ON SOME NEW REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS IN LATTICED DOMAINS

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Abstract

In this paper some new representations of holomorphic functions in latticed domains are obtained by means of conformal mapping singular integral equation method.

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Various problems of Mathematical Physics are reduced to the Dirichlet planar problem for the Laplace equation in a latticed domains. It is well known that this problem is closely connected with the boundary value problems for the holomorphic functions. For example the theory of hydroturbins, the 3D motion of particles in a torus, the wave propagation in the sells (the size of sell is so small, that the whole domain we can consider as the infinite) [2,6,7,9].

At first, let us recall some notations from the theory of doubly-periodic functions [1, 3, 4, 6].

Consider a complex z-plane ($\mathbb{C}$), and two numbers $\omega_1, \omega_2 > 0$.

Definition 1 *A set* $D \subset \mathbb{C}$ *is called* doubly-periodic set *if* $z \in \mathbb{C}$ *implies* $z + 2m\omega_1 + 2ni\omega_2 \in D$, $m, n = 0, \pm 1, \ldots$ *points* z *and* $z + 2m\omega_1 + 2ni\omega_2$ *are called the congruent points.*

Definition 2 *The interior of the parallelogram with the vertexes* $0, 2\omega_1, 2\omega_1 + 2i\omega_2, 2i\omega_2$ *and with sides* $[0\omega_1]$ *and* $[0, 2i\omega_2]$ *is called the* fundamental parallelogram *[1,3,4, 6]. The interior domain of the fundamental parallelogram we denote by* S_{00}.

On the figures 1 and 2 examples of doubly-periodic sets are given.

We also use definitions introduced in [5].

Definition 3 *The function $\Phi(z)$ defined in the doubly-periodic domain is called* exponentially doubly-quasi periodic *if the following conditions are fulfilled*

$$\Phi(z+2\omega_1)=\Phi(z)\exp(P_{k_1}(z)),\ \Phi(z+2i\omega_2)=\Phi(z)\exp(Q_{k_2}(z)),$$

where P_{k_1} and Q_{k_2} are the definite polynomials of the k_1 and k_2 orders respectively.

This class of functions we denote by $P_e(k)$,$k=\max(k_1,k_2)$ [5].

Example. The Weierstrass "sigma-function" $\sigma(z)$ is of the class $P_e(1)$. Indeed

$$\sigma(z+2\omega_1)=\sigma(z)\exp(\delta_1(z+\omega_1)+\pi i),\sigma(z+2i\omega_2)=\sigma(z)\exp(\delta_2(z+i\omega_2)+\pi i),$$

where δ_1 and δ_2 are the definite constants [1,4,6].

Definition 4 *A function $F(z)$ defined in the doubly-periodic domain D is called* polynomially doubly quasi-periodic of k-order *with the periods $2\omega_1$ and $2i\omega_2$ if the following conditions are fulfilled*

$$F(z+2\omega_1)=F(z)+P_{k_1}(z),\quad F(z+2i\omega_2)=F(z)+Q_{k_2}(z),\ \ z\in D,$$

where P_{k_1}, Q_{k_2}, are the definite polynomials of degree k_1 and k_2 respectively. This polynomials we call the proper polynomials *of the function $F(z)$.*

This class of functions we denote by $\mathcal{P}(k)$, $k=\max(k_1,k_2)$ [5]. The class $\mathcal{P}(0)$ is the class of *doubly quasi-periodic* functions.
This class of functions we denote by $P_e(k)$,$k=\max(k_1,k_2)$ [5].

Example. The Weierstrass "zeta-function" $\zeta(z)$ is doubly quasi-periodic function [1,4,6]

$$\zeta(z+2\omega_1)=\zeta(z)+\delta_1,\quad \zeta(z+2i\omega_2)=\zeta(z)+\delta_2.$$

In a complex z-plane let us consider the doubly-periodic line L which is a union of a countable number of smooth non-intersected contours L^j_{mn} $j=1,2,\ldots,k,\ m,n=0,\pm1,\ldots$ doubly-periodically distributed with periods $2\omega_1$ and $2i\omega_2$ in the whole z-plane

$$L=\bigcup_{m,n=-\infty}^{\infty}L_{mn},$$

$$L_{mn}=\bigcup_{j=1}^{k}L^j_{mn},\quad \mathrm{L}^{J_1}_{mn}\bigcap L^{j_2}_{mn}=\varnothing,\quad j_1\neq j_2.$$

In the sequel we will consider the case, when the contours L_{mn} are closed.

By S^+ we denote an infinite region bounded by the contour L. The positive direction on L will be taken such that S^+ remains on the left.

The union of the finite domains S_{mn}^- contained in every L_{mn},$m, n = 0, \pm 1, \dots$ respectively will be denoted by S^-.

We consider the case when L_{00} is simple closed contour symmetric with respect to the axis $x = \omega_1$ and $y = i\omega_2$. Let us consider the following problem

Problem 1 *To find the real function $u(x, y)$,$u(x, y) \geq 0$, sub-harmonic in S^+, continuous in $S^- + L$, having simple zeros at the finite number of points b_i, $i = 1, 2, \dots, l$ of the area S_{00} and satisfying the boundary condition*

$$u = f(t), \quad t \in L, \tag{1}$$

where $f(t)$,$f(t) > 0$ is real function given on L satisfies the H condition on L_{00}, of $P_e(1)$class and symmetric with respect to the axis $y = i\omega_2$. $u(x, y)$ is also assumed to be symmetric and to belong to the class $P_e(1)$. We admit that proper polynomials of $f(t)$ are

$$P(z) = Re\,\alpha_1 z + \beta_1,$$
$$Q(z) = Re\,\alpha_2 z + \beta_2,$$

$\alpha_1, \beta_1, \alpha_2, \beta_2$ are given constants.

With this problem we will consider three auxiliary problems

Problem 2 *To find the real doubly-periodic function u^* harmonic in S^+ continuous in $S^- + L$,symmetric with respect to the axis $y = i\omega_2$ and satisfying the boundary condition*

$$u^* = f_0(t), \quad t \in L, \tag{2}$$

where $f_0(t)$ is real symmetric doubly-periodic function given on L satisfies the H condition on L_{00}

Problem 3 *To find the real function U^* sub-harmonic in S^+ continuous in $S^- + L$, of the class $P_e(1)$, symmetric with respect to the axis $y = i\omega_2$ satisfying the boundary condition (1) and the condition $U^*(x, y) > 0$.*

Problem 4 *To find a function $\Psi(z) = u_1 + iv_1$ of $P_e(1)$ class holomorphic in S^+ and continuous from the left everywhere on L,satisfying the following condition $|\Psi(t)| = f(t)$ $t \in L$.*

Let us begin with the Problem 2.

We will find u^* as the real part of the sectionally holomorphic doubly periodic function $\Phi(z)$.

We assume that the contour L_{00} is smooth and the angle between its tangent and some constant direction satisfies the H condition.

Let $z = w(z^*)$ be conformal mapping of the upper z^* plane on the upper half of the rectangle S_{00} with the following correspondence of points:to the points $\omega_1 + 2i\omega_2$, $\omega_1 + i\omega_2$ corresponds the points 0, ∞ consequently.

This mapping is given by the Shvartz-Kristoffel [1,4,6] formula.

$$w(z^*) = C \int_0^{z^*} \frac{dt}{\sqrt{(1-t^2)\,(1-k^2t^2)}} + \omega_1 + i\omega_2,$$

C and k are definite constants.

Continue this mapping analytically we obtain the mapping of the fundamental parallelogram on z plane. To the line L_{00} in z plane corresponds the line L^* in z^* plane.

The inverse transformation $z^* = w^{-1}(z)$ is an elliptic function with zeros at ω_1,$3\omega_1 + 2i\omega_2$ and with poles at $\omega_1 + i\omega_2$,$2\omega_1 + 3i\omega_2$.The periods of this function are $4\omega_1$ and $2i\omega_2$ [1,4,6] and

$$z^* = C^* \frac{\sigma(z-\omega_1)\sigma(z-3\omega_1-2i\omega_2)}{\sigma(z-\omega_1-i\omega_2)\sigma(z-2\omega_1-3i\omega_2)},$$

where σ is the Weierstrass "sigma-function" [1,4,9], C^* is the definite constant.

It is known from the theory of conformal mapping that, under the assumed conditions the functions, $w(z^*)$, $w^{-1}(z)$, are continuous from the left on L^* and L_{00} respectively.

It is obvious that if $f_0(t)$ satisfy the H condition on L_{00}, then the corresponding function obtained by the mapping $w^{-1}(z)$ will satisfy the H condition on L^* with the same Holder index.

So mapping the region S_{00} conformally on the plane and taking into the account that the function $\Phi(z)$ is doubly periodic and symmetric one arrives at exactly the same problem, but for the finite region S^* bounded by the line L^*, the boundary condition for the letter problem is given by the same formula if $f_0(t)$ is understood to the function $f_0(w(\tau))$ of points on the contour L^*.

Hence conclusions obtained for the case of region S^* may be directly transferred to the case of region S^-.

Using results of Muskhelishvili [8] the solution of the problem 2 in the plane z^* will be given by

$$u^*(z^*) = \text{Re } \Phi^*(z^*),$$

where

$$\Phi^*(z^*) = \frac{1}{\pi i}\int_{L_{00}} \frac{\varphi(t)}{t - z^*}dt + iC_0, \tag{3}$$

$\varphi(t)$ is unknown real function satisfying the integral equation

$$\varphi(t_0) + \mathrm{Re}\left\{\frac{1}{\pi i}\int_{L_{00}} \frac{\varphi(t)}{t - t_0}dt\right\} = f_0(t_0) + C_0, \quad t_0 \in L_{00}, \tag{4}$$

C_0 is the definite real constant.

This integral equation always has the unique solution [8]. So the initial problem is solvable.

Using the representation (3) and the transformation $t^* = w^{-1}(t)$ we obtain

$$\Phi(z) = \frac{1}{\pi i}\int_{L_{00}} \frac{\varphi(t)[w^{-1}(t)]'dt}{w^{-1}(t) - w^{-1}(z)} + iC_0, \tag{5}$$

Taking into the account symmetricity of the line L_{00} and of the function $\Phi^*(z^*)$ [8] and doubly-periodicity of the function

$$\frac{[w^{-1}(t)]'dt}{w^{-1}(t) - w^{-1}(z)}$$

we obtain the representation of the function $\Phi(z)$ in z plane in the following form

$$\Phi(z) = \frac{1}{\pi i}\int_{L_{00}} \varphi(t)\,[\zeta(t - z + \omega_1 + i\omega_2) + \zeta(t + z - \omega_1 - i\omega_2)]dt + iC_0, \tag{6}$$

where $\zeta(t - z)$ is the Weierstrass “zeta-function” and the integral equation (4) transforms to the following integral equation

$$\begin{aligned}&\varphi(t_0) + \mathrm{Re}\left\{\frac{1}{\pi i}\int_{L_{00}} \varphi(t)\,[\zeta(t - t_0 + \omega_1 + i\omega_2) + \zeta(t + t_0 - \omega_1 - i\omega_2)]dt\right\}\\ &= f_0(t_0) + C_0, \quad t_0 \in L_{00}.\end{aligned} \tag{7}$$

using the results of Muskhelishvili [8] we can conclude that the integral equation (7) always has the solution. Hence, the solution of the Problem 2 is uniquely determined.

Now we consider the Problem 4. In S^+ let us consider the function

$$\Psi^*(z) = \ln \Psi(z),$$

where ln is the definite branch of this function. The function $\Psi^*(z)$ is holomorphic in S^+, of the class $\mathcal{P}(1)$ and satisfies the following boundary condition

$$\mathrm{Re}\ \Psi^*(t) = \ln f(t), \quad t \in L. \tag{8}$$

According to the previous results and results of the author [5]we obtain the representation of the function $\Psi^*(z)$ in the form

$$\begin{aligned}\Psi^*(z) = \frac{1}{\pi i}\int_{L_{00}} \varphi(t)\left[\zeta(t - z + \omega_1 + i\omega_2) + \zeta(t + z - \omega_1 - i\omega_2)\right]dt \\ + A\ln\sigma(z - \omega_1 - i\omega_2) + \overline{A\ln\sigma(\overline{z - \omega_1 - i\omega_2})} \\ + D\zeta(z - \omega_1 - i\omega_2) + \overline{D\zeta(\overline{z - \omega_1 - i\omega_2})}) \\ + B(z - \omega_1 - i\omega_2)^2 + C_1(z - \omega_1 - i\omega_2) + iC_0, \qquad (9)\end{aligned}$$

where $\zeta(t-z)$ is the Weierstrass "zeta-function", C_0 is the definite real constant,

$$A = \frac{\alpha_1 i\omega_2 - \alpha_2\omega_1}{2\pi i},$$

$$B = \frac{\alpha_2\delta_1 - \alpha_1\delta_2}{4\pi i},$$

$$D = \frac{\beta_1 i\omega_2 - \beta_2\omega_1 - (\alpha_1 - \alpha_2)i\omega_2\omega_1}{2\pi i},$$

$$C_1 = \frac{\beta_2\delta_1 - \beta_1\delta_2 - i\omega_2\delta_1\alpha_2 + \omega_1\delta_2\alpha_1}{\pi i},$$

$\varphi(t)$ is unknown real function of H class, satisfying the integral equation

$$\begin{aligned}\varphi(t_0) + \mathrm{Re}\left\{\frac{1}{\pi i}\int_{L_{00}} \varphi(t)\left[\zeta(t - t_0 + \omega_1 + i\omega_2) + \zeta(t + t_0 - \omega_1 - i\omega_2)\right]dt\right\} \\ = \ln f^*(t_0) + C_0, \quad t_0 \in L, \qquad (10)\end{aligned}$$

where

$$\begin{aligned}f^*(t_0) = f(t) - A\ln\sigma(z - \omega_1 - i\omega_2) - \overline{A\ln\sigma(\overline{z - \omega_1 - i\omega_2})} \\ - D\zeta(z - \omega_1 - i\omega_2) - \overline{D\zeta(\overline{z - \omega_1 - i\omega_2})}) \\ - B(z - \omega_1 - i\omega_2)^2 - C_1(z - \omega_1 - i\omega_2). \qquad (11)\end{aligned}$$

This integral equation has the unique solution.Finding $\varphi(t)$ the solution of the Problem 4 will be given by

$$\Psi(z) = exp\Psi^*(z),$$

and the solution of the Problem 3 will be given by $U^*(z) = |\Psi(z)|$.

By means of the Problem 3 we now solve the initial problem .

For simplicity we assume that the function $u(x,y)$ has simple zeros at the points $a_1 + ia_2$, $2i\omega_2 + a_1 - ia_2$..

Consider the function

$$\Phi^*(z) = \ln \Phi_1(z) + \ln \sigma(z - a_1 - ia_2)\,\sigma(z - 2i\omega_2 - a_1 + ia_2), \tag{12}$$

where $\Phi_1(z)$ is holomorphic function of $P_e(1)$class with the same zeros as $u(z)$, subject the condition

$$u(z) = |\Phi_1(z)|. \tag{13}$$

The function $\Phi^*(z)$ is of the class $P_e(1)$ with the proper polynomials

$$P(z) = \mathrm{Re}(\alpha_1 + 4\delta_1)z + \beta_1 - 2\delta_1(2a_1 - 2\omega_1 + i\omega_2),$$
$$Q(z) = \mathrm{Re}(\alpha_2 + 4\delta_2)z + \beta_2 - 4\delta_1(a_1 - i\omega_2),$$

According to the formulas (9), (12), (13) the solution of the Problem 1 is given by

$$u = |\Phi_1(z)| = \left|\frac{\exp \Phi^*(z)}{\sigma(z - a_1 - ia_2)\,\sigma(z - 2i\omega_2 - a_1 + ia_2)}\right|,$$

where $\Phi^*(z)$ is the solution of the Problem 4.

Let us check the uniqueness of the Problem 1.

Let u_1 and u_2 be two possible solutions of the Problem 1.

The function

$$\Phi_0(z) = \ln \Phi_1(z) - \ln \Phi_2(z),$$

where $u_1 = |\Phi_1(z)|$, $u_2 = |\Phi_2(z)|$, will be doubly-periodic with no poles, satisfying the condition $\mathrm{Re}\,\Phi_0(z) = 0$. Hence $\Phi_0(z) = C$, C is the constant [1,4,6,8].

In some particular cases the solution of the Problem 3 and hence of the Problem 1 can be written effectively:

1) In the case of L_{00} is a rectangle $ABCD$ (Fig.1), we immediately obtain

$$\begin{aligned} U^*(z) = \Bigg| \exp\Bigg\{ &\frac{1}{\pi i}\int_{BC\cup AD} f^*(t)\,[\zeta(t - t_0) + \zeta(t + t_0)]dt \\ &+ \frac{1}{\pi}\int_{AB\cup CD} f^*(t)\,[\zeta(it - it_0) + \zeta(it + it_0)]dt \\ &+ A\ln\sigma(z - \omega_1 - i\omega_2) + \overline{A\ln\sigma(\overline{z - \omega_1 - i\omega_2})} \\ &+ D\zeta(z - \omega_1 - i\omega_2) + \overline{D\zeta(\overline{z - \omega_1 - i\omega_2})}) \\ &+ B(z - \omega_1 - i\omega_2)^2 + C_1(z - \omega_1 - i\omega_2) + iC_0 \Bigg|, \end{aligned}$$

where BC and AD are horizontal sides of the rectangle.

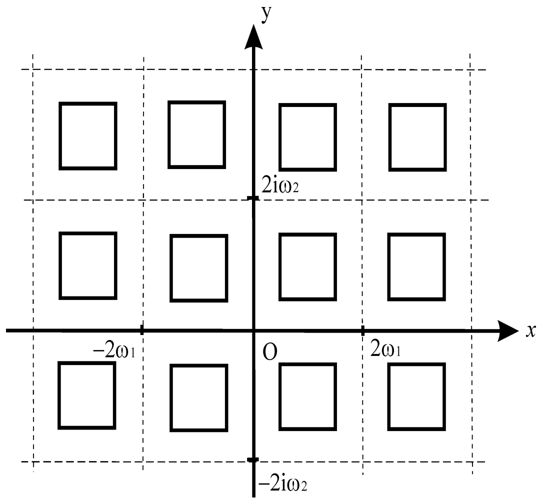

Figure 1: The case of rectangle.

2) In the case of L_{00} is the circle with the sufficiently small radius $r = \varepsilon$ and with the center (ω_1, ω_2) (Fig. 2). Using the behavior of conformal mapping and the Shvartz formula [5,8] we obtain

$$U^*(z) = \Big| \exp\left\{ \frac{1}{\pi i} \int_{L_{00}} f^*(t) \left[\zeta(t - t_0) + \zeta(t + t_0) \right] dt \right. \\ + A \ln \sigma(z - \omega_1 - i\omega_2) + \overline{A \ln \sigma(\overline{z - \omega_1 - i\omega_2})} \\ + D\zeta(z - \omega_1 - i\omega_2) + \overline{D\zeta(\overline{z - \omega_1 - i\omega_2})}) \\ + B(z - \omega_1 - i\omega_2)^2 + C_1(z - \omega_1 - i\omega_2) + iC_0 \Big| .$$

Note. The doubly-periodic problems for the doubly-periodic areas was considered in [3] and [10].

Note. In the case of $\alpha_1 = \alpha_2 = 0, \beta_1 = \beta_2 = \beta$,where β is a real constant, the function Re $\Psi(z)$ will be harmonic of the class $P_e(0)$

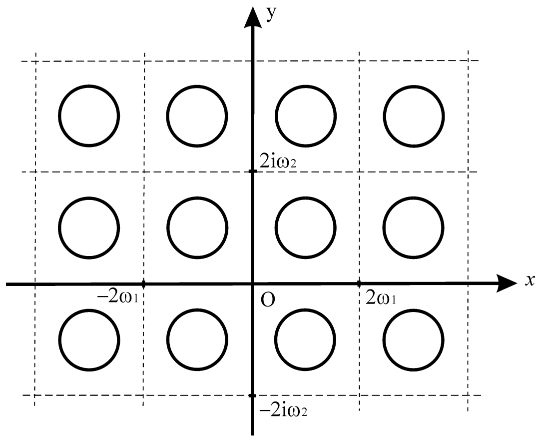

Fig.1. The case of $\mathrm{Im}\, 2\omega_1=0$; $\mathrm{Re}\, 2i\omega_2=0$.
L_{00} is the circle.

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