ON APPLICATION OF I. VEKUA'S METHOD FOR NON-LINEAR SHALLOW CYLINDRICAL SHELLS

B. Gulua
Iv. Javakhishvili Tbilisi State University
0186 University Street 2, Tbilisi, Georgia

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Abstract

In the present paper we consider the geometrically non-linear shallow cylindrical shells, when components of the deformation tensor have non-linear terms. By means of I. Vekua method two-dimensional problems is obtained. Using the method of the small parameter approximate solutions of I. Vekua's equations for approximations $N = 0$ and $N = 1$ is constructed. The small parameter $\varepsilon = h/R$, where $2h$ is the thickness of the shell, R is the radius of the cylinder. Concrete problem is solved, when components of external force are constants.

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The refined theory of shells is constructed by reduced the three- dimensional problems of the theory of elasticity to the two-dimensional problems. I.Vekua had obtained the equations of shallow shells [1],[2],[3]. It means that the interior geometry of the shell does not vary in thickness. This method for non-shallow shells in case of geometrical and physical non-linear theory was generalized by T.Meunargia [4].

In the present paper we consider the system of equilibrium equations of the two-dimensional geometrically non-linear shallow cylindrical shells which is obtained from the three-dimensional problems of the theory of elasticity for isotropic and homogeneous shell by the method of I.Vakua.

The system of equilibrium equations of the two-dimensional geometrically non-linear shallow cylindrical shells may be written in the following form (approximation $N = 0$):

$$\begin{aligned}
&\partial_1 \overset{(0)}{\sigma}_{11} + \partial_2 \overset{(0)}{\sigma}_{21} + \varepsilon \overset{(0)}{\sigma}_{13} + \overset{(0)}{F}_1 = 0,\\
&\partial_1 \overset{(0)}{\sigma}_{12} + \partial_2 \overset{(0)}{\sigma}_{22} + \overset{(0)}{F}_2 = 0,\\
&\partial_1 \overset{(0)}{\sigma}_{13} + \partial_2 \overset{(0)}{\sigma}_{23} - \varepsilon \overset{(0)}{\sigma}_{11} + \overset{(0)}{F}_3 = 0,
\end{aligned} \tag{1}$$

$$\overset{(0)}{\boldsymbol{F}} = \overset{(0)}{\boldsymbol{\Phi}} + \frac{1}{2h}\left[(1+\varepsilon)\overset{(+)}{\boldsymbol{\sigma}}_3 - (1-\varepsilon)\overset{(-)}{\boldsymbol{\sigma}}_3\right],$$

$$\left(\overset{(0)}{\sigma}_{ij},\ \overset{(0)}{\boldsymbol{\Phi}}\right) = \frac{1}{2h}\int\limits_{-h}^{h}(1+\varepsilon)\,(\sigma_{ij},\ \boldsymbol{\Phi})dx_3.$$

$$\boldsymbol{\sigma}_3(x_1, x_2, \pm h) = \overset{(\pm)}{\boldsymbol{\sigma}}_3,$$

where $\boldsymbol{\Phi}$ is an external force, σ_{ij} - covariant components of the stress tensor, x_1 and x_2 - isometric coordinates on the cylindrical surface, x_3 - the thickness coordinate, R - the radius of the middle surface of the cylinder.

Hook's law have the form:

$$\begin{aligned}
\overset{(0)}{\sigma}_{11} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right](1 + \partial_1 u_1 + \varepsilon u_3) \\
&+\mu\left[\left(2(\partial_1 u_1 + \varepsilon u_3) + (\partial_1\mathbf{u})^2\right)(1 + \partial_1 u_1 + \varepsilon u_3)\right. \\
&\left.+(\partial_1 u_2 + \partial_2 u_1 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})\partial_2 u_1\right], \\
\overset{(0)}{\sigma}_{12} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right]\partial_1 u_2 \\
&+\mu\left[\left(2(\partial_1 u_1 + \varepsilon u_3) + (\partial_1\mathbf{u})^2\right)\partial_1 u_2\right., \qquad (2) \\
&\left.+(\partial_1 u_2 + \partial_2 u_1 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})(1 + \partial_2 u_2)\right] \\
\overset{(0)}{\sigma}_{13} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right](\partial_1 u_3 - \varepsilon u_1) \\
&+\mu\left[\left(2(\partial_1 u_1 + \varepsilon u_3) + (\partial_1\mathbf{u})^2 + 1\right)(\partial_1 u_3 - \varepsilon u_1)\right. \\
&\left.+\ (\partial_1 u_2 + \partial_2 u_1 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})\partial_2 u_3\right], \\
\overset{(0)}{\sigma}_{21} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right]\partial_2 u_1 \\
&+\mu\left[(\partial_2 u_1 + \partial_1 u_2 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})(1 + \partial_1 u_1 + \varepsilon u_3)\right. \\
&\left.+(2\partial_2 u_2 + (\partial_2\mathbf{u})^2)\partial_2 u_1\right], \\
\overset{(0)}{\sigma}_{22} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right](1 + \partial_2 u_2) \\
&+\mu\left[(\partial_2 u_1 + \partial_1 u_2 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})\partial_1 u_2 + (2\partial_2 u_2 + (\partial_2\mathbf{u})^2)(1 + \partial_2 u_2)\right], \\
\overset{(0)}{\sigma}_{23} &= \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right]\partial_2 u_3 \\
&+\mu\left[(\partial_2 u_1 + \partial_1 u_2 + \partial_1\mathbf{u}\cdot\partial_2\mathbf{u})(\partial_1 u_3 - \varepsilon u_1)\right. \\
&\left.+(1 + 2\partial_2 u_2 + (\partial_2\mathbf{u})^2)\partial_2 u_3\right], \\
\overset{(0)}{\sigma}_{31} &= \mu\left[(\partial_1 u_3 - \varepsilon u_1)(1 + \partial_1 u_1 + \varepsilon u_3) + \partial_2 u_1\partial_2 u_3\right], \\
\overset{(0)}{\sigma}_{32} &= \mu\left[(\partial_2 u_3 + \partial_1 u_2\partial_1 u_3 + \partial_2 u_2\partial_2 u_3 - \varepsilon u_1\partial_1 u_2\right],
\end{aligned}$$

$$\overset{(0)}{\sigma}_{33} = \lambda\left[\partial_1 u_1 + \partial_2 u_2 + \varepsilon u_3 + \frac{1}{2}\left((\partial_1\mathbf{u})^2 + (\partial_2\mathbf{u})^2\right)\right] + \mu\left((\partial_1 u_3 - \varepsilon u_1)^2 + (\partial_2 u_3)^2\right),$$

where

$$\partial_1\mathbf{u}\cdot\partial_1\mathbf{u} = \frac{1}{2}\left[(\partial_1 u_1)^2 + (\partial_1 u_2)^2 + (\partial_1 u_3)^2 + \varepsilon^2(u_1^2 + u_3^2)\right],$$

$$\partial_2\mathbf{u}\cdot\partial_2\mathbf{u} = \frac{1}{2}\left[(\partial_2 u_1)^2 + (\partial_2 u_2)^2 + (\partial_2 u_3)^2\right],$$

$$\partial_1\mathbf{u}\cdot\partial_2\mathbf{u} = \frac{1}{2}\left[\partial_1 u_1\cdot\partial_2 u_1 + \partial_1 u_2\cdot\partial_2 u_2 + \partial_1 u_3\cdot\partial_2 u_3 + \varepsilon u_3\partial_2 u_3 - \varepsilon u_1\partial_2 u_1\right],$$

$$\mathbf{u} = \frac{1}{2h}\int_{-h}^{h}\mathbf{u}' dx_3.$$

Here u_i' are the components of the displacement vector, λ and μ - Lame's constants.

Let us use the method of the small parameter. The same method has been also used for spherical and cylindrical shallow shells [5],[6].

Let us construct the solutions of the form:

$$u_i = \sum_{k=0}^{\infty}\overset{(k)}{u}_i\varepsilon^k, \tag{3}$$

Formal substitution of (3) into (2) and (1) shows the series (3) may satisfy equations (1) if the following equations are fulfilled:

$$\begin{aligned} \mu\Delta\overset{(k)}{u}_1 &+ (\lambda+\mu)\partial_1\overset{(k)}{\theta} = \overset{(k)}{X}_1,\\ \mu\Delta\overset{(k)}{u}_2 &+ (\lambda+\mu)\partial_2\overset{(k)}{\theta} = \overset{(k)}{X}_2,\\ \mu\Delta\overset{(k)}{u}_3 &= \overset{(k)}{X}_3 \ (k=1,2,...), \end{aligned} \tag{4}$$

where

$$(k = 0,1,2,...;\ \ \overset{(k)}{u}_i = 0,\ \text{ if } k<0;).$$

For each fixed k equations (4) coincide with equations of plane theory of elasticity and Poisson. The right parts of equations (4) are well-known quantities, defined by functions $\overset{(1)}{\boldsymbol{u}}, \overset{(2)}{\boldsymbol{u}}, ..., \overset{(k-1)}{\boldsymbol{u}}$.

The complex form of the system (4) is:

$$\begin{aligned} &\mu\Delta\overset{(k)}{u}_+ + 2(\lambda+\mu)\partial_{\bar z}\overset{(k)}{\theta} = \overset{(k-1)}{X}_+,\\ &\mu\Delta\overset{(k)}{u}_3 = \overset{(k-1)}{X}_3, \end{aligned} \tag{5}$$

where $z = x_1 + ix_2$, $\overset{(k)}{u}_+ = \overset{(k)}{u}_1 + i\,\overset{(k)}{u}_2$, $\overset{(k)}{X}_+ = \overset{(k)}{X}_1 + i\,\overset{(k)}{X}_2$.

The general solutions of this system are written as following [7]:

$$\begin{cases} 2\mu\,\overset{(k)}{u}_+ = æ\,\overset{(k)}{\varphi}(z) - z\,\overline{\overset{(k)}{\varphi}{}'(z)} - \overline{\overset{(k)}{\psi}(z)} + \overset{(k)}{u}_{+p}, \\ 2\mu\,\overset{(k)}{u}_3 = \overset{(k)}{f}(z) + \overline{\overset{(k)}{f}(z)} + \overset{(k)}{u}_{3p}, \end{cases}$$

where $æ = \frac{\lambda+3\mu}{\lambda+\mu}$, $\overset{(k)}{\varphi}(z)$, $\overset{(k)}{\psi}(z)$ and $\overset{(k)}{f}(z)$ are any analytic functions of complex variable z, $\overset{(k)}{u}_{+p}$ and $\overset{(k)}{u}_{3p}$ - particular solutions of the system (5).

Let us solve the problem when the middle surface of the body after developing on the plane, is The circle with the radius r_0 and consider the concrete problem, when the components of external force are constant $\overset{(0)}{F}_i = P_i = const$. The boundary conditions are:

$$u_r = 0, \;\; u_\theta = 0, \;\; u_3 = 0, \;\; (z = re^{i\theta}, \;\; |z| = r_0).$$

For $\overset{(1)}{u}_+$ and $\overset{(1)}{u}_3$ we have:

$$\begin{aligned} 2\mu\,\overset{(1)}{u}_+ &= \frac{\mu}{\lambda+3\mu}(z\bar{z} - r_0^2)P_+, \\ 2\mu\,\overset{(1)}{u}_3 &= (z\bar{z} - r_0^2)\frac{P_3}{2}. \end{aligned}$$

The system of equilibrium equations for approximation $k = 2$ are

$$\begin{aligned} &\mu\Delta\,\overset{(2)}{u}_+ + 2(\lambda+\mu)\partial_{\bar{z}}\,\overset{(2)}{\theta} = \overset{(1)}{X}_+, \\ &\mu\Delta\,\overset{(2)}{u}_3 = \overset{(1)}{X}_3, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \overset{(1)}{X}_+ &= -(2\lambda+3\mu)\frac{\partial\overset{(1)}{u}_3}{\partial\bar{z}} - 3\mu\frac{\partial\overset{(1)}{u}_3}{\partial z} - 4\mu\frac{\partial}{\partial z}\left(\frac{\partial\overset{(1)}{u}_+}{\partial z}\frac{\partial\overline{\overset{(1)}{u}_+}}{\partial\bar{z}} + \frac{\partial\overset{(1)}{u}_3}{\partial z}\frac{\partial\overline{\overset{(1)}{u}_3}}{\partial\bar{z}}\right) \\ &- 2(\lambda+\mu)\frac{\partial}{\partial\bar{z}}\left(\frac{\partial\overset{(1)}{u}_+}{\partial z}\frac{\partial\overline{\overset{(1)}{u}_+}}{\partial\bar{z}} + \frac{\partial\overline{\overset{(1)}{u}_+}}{\partial z}\frac{\partial\overset{(1)}{u}_+}{\partial\bar{z}} + 2\frac{\partial\overset{(1)}{u}_3}{\partial z}\frac{\partial\overline{\overset{(1)}{u}_3}}{\partial\bar{z}}\right), \\ \overset{(1)}{X}_3 &= \frac{2\lambda+3\mu}{2}\left(\frac{\partial\overset{(1)}{u}_+}{\partial z} + \frac{\partial\overline{\overset{(1)}{u}_+}}{\partial\bar{z}}\right) + \frac{3\mu}{2}\left(\frac{\partial\overline{\overset{(1)}{u}_+}}{\partial z} + \frac{\partial\overset{(1)}{u}_+}{\partial\bar{z}}\right). \end{aligned}$$

The boundary conditions have the form

$$\begin{cases} \overset{(2)}{u}_r + i\overset{(2)}{u}_\theta = 0, \quad |z| = r_0, \\ \overset{(2)}{u}_3 = 0, \quad |z| = r_0. \end{cases} \tag{7}$$

The general solutions of the system (6) are:

$$\begin{cases} 2\mu\overset{(2)}{u}_+ = æ\overset{(2)}{\varphi}(z) - z\overline{\overset{(2)}{\varphi}{}'(z)} - \overline{\overset{(2)}{\psi}(z)} - \dfrac{2\lambda+3\mu}{16(\lambda+2\mu)}P_3 z^2\bar{z} \\ -\dfrac{3(\lambda+2\mu)}{32(\lambda+2\mu)}P_3 z\bar{z}^2 + \dfrac{\lambda+\mu}{32(\lambda+2\mu)}P_3 z^3, \\ 2\mu\overset{(2)}{u}_3 = \overset{(2)}{f}(z) + \overline{\overset{(2)}{f}(z)} - \dfrac{(2\lambda+3\mu)\overline{P_+} + 3\mu P_+}{16(\lambda+3\mu)} z^2\bar{z} \\ -\dfrac{(2\lambda+3\mu)P_+ + 3\mu\overline{P_+}}{16(\lambda+3\mu)} z\bar{z}^2. \end{cases} \tag{8}$$

Functions $\overset{(2)}{\varphi}(z)$, $\overset{(2)}{\psi}(z)$ and $\overset{(2)}{f}(z)$ are introduced by series:

$$\overset{(2)}{\varphi}(z) = \sum_{n=1}^{\infty} \overset{(2)}{a}_n z^n, \quad \overset{(2)}{\psi}(z) = \sum_{n=0}^{\infty} \overset{(2)}{b}_n z^n, \quad \overset{(2)}{f}(z) = \sum_{n=0}^{\infty} \overset{(2)}{c}_n z^n. \tag{9}$$

By substituting (8), (9) into (7) we obtain:

$$\overset{(2)}{a}_1 = \frac{(\lambda+\mu)(2\lambda+3\mu)}{32\mu(\lambda+2\mu)} r_0^3 P_3, \quad \overset{(2)}{a}_3 = \frac{(\lambda+\mu)^2}{32(\lambda+2\mu)(\lambda+3\mu)} P_3,$$

$$\overset{(2)}{b}_1 = \frac{3\mu}{8(\lambda+2\mu)} r_0^2 P_3, \quad \overset{(2)}{c}_1 = \frac{(2\lambda+3\mu)\overline{P_+} + 3\mu P_+}{16(\lambda+3\mu)} r_0^2.$$

For the components of the displacement vector $\overset{(2)}{u}_+$ and $\overset{(2)}{u}_3$ we get:

$$\begin{cases} 2\mu\overset{(2)}{u}_+ = \dfrac{2\lambda+3\mu}{16(\lambda+2\mu)} P_3(r_0^2 z - z^2\bar{z}) + \dfrac{3\mu P_3}{8(\lambda+3\mu)}(r_0^2\bar{z} - z\bar{z}^2), \\ 2\mu\overset{(2)}{u}_3 = \dfrac{(2\lambda+3\mu)\overline{P_+} + 3\mu P_+}{16(\lambda+3\mu)}(r_0^2 z - z^2\bar{z}) - \dfrac{(2\lambda+3\mu)P_+ + 3\mu\overline{P_+}}{16(\lambda+3\mu)}(r_0^2\bar{z} - z\bar{z}^2). \end{cases} \tag{10}$$

The problem will be solved when the middle surface of the body after developing on the plane, is The circular ring with the radiuses R_1 and R_2.

Let us consider I.Vekua approximation $N = 1$. Then the displacement vector has the following formula:

$$\mathbf{u}'(x_1, x_2, x_3) = \mathbf{u}(x_1, x_2,) + \frac{x_3}{h}\mathbf{v}(x_1, x_2)$$

$$\mathbf{u}=\frac{1}{2h}\int\limits_{-h}^{h}\mathbf{u}^{'}dx_3,\quad \mathbf{v}=\frac{3}{2h^2}\int\limits_{-h}^{h}\mathbf{u}^{'}x_3dx_3,$$

The system of equilibrium equations may be written in the following form:

$$\begin{array}{ll}\partial_\alpha \overset{(0)}{\sigma}_{\alpha 1}+\varepsilon \overset{(0)}{\sigma}_{13}+F_1=0, & \partial_\alpha \overset{(1)}{\sigma}_{\alpha 1}-3\overset{(0)}{\sigma}_{31}+\varepsilon \overset{(1)}{\sigma}_{13}+F_4=0,\\ \partial_\alpha \overset{(0)}{\sigma}_{\alpha 2}+F_2=0, & \partial_\alpha \overset{(1)}{\sigma}_{\alpha 2}-3\overset{(0)}{\sigma}_{32}+F_5=0,\\ \partial_\alpha \overset{(0)}{\sigma}_{\alpha 3}-\varepsilon \overset{(0)}{\sigma}_{11}+F_3=0, & \partial_\alpha \overset{(1)}{\sigma}_{\alpha 3}-3\overset{(0)}{\sigma}_{33}-\varepsilon \overset{(1)}{\sigma}_{11}+F_6=0,\end{array}\tag{11}$$

where

$$\overset{(m)}{\sigma}_{ij}=\frac{2m+1}{2h}\int\limits_{-h}^{h}\sigma_{ij}P_m\left(\frac{x_3}{h}\right)dx_3,\quad (m=0,1).$$

If we use the method of the small parameter

$$\begin{array}{l}u_3=\sum\limits_{k=0}^{\infty}\overset{k}{u}_3\varepsilon^k,\quad u_\alpha=\sum\limits_{k=0}^{\infty}\overset{k}{u}_\alpha\varepsilon^k\ \ (\alpha=1,2),\\ v_3=\sum\limits_{k=0}^{\infty}\overset{k}{v}_3\varepsilon^k,\quad v_\alpha=\sum\limits_{k=0}^{\infty}\overset{k}{v}_\alpha\varepsilon^k\ \ (\alpha=1,2).\end{array}$$

from (11) we obtain the systems of equilibrium equations in components of the displacement vector which for any k have the following form:

$$\begin{array}{lll}\mu\Delta \overset{k}{u}_1 & + & (\lambda+\mu)\partial_1\overset{k}{\theta}+\lambda\partial_1\overset{k}{v}_3=\overset{k}{X}_1,\\ \mu\Delta \overset{k}{u}_2 & + & (\lambda+\mu)\partial_2\overset{k}{\theta}+\lambda\partial_2\overset{k}{v}_3=\overset{k}{X}_2,\\ \mu\Delta \overset{k}{v}_3 & - & 3\left[\lambda\overset{k}{\theta}+(\lambda+2\mu)\overset{k}{v}_3\right]=\overset{k}{X}_3,\end{array}\tag{12}$$

$$\begin{array}{lll}\mu\Delta \overset{k}{v}_1 & + & (\lambda+\mu)\partial_1\overset{k}{\Theta}-3\mu(\partial_1\overset{k}{u}_3+\overset{k}{v}_1)=\overset{k}{X}_4,\\ \mu\Delta \overset{k}{v}_2 & + & (\lambda+\mu)\partial_2\overset{k}{\Theta}-3\mu(\partial_2\overset{k}{u}_3+\overset{k}{v}_2)=\overset{k}{X}_5,\\ \mu\Delta \overset{k}{u}_3 & + & \mu\overset{k}{\Theta}=\overset{k}{X}_6,\\ & & (k=1,2,...),\end{array}\tag{13}$$

where the quantities $\overset{k}{X}_i$, $(i=1,...,6)$ are expressed by the known functions and

$$\overset{(k)}{\theta}=\left(\frac{\partial \overset{(k)}{u}_+}{\partial z}+\frac{\partial \overline{\overset{(k)}{u}_+}}{\partial \bar z}\right),\quad \overset{(k)}{\vartheta}=\left(\frac{\partial \overset{(k)}{v}_+}{\partial z}+\frac{\partial \overline{\overset{(k)}{v}_+}}{\partial \bar z}\right),$$

$$\overset{(k)}{u}_{+} = \overset{(k)}{u}_{1} + i\,\overset{(k)}{u}_{2}, \quad \overset{(k)}{v}_{+} = \overset{(k)}{v}_{1} + i\,\overset{(k)}{v}_{2}.$$

The left parts of systems (12) and (13) coincide with equations of I.Vekua which is obtained for prismatic shells in case of approximation $N = 1$.

The problems when the middle surface of the body after developing on the plane are the circle with radius R_0 or the circular ring with the radiuses R_1 and R_2 will be solved.

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