent after extension with another linear bounded operator $T : Y_1 \to Y_2$ (also acting between Banach spaces) if there exist Banach spaces Z_1 and Z_2, and boundedly invertible linear operators E and F, such that $\mathrm{diag}\,[T, I_{Z_1}] = E \ \mathrm{diag}\,[S, I_{Z_2}] \ F$, where I_{Z_i} represents the identity operator in the Banach space Z_i, $i = 1, 2$.

Remark 7.2 It is clear that if T is equivalent after extension with S, then T and S have the same Fredholm regularity properties (i.e., the properties that directly depend on the kernel and on the image of the operators).

Lemma 7.3 *Let $\Phi \in \mathcal{G}[L^\infty(\mathbb{R})]^{N\times N}$. Then $\mathfrak{D}_\Phi$ is equivalent after extension with $W_{\Phi\widetilde{\Phi^{-1}}}$.*

proof. This lemma has its roots in the Gohberg-Krupnik-Litvinchuk identity [8, 10], from which with additional equivalence after extension operator relations it is possible to find invertible and bounded linear operators E and F such that

$$\mathfrak{D}_\Phi = E \,\mathrm{diag}\left[W_{\Phi\widetilde{\Phi^{-1}}},\ I_{[L^2_+(\mathbb{R})]^N}\right]\ F\ . \tag{7.1}$$

A technique about how to construct such equivalence after extension relation is described in [4].

We are now in conditions to present the main theorem of the present work.

Theorem 7.4 *Let $\Phi \in \mathcal{G}PAP^{N\times N}$, and assume that $\Phi_\ell\widetilde{\Phi_r^{-1}}$ admits a right AP factorization, then the operator $\mathfrak{D}_\Phi$ is Fredholm if and only if*

(i) $\Phi_\ell\widetilde{\Phi_r^{-1}}$ *admits a canonical right AP factorization, i.e.,* $k(\Phi_\ell\widetilde{\Phi_r^{-1}}) = (0,\dots,0)$,

(ii) $\mathrm{sp}[\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset$,

(iii) $\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty,0] = \emptyset\ , \quad x \in \mathbb{R}.$

In addition, when in the presence of the Fredholm property

$$\begin{aligned}\mathrm{Ind}\mathfrak{D}_\Phi \;=\; & -\sum_\ell \mathrm{ind}_\ell[\det\Xi] - \mathrm{ind}[\det\Theta]\\ & -\sum_{x\in\Delta_\Phi}\sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{2\pi}\arg\xi_k(x)\right\}\right)\\ & -\sum_{k=1}^{N}\left(\frac{1}{2} - \left\{\frac{1}{2} - \frac{1}{\pi}\arg\eta_k\right\}\right)\ ,\end{aligned}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ is a corresponding factorization in the sense of (6.1) for the invertible matrix-valued PAP function $\Phi\widetilde{\Phi^{-1}}$ which appears in the formula (7.1), $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, and η_k are the eigenvalues of the matrix $\mathbf{d}(\Phi_\ell\widetilde{\Phi_r^{-1}})$.

proof. We will first prove the "if part" of the theorem. Set $\Psi := \Phi\widetilde{\Phi^{-1}}$ for the notation simplification of the further reasoning. Direct computations lead to $\Psi_\ell = \Phi_\ell\widetilde{\Phi_r^{-1}}$ and $\Psi_r = \Phi_r\widetilde{\Phi_\ell^{-1}}$. From here, it is also of importance to observe that

$$\Psi_\ell = \widetilde{\Psi_r^{-1}} \,. \tag{7.2}$$

From the hypothesis of the theorem (cf. condition (i) of the present theorem) we have that Ψ_ℓ admits a canonical right AP factorization. Employing the formula (7.2) we deduce that Ψ_r also admits a canonical right AP factorization. Set $\Lambda := \mathbf{d}(\Psi_\ell)$. From the condition (ii) of the present theorem we derive that $\mathrm{sp}[\Lambda^2] \cap (-\infty, 0] = \emptyset$. In fact, as far as we know that Ψ_ℓ admits a canonical right AP factorization, we can write it in the following normalized way:

$$\Psi_\ell = \Pi_-\Lambda\Pi_+ \,, \tag{7.3}$$

where $\Pi_\pm$ have the same factorization properties as the original lateral factors of the canonical factorization but with $M(\Pi_\pm) = I$. Thus, (7.3) allows

$$\Psi_r = \widetilde{\Psi_\ell^{-1}} = \widetilde{\Pi_+^{-1}}\Lambda^{-1}\widetilde{\Pi_-^{-1}} \,,$$

which in particular shows that $\mathbf{d}(\Psi_r) = \Lambda^{-1}$, and hence $\mathbf{d}^{-1}(\Psi_r) = \Lambda$. Consequently $\Lambda^2 = \mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)$ and the condition (ii) of the present theorem is equivalent to $\mathrm{sp}[\mathbf{d}^{-1}(\Psi_r)\mathbf{d}(\Psi_\ell)] \cap (-\infty, 0] = \emptyset$.

Condition (iii) allows us to conclude that

$$\mathrm{sp}[\Psi^{-1}(x-0)\Psi(x+0)] \cap (-\infty, 0] = \emptyset.$$

Altogether, we can conclude from the Proposition 6.3 that W_Ψ is a Fredholm operator. Employing the above introduced notion of the equivalence after extension we obtain that $\mathfrak{D}_\Phi$ is a Fredholm operator (cf. Lemma 7.3 and Remark 7.2). Thus the "if part" is proved.

Now we will proceed to prove the "only if" part. Assume that $\Phi \in \mathcal{G}PAP^{N\times N}$, with $\Phi_\ell\widetilde{\Phi_r^{-1}} = \Psi_\ell$ admitting a right AP factorization, and with $\mathfrak{D}$ being a Fredholm operator. Thus, as before, Ψ_r also admits a right AP factorization. In addition, from the formula (7.1) we conclude that W_Ψ is also a Fredholm operator. From the Proposition 6.3 (i) we deduce that Ψ_ℓ and Ψ_r admit a canonical right AP factorization. Moreover the three conditions (i)–(iii) of the Proposition 6.3 are satisfied for the function Ψ. Now, reasoning in a very similar way as in the "if part" we reach to the fact that $\Phi_\ell \widetilde{\Phi_r^{-1}}$ admits a canonical right AP factorization (recall that $\Psi_\ell = \Phi_\ell \widetilde{\Phi_r^{-1}}$),

$$\mathrm{sp}[\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})] \cap i\mathbb{R} = \emptyset ,$$

and

$$\mathrm{sp}[\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)] \cap (-\infty, 0] = \emptyset .$$

Hence the "only if" part is proved.

As about the index formula, by using the formula (7.1) we obtain that $\mathrm{Ind}\mathfrak{D}_\Phi = \mathrm{Ind}W_{\Phi\widetilde{\Phi^{-1}}}$. Therefore, from (6.3), one obtains that

$$\begin{aligned} \mathrm{Ind}\mathfrak{D}_\Phi = & - \sum_\ell \mathrm{ind}_\ell(\det \Xi) - \mathrm{ind}[\det \Theta] \\ & - \sum_{x \in \Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) \\ & - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \tau_k \right\} \right) , \end{aligned} \tag{7.4}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$ is a corresponding factorization in the sense of (6.1) for the invertible PAP function $\Phi\widetilde{\Phi^{-1}}$ which appears in the formula (7.1), and which is always possible due to the Proposition 6.1, $\xi_k(x)$ are the eigenvalues of the matrix function $\Phi(-x+0)\Phi^{-1}(x-0)\Phi(x+0)\Phi^{-1}(-x-0)$, and τ_k are the eigenvalues of the matrix $\mathbf{d}^{-1}(\Phi_r \widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}})$. As we already know that $\mathbf{d}^{-1}(\Phi_r \widetilde{\Phi_\ell^{-1}})\mathbf{d}(\Phi_\ell \widetilde{\Phi_r^{-1}}) = \Lambda^2$, then the formula (7.4) simplifies to the following one:

$$\begin{aligned} \mathrm{Ind}\mathfrak{D}_\Phi = & - \sum_\ell \mathrm{ind}_\ell(\det \Xi) - \mathrm{ind}[\det \Theta] \\ & - \sum_{x \in \Delta_\Phi} \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{2\pi} \arg \xi_k(x) \right\} \right) \\ & - \sum_{k=1}^{N} \left(\frac{1}{2} - \left\{ \frac{1}{2} - \frac{1}{\pi} \arg \eta_k \right\} \right) , \end{aligned}$$

where $\Phi\widetilde{\Phi^{-1}} = \Theta\Xi$, $\xi_k(x)$ are as above and η_k are the eigenvalues of the matrix Λ.

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