ON THE EXISTENCE AND NONEXISTENCE OF GLOBAL SOLUTIONS OF THE CHARACTERISTIC CAUCHY AND DARBOUX PROBLEMS FOR THE MULTIDIMENSIONAL NONLINEAR WAVE EQUATIONS

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Abstract

For the multidimensional nonlinear wave equations the characteristic Cauchy and Darboux problems are considered in the conical domains. Depending on the exponent of nonlinearity and on the space dimension of the equation, the problem of the existence and nonexistence of global solutions of the characteristic Cauchy or Darboux problem is investigated. The question of the local solvability of these problems is also considered.

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Let us consider a nonlinear wave equation of the type

$$\square u := \frac{\partial^2 u}{\partial t^2} - \Delta u = \lambda \mu(u) + F, \tag{1}$$

where μ is the given nonlinear function, $\mu(0) = 0$, F is the given and u is an unknown real functions; $\lambda \neq 0$ is the given real constant, $\Delta = \sum\limits_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$, $n \geq 2$.

For (1) we consider the Cauchy characteristic problem on finding in a truncated light cone of the future $D_T : |x| < t < T$, $x = (x_1, ..., x_n)$, $T = const > 0$, a solution $u(x,t)$ of that equation by the boundary condition

$$u|_{S_T} = f, \tag{2}$$

where f is the given real function on the characteristic cone surface S_T : $t = |x|, t \leq T$. When considering the case $T = +\infty$ we assume that $D_\infty : t > |x|$.

For (1) we also consider the Darboux problem on finding in a half of a truncated light cone $D_T^+ = D_T \cap \{x_n > 0\}$ a solution $u(x,t)$ of that equation under the boundary conditions

$$\left.\frac{\partial u}{\partial x_n}\right|_{S_T^0} = 0, \;\; u|_{S_T^+} = g, \tag{2+}$$

where $S_T^0 = \partial D_T^+ \cap \{x_n = 0\}$ and g is a given real function on $S_T^+ = S_T \cap \{x_n \geq 0\}$.

Note that the question on the existence or nonexistence of a global solution of the Cauchy problem for semilinear equation of type (1) with initial conditions $u|_{t=0} = u_0$, $\left.\frac{\partial u}{\partial t}\right|_{t=0} = u_1$ has been studied in [1]-[12].

Below we consider the cases when in equation (1)

$$\mu(u) = |u|^\alpha, \;\; \alpha = const > 0 \tag{3}$$

or

$$\mu(u) = |u|^\alpha \, sign\ u, \;\; \alpha = const > 0, \, \alpha \neq 1. \tag{4}$$

As for characteristic Cauchy and Darboux problems in a linear case, that is, when the right-hand side of (1) does not involve the nonlinear summand $\lambda\mu(u)$, these problems are, as it is known, formulated correctly, and the global solvability takes place in the corresponding spaces of functions [13]-[21].

We have showed, that under certain conditions on the degree of nonlinearity and the functions F and f, g these problems have or have not global solutions, though nevertheless these problems always are locally solvable [22]-[23].

Before introducing the definition of a weak generalized solution of problem (1), (2) in case $T = \infty$ note that, if $u \in C^2\left(\overline{D}_\infty\right)$ is a classical solution of problem (1), (2), then multiplying both parts of equation (1) by an arbitrary function $\varphi \in C^1\left(\overline{D}_\infty\right)$, finite on variable $r = \left(t^2 + |x|^2\right)^{\frac{1}{2}}$, that is, equal to zero of sufficient large r, after integration by parts we obtain

$$\int\limits_{\partial D_\infty} \frac{\partial u}{\partial N}\varphi ds - \int\limits_{D_\infty} u_t\varphi_t dxdt + \int\limits_{D_\infty} \nabla_x u \nabla_x \varphi dxdt$$

$$= \lambda \int\limits_{D_\infty} \mu(u)\varphi dxdt + \int\limits_{D_\infty} F\varphi dxdt, \tag{5}$$

where $\frac{\partial}{\partial N}=\nu_0\frac{\partial}{\partial t}-\sum\limits_{i=1}^{n}\nu_i\frac{\partial}{\partial x_i}$ is the derivative with respect to the conormal, $\nu=(\nu_1,...,\nu_n,\nu_0)$ is the unit vector of the outer normal to $S_\infty=\partial D_\infty$: $t=|x|$, $\nabla_x=\left(\frac{\partial}{\partial x_1},...,\frac{\partial}{\partial x_n}\right)$.

Taking into account that $S_\infty : t=|x|$ is the characteristic manifold on which $\frac{\partial}{\partial N}$ is the interior differential operator and by virtue of (2) equality (5) takes the form

$$-\int\limits_{D_\infty} u_t\varphi_t dxdt+\int\limits_{D_\infty}\nabla_x u\nabla_x\varphi dxdt$$

$$=\lambda\int\limits_{D_\infty}\mu(u)\varphi dxdt+\int\limits_{D_\infty}F\varphi dxdt-\int\limits_{\partial D_\infty}\frac{\partial f}{\partial N}\varphi ds. \tag{6}$$

Equality (6) can be set as a base of definition for a weak generalized solution of problem (1), (2).

Definition 1. Let $F\in\tilde{L}_{2,loc}(D_\infty)$ and $f\in\widetilde{W}^1_{2,loc}(\partial D_\infty)$. A function $u\in\tilde{L}_{\alpha,loc}(D_\infty)\cap\widetilde{W}^1_{2,loc}(D_\infty)$ is called a weak generalized solution of problem (1), (2) in domain D_∞, if for any function $\varphi\in C^1\left(\overline{D}_\infty\right)$, finite on variable $r=\left(t^2+|x|^2\right)^{\frac{1}{2}}$, the integral equality (6) is fulfilled. Such solution we shall call also a global solution of problem (1), (2) in domain D_∞.

Here the space $\tilde{L}_{2,loc}(D_\infty)$ $(\widetilde{W}_{2,loc}(\partial D_\infty))$ consists of functions $F(f)$, which restriction on set $D_\tau=D_\infty\cap\{t<\tau\}$ $(S_\tau=\partial D_\infty\cap\{t<\tau\})$ for any $\tau>0$ belongs to the space $L_2(D_\tau)(W^1_2(S_\tau))$. The spaces $\tilde{L}_{\alpha,loc}(D_\infty)$ and $\widetilde{W}^1_{2,loc}(D_\infty)$ are similarly defined (here $W^1_2(\Omega)$ is known Sobolev space).

Definition 2. Let $F\in L_2(D_T)$ and $f\in W^1_2(S_T)$. A function $u\in L_\alpha(D_T)\cap W^1_2(D_T)$ is called a weak generalized solution of problem (1), (2) in domain D_T, if for any function $\varphi\in C^1\left(\overline{D}_T\right)$ such that $\varphi|_{t=T}=0$ the integral equality

$$-\int\limits_{D_T} u_t\varphi_t dxdt+\int\limits_{D_T}\nabla_x u\nabla_x\varphi dxdt$$

$$=\lambda\int\limits_{D_T}\mu(u)\varphi dxdt+\int\limits_{D_T}F\varphi dxdt-\int\limits_{S_T}\frac{\partial f}{\partial N}\varphi ds \tag{7}$$

is fulfilled.

Theorem 1. Let

$$F\in\tilde{L}_{2,loc}(D_\infty),\ \lambda F|_{D_\infty}\geq 0 \tag{8}$$

and

$$f \in \widetilde{W}^1_{2,loc}(\partial D_\infty),\ \lambda f|_{\partial D_\infty} \geq 0,\ \ \lambda\ \frac{\partial f}{\partial r}\bigg|_{\partial D_\infty} \geq 0. \tag{9}$$

Then if $\mu(u) = |u|^\alpha$ and

$$1 < \alpha \leq \frac{n+1}{n-1}, \tag{10}$$

then there does not exist a weak generalized solution $u \in \tilde{L}_{\alpha,loc}(D_\infty) \cap \widetilde{W}^1_{2,loc}(D_\infty)$ of problem (1), (2) in domain D_∞ (which is nontrivial in case $F = 0$ and $f = 0$).

Below, in remarks 1-3, for simplicity we assume that $\lambda > 0$.

Remark 1. In case $0 < \alpha < 1$ and $\mu(u) = |u|^\alpha$ problem (1), (2) may have more than one global solution. For example, for $F = 0$ and $f = 0$ conditions (8) and (9) are fulfilled, but problem (1), (2) may, besides a trivial solution, have an infinite set of global independent solutions $u_\sigma(x,t) \in \tilde{L}^1_{2,loc}(D_\infty) \cup \widetilde{W}^1_{2,loc}(D_\infty)$, depending on parameter $\sigma \geq 0$, and given by the formula

$$u_\sigma(x,t) = \begin{cases} \beta\left[(t-\sigma)^2 - |x|^2\right]^{\frac{1}{1-\alpha}},\ t > \sigma + |x|, \\ 0,\ |x| \leq t \leq \sigma + |x|, \end{cases}$$

where $\beta = \lambda^{\frac{1}{1-\alpha}}\left[\frac{4\alpha}{(1-\alpha)^2} + \frac{2(n+1)}{1-\alpha}\right]^{-\frac{1}{1-\alpha}}$.

Remark 2. The conclusion of Theorem 1 fails to be valid if $\alpha > \frac{n+1}{n-1}$ and the second of conditions (9), i.e. condition $f|_{\partial D_\infty} \geq 0$ is violated. Indeed, function

$$u(x,t) = -\epsilon(1 + t^2 - |x|^2)^{\frac{1}{1-\alpha}},\ \epsilon = const > 0,$$

is the global classical, and hence, generalized solution of problem (1), (2) for $f = -\epsilon\big(\frac{\partial f}{\partial r}\big|_{\partial D_\infty} = 0\big)$ and $F = [2\epsilon\frac{n+1}{\alpha-1} - 4\epsilon\frac{\alpha}{(\alpha-1)^2}\frac{t^2-|x|^2}{1+t^2-|x|^2} - \lambda\epsilon^\alpha](1 + t^2 - |x|^2)^{\frac{\alpha}{1-\alpha}}$, where, as it can be easily verified, $F|_{D_\infty} \geq 0$ if $\alpha > \frac{n+1}{n-1}$ and $0 < \epsilon \leq \{\frac{2}{\lambda}[\frac{n+1-\frac{2\alpha}{\alpha-1}}{\alpha-1}]\}^{\frac{1}{\alpha-1}}$. Note that inequality $n + 1 - \frac{2\alpha}{\alpha-1} > 0$ is equivalent to inequality $\alpha > \frac{n+1}{n-1}$.

Remark 3. The conclusion of Theorem 1 also fails to be valid if only the third of conditions (9), i.e. condition $\frac{\partial f}{\partial r}\big|_{\partial D_\infty} \geq 0$ is violated. Indeed, function

$$u(x,t) = \beta[(t+1)^2 - |x|^2]^{\frac{1}{1-\alpha}},$$

where $\beta = \lambda^{\frac{1}{1-\alpha}}[\frac{4\alpha}{(1-\alpha)^2} + \frac{2(n+1)}{1-\alpha}]^{\frac{1}{1-\alpha}}$, is the global classical solution of problem (1), (2) for $F = 0$ and $f = u|_{\partial D_\infty} = \beta[(t+1)^2 - t^2]^{\frac{1}{1-\alpha}} > 0$.

Now consider question about local solvability of problem (1), (2) in case of homogeneous boundary condition (2).

Theorem 2. Let $\mu(u) = |u|^\alpha$, $\lambda \neq 0$, or $\mu(u) = |u|^\alpha sign\ u$ but $\lambda > 0$. If $f = 0$, $F \in \tilde{L}_{2,loc}(D_\infty)$ and $1 < \alpha < \frac{n+1}{n-1}$, then there exists the positive number $T_0 = T_0(F)$ such that for $T < T_0$ problem (1), (2) in domain D_T has at least one weak generalized solution $u \in L_\alpha(D_T) \cup W_2^1(D_T)$.

The estimate of number $T_0 = T_0(F)$, which is presented in Theorem 2, is given.

In the case $\mu(u) = |u|^\alpha sign\ u$ and $\lambda < 0$ there is the global solvability of problem (1), (2) in the following sense: let $f = 0$, then for every $F \in \tilde{L}_{2,loc}(D_\infty)$ and any $T > 0$ problem (1), (2) in domain D_T has at least one weak generalized solution $u \in L_\alpha(D_T) \cup W_2^1(D_T)$.

However if $\mu(u) = |u|^\alpha sign\ u$ but $\lambda > 0$, then there exists function $F \in \tilde{L}_{2,loc}(D_\infty)$ $(f = 0)$ and the positive number $T_0 = T_0(F)$ such that for $T \geq T_0$ problem (1), (2) can not have a solution in domain D_T.

Analogous results for the Darboux problem (1), (2+) are valid.

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