ELLIPTIC SYSTEMS ON THE PLANE

G. Akhalaia, T. Makatsaria, N. Manjavidze

I. Vekua Institute of Applied Mathematics,
Iv. Javakhishvili Tbilisi State University
0143 University Street 2, Tbilisi, Georgia

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Abstract

On the basis of generalizing Cauchy's integral formulae the boundary value problems with discontinuous matrix coefficients for general elliptic systems of first order on the plane are solved. The necessary and sufficient conditions for the solvability and the index formulae of these problems in the weighted classes are established. Sufficiently wide classes of special (degenerate in point) differential equations are also studied.

Key words and phrases: Elliptic systems, Cauchy's integral formulae, Boundary value problems, Index formulae.
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1 Introduction

The first order linear system of partial differential equations

$$\frac{\partial u}{\partial x} = A(x,y)\,\frac{\partial u}{\partial y} + B(x,y)\,u(x,y) + F(x,y), \tag{1}$$

where $u = u(u_1, u_2, \ldots, u_n)$ is $2n$-desired vector, A, B are given real $2n \times 2n$-matrices, depending on two variables x, y, F is given real $2n$-vector, is said to be elliptic in domain D, if

$$\det(A - \lambda I) \neq 0, \tag{2}$$

for all real λ and for all points $(x, y) \in D$; I is a unit matrix. In other words the system is elliptic in some plane domain D if and only if the matrix A has no real characteristic numbers in D.

As it is well-known, when $n = 1$ in case of sufficient smoothness of the coefficients of (1), after corresponding changing of variables we can reduce the system to one complex equation

$$\partial_{\overline{z}} w + A_1 w + B_1 \overline{w} = F \quad \left(\partial_{\overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big)\right). \tag{3}$$

At present this equation is called Carleman-Vekua equation.

Remark. Some important generalizations in other directions see below in Section 3.

The complete theory of functions, satisfying this equation, the theory of generalized analytic functions was constructed by I. Vekua [13] Later on B. Bojarski has shown, that the methods of the theory of generalized analytic functions admit far-going generalizations on case of elliptic system of first order in complex form which has the following form

$$\partial_{\overline{z}} w - Q(z)\partial_z w + Aw + B\overline{w} = 0, \quad \Big(\partial_z = \frac{1}{2}(\partial x - i\,\partial y)\Big), \tag{4}$$

where $Q(z), A(z), B(z)$ are given square matrices of order n, $Q(z)$ is a matrix of special quasi-diagonal form [1], $Q(z) \in W_p^1(\mathbb{C})$, $p > 2$, $|q_{ii}| \leq q_0 < 1$, $Q(z) \equiv 0$ outside of some circle, A, B are bounded measurable matrices. (The notation $A \in K$, where A is a matrix and K is some class of functions, means that every element $A_{\alpha\beta}$ of A belongs to K).

The regular solutions of the equation (4) are called the generalized analytic vectors. In case $A = B = 0$ such solutions are called the Q-holomorphic vectors.

In the works of B. Bojarski by the full analogy with the theory of generalized analytic functions are given the formulae of general representations. On this basis the boundary value problems of linear conjugation and Riemann-Hilbert boundary value problem with Hölder-continuous coefficients are considered. These results of B. Bojarski and some further development of the theory of generalized analytic vectors are presented in the monograph [2].

The present paper first of all deals with discontinuous problems of the theory of analytic vectors. By analogy with the case of analytic functions [3,4] under these problems we mean the problems, where desired vectors in considered domains have angular boundary values and conditions continuous are to be fulfilled only almost everywhere on Γ. In addition given coefficients of the boundary conditions are to be piecewise continuous matrices. In the second part of our paper sufficiently wide classes of special (degenerate at point) differential equations are studied.

2 The solvability of the Problem (V)

Differential boundary value problem is such boundary value problem for which the boundary condition contains the boundary values of derivatives of the desired functions. In the theory of differential boundary value problems for holomorphic functions an integral representation formula constructed by I. Vekua (1942) [12] plays important role.

Let D be a finite domain bounded by a simple smooth curve Γ, $0 \in D$, let $\Phi(z)$ be holomorphic in D. Suppose the derivative of order m $(m \geq 1)$ of $\Phi(z)$ has boundary values on Γ satisfying Hölder-condition. Then $\Phi(z)$ can be represented by the formula

$$\Phi(z) = \int_\Gamma \mu(t)\Big(1 - \frac{z}{t}\Big)^{m-1} \ln\Big(1 - \frac{z}{t}\Big) ds + \int_\Gamma \mu(t) ds + ic, \tag{5}$$

where $\mu(t)$ is a real-valued function, $\mu(t) \in H(\Gamma)$ and c is a real constant; $\mu(t)$ and c are uniquely determined by $\Phi(z)$.

This representation gave I. Vekua the possibility to study the differential boundary value problem for holomorphic functions in Hölder-classes.

We introduce the suitable classes of generalized analytic vectors and for the elements of these classes the analog of I. Vekua representations, which allow us the investigation of discontinuous differential boundary value problems in these classes.

Denote by $E_p(D,Q)$, $p \geq 1$, $Q(z) \in W^1_{p_0}(\mathbb{C})$, $p_0 > 2$, the class of Q-holomorphic vectors in D satisfying the conditions

$$\int_{\delta_{kr}} |w_k(z)|^p \, |dz| \leq c, \quad k = 1, 2, \ldots, n, \tag{6}$$

where c is a constant, $\delta_{k\Gamma}$ is the image of the circumference $|\zeta| = r$, $r < 1$, under quasi-conformal mapping

$$\zeta = \omega_k(s_k(z)) \tag{7}$$

of unit circle $|\zeta| < 1$ onto D, ω_k is a schlicht analytic function in the domain $s_k(D)$, $s_k(z_)$ is a fundamental homeomorphism of the Beltrami equation

$$\partial_{\overline{z}} S - q_{kk}(z) \, \partial_z S = 0, \quad k = 1, 2, \ldots, n, \tag{8}$$

q_{kk} are the main diagonal elements of the matrix Q.

By $E_{m,p}(D,Q)$ denote the class of Q-holomorphic vectors satisfying the inequalities

$$\int_{\delta_{kr}} \left| \frac{\partial^m w_k(z)}{\partial z^m} \right|^p |dz| \leq c, \quad k = 1, 2, \ldots, n, \tag{9}$$

where c is a constant and δ_{kr} denotes the same.

By $E_{m,p}(D,Q,\rho)$ denote the class of the vectors $w(z)$ belonging to the class $E_{m,\lambda}(D,Q)$ for some $\lambda > 1$ such that the boundary values of the vector $\partial^m w / \partial z^m$ belong to the class $L_p(\Gamma, \rho)$.

If $w(z)$ is a Q-holomorphic vector from $E_{m,p}(D,Q,\rho)$, $Q(z)\in W^m_{p_0}(\mathbb{C})$, $p_0>2$. Then the analogous formula of (5) holds.

$$w(z)=\int_\Gamma \left[I-\zeta^{-1}(t)\right]^{m-1}\ln\left[I-\zeta(z)\,\zeta^{-1}(t)\right]\left[I+Q(t)\overline{t'}^{\,2}\right]\mu(t)\,ds + \int_\Gamma M(t)\,\mu(t)ds+iC, \tag{10}$$

where $C=\operatorname{Im}w(0)$, $M(t)=\operatorname{diag}[M_1(t),\dots,M_n(t)]$ is a definite real continuous diagonal matrix depending only on Q and Γ; the real vector $\mu(t)\in L_p(\Gamma,\rho)$ is defined uniquely by the vector $w(z)$. By $\ln[I-\zeta(z)\,\zeta^{-1}(t)]$ we mean the branch on the plane, cut along the curve l_t (l_t connects the point t on Γ with the point $z=\infty$ and lies outside of D) which is zero-matrix at the point $z=0$.

$E_{m,p}(D,Q,A,B,\rho)$ is the subclass of the class $E_{m,\lambda}(D,Q,A,B)$ for some $\lambda>1$ containing vectors whose angular boundary values $\partial^m w/\partial z^m$ belong to $L_p(\Gamma,\rho)$.

The following formula holds [13]:

$$w(z)=\Phi(z)+\int_D\left[\Gamma_1(z,t)\,\Phi(t)+\Gamma_2(z)\,\overline{\Phi(t)}\,\right]dt+\sum_{k=1}^{N}c_kW_k(z), \tag{11}$$

where $\Phi(z)$ is a Q-holomorphic vector, c_k are real constants, $\{W_k(z)\}$ ($k=1,\dots,N$) is a complete system of linearly independent solutions of the Fredholm equation

$$Kw\equiv w(z)-\frac{1}{\pi}\int_D V(t,z)\left[A(t)\,w(t)+B(t)\,\overline{w(t)}\,\right]d\sigma_t. \tag{12}$$

$W_k(z)$ turn out to be continuous vectors in the whole plane vanishing at infinity; the kernels $\Gamma_1(z,t)$ and $\Gamma_2(z,t)$ satisfy the system of integral equations

$$\begin{aligned}
&\Gamma_1(z,t)+\frac{1}{\pi}V(t,z)\,A(t)+\frac{1}{\pi}\int_D V(\tau,z)\left[A(\tau)\,\Gamma_1(\tau,t)+B(\tau)\overline{\Gamma_2(\tau,t)}\right]d\sigma_\tau\\
&\qquad=-\frac{1}{2}\sum_{k=1}^{N}\left\{v_k(z),\overline{v}_k(t)\right\},\\
&\Gamma_2(z,t)+\frac{1}{\pi}V(t,z)\,A(t)+\frac{1}{\pi}\int_D V(\tau,z)\left[A(\tau)\,\Gamma_2(\tau,t)+B(\tau)\overline{\Gamma_1(\tau,t)}\right]d\sigma_\tau\\
&\qquad=-\frac{1}{2}\sum_{k=1}^{N}\left\{v_k(z),\overline{v}_k(t)\right\},
\end{aligned} \tag{13}$$

where $v_k(z) \in L_p(\overline{D})$ $(k = 1, \ldots, N)$ form a system of linearly independent solutions of the Fredholm integral equation

$$v(z) + \frac{\overline{A'(z)}}{\pi} \int_D \overline{V'(z,t)}\, v(t)\, d\sigma_t + \frac{\overline{B'(z)}}{\pi} \int_D V'(z,t)\, \overline{v(t)}\, d\sigma_t = 0. \qquad (14)$$

In (13) the curly bracket $\{v, w\}$ means a diagonal product of the vectors v and w, the matrix $V(t,z)$ is generalized Cauchy kernel for the equation (4) in case $A(z) \equiv B(z) \equiv 0$. $\Phi(z)$ in (11) has to satisfy the following conditions

$$\operatorname{Re} \int_D \Phi(z)\, v_k(z)\, d\sigma_z = 0, \quad k = 1, \ldots, N. \qquad (15)$$

Note that generally speaking, the Liouville theorem in not true for solutions of the equation (4). This explains the appearence of the constants c_k in the representation formula (11) and the conditions (15).

From (11) we have

$$w(z) = \Phi(z) + h(z), \qquad (16)$$

where $\Phi(z) \in E_{m,p}(D, Q, \rho)$ and $h(z) \in H^m(D)$, $W_k(z) \in H^m(D)$.

In the section we consider differential boundary value problem of linear conjugation type for generalized analytic vectors, i.e. the boundary condition contains the boundary values of desired vector and its derivatives on both sides of jump line.

Let Γ be a smooth simple curve. Denote by $D^+(D^-)$ the finite (infinite) domain which is bounded by Γ. Suppose $0 \in D^+$. Consider the pair of equations

$$\frac{\partial w}{\partial \bar{z}} - Q_+(z)\frac{\partial}{\partial z} + A_+(z)\, w(z) + B_+(z)\, \overline{w(z)} = 0 \quad \text{in} \quad D^+ \qquad (17)$$

and

$$\frac{\partial w}{\partial \bar{z}} - Q_-(z)\frac{\partial}{\partial z} + A_-(z)\, w(z) + B_-(z)\, \overline{w(z)} = 0 \quad \text{in} \quad D^-, \qquad (18)$$

where $Q_+ \in W_p^l(\mathbb{C})$, $Q_- \in W_p^m(\mathbb{C})$, $A_+, B_+ \in H^{l-1}(D^+)$, $A_-, B_- \in H^{m-1}(D)$, $A_- = B_- = 0$ in certain neighborhood of $z = \infty$. By $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$ we denote the class of solutions of equations (17) and (18) respectively, belonging to the class $E_{l,p}(D^+, Q_+, A_+, B_+, \rho)$ $[E_{m,p}(D^-, Q_-, A_-, B_-, \rho)]$ in the domain $D^+\,[D^-]$. The classes $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, 0, 0, \rho)$ will be denoted by $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$.

Problem (V). Find a vector $w(z)$ of the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm)$ satisfying the boundary condition

$$\sum_{k=0}^{l} \left[a_k(t) \left(\frac{\partial^k w}{\partial t^k} \right)^+ + b_k(t) \overline{\left(\frac{\partial^k w}{\partial t^k} \right)^+} \right]$$

$$+\sum_{k=o}^{m}\left[c_k(t)\left(\frac{\partial^k w}{\partial t^k}\right)^{-}+d_k(t)\overline{\left(\frac{\partial^k w}{\partial t^k}\right)^{-}}\,\right]=f(t), \tag{19}$$

almost everywhere on Γ, where $a_k(t), b_k(t), c_k(t), d_k(t)$ are given piecewise continuous square matrices of order k, and $f(t)$ is given vector of the class $L_p(\Gamma,\rho)$.

Boundary condition can also contain integral term, which we omit for the sake of simplicity.

First we consider this problem in case $A_\pm = B_\pm = 0$, i.e. in the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$. For vectors of this class the following representation formula

$$w(t)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_+(z,\tau,l)\,d\zeta_+(\tau)\,\mu(\tau), & z\in D^+\\ -\dfrac{1}{2\pi i}\displaystyle\int_\Gamma S_-(z,\tau,m)\,d\zeta_-(\tau)\,\mu(\tau), & z\in D^-\end{cases} \tag{20}$$

holds, the kernels $S_+(z,t,l)$ and $S_-(z,t,m)$ are represented by the matrices $\zeta_+\,[\zeta_-]$ respectively. They are fundamental matrices for $Q_+(z)\,[Q_-(z)]$, $\mu(t)$ is the solution of the equation

$$N_\mu = (\dot{D})^l\big(\zeta_+^l(t)\,\Phi_+(t)\big)-\zeta_-^m\,\dot{D}^m\,\Phi_-(t) \quad \text{in} \quad L_p(\Gamma,\rho), \tag{21}$$

where

$$\begin{aligned}\dot{D}\,f(z)&=\alpha(z)\,f_{\overline{z}}(z)+\beta(z)\,f_z(z),\\ \alpha(z)&=-\overline{\zeta_{\overline{z}}(z)}\big[\zeta_z(z)\,\overline{\zeta_z(z)}-[\zeta_{\overline{z}}(z)\,\overline{\zeta_{\overline{z}}(z)}\big]^{-1},\\ \beta(z)&=-\zeta_z(z)\big[\zeta_z(z)\,\overline{\zeta_z(z)}-[\zeta_{\overline{z}}(z)\,\overline{\zeta_{\overline{z}}(z)}\big]^{-1}.\end{aligned} \tag{22}$$

Substituting the representation (20) into the boundary condition for desired vector $\mu(t)$ we obtain the following system of singular integral equations

$$K_\mu = K_1\mu+\overline{K_2}\mu = 2f(t), \tag{23}$$

where

$$K_s\mu = A_s(t)\,\mu(t)+\frac{B_s(t)}{\pi i}\int_\Gamma\frac{\mu(\tau)d\tau}{\tau-t}+\int_\Gamma k_s(t,\tau)\,\mu(\tau)\,d\tau \tag{24}$$
$$(s=1,2),$$

and

$$
\begin{aligned}
A_1(t) &= a_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l \zeta_+^{-l}(t) - c_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m \zeta_-^{-m}(t),\\
A_2(t) &= \overline{b_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l}\, \zeta_+^{-l}(t) - \overline{d_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m}\, \zeta_-^{-m}(t),\\
B_1(t) &= a_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l \zeta_+^{-l}(t) + c_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m \zeta_-^{-m}(t),\\
B_2(t) &= \overline{b_l(t)\Big(\frac{\partial}{\partial t}\,\zeta_+(t)\Big)^l}\, \zeta_+^{-l}(t) + \overline{d_m(t)\Big(\frac{\partial}{\partial t}\,\zeta_-(t)\Big)^m}\, \zeta_-^{-m}(t),
\end{aligned}
\tag{25}
$$

$k_s(\tau, t)$ are certain matrices with weak singularities.

In general case the problem (19) is to be considered in the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$, and we use the integral formula

$$
\begin{aligned}
w_\pm(z) = \Phi_\pm(z) &+ \int_\Gamma \big[\Gamma^1_\pm(z,\tau)\,\Phi_\pm(\tau) + \Gamma^2_\pm(z,\tau)\,\overline{\Phi(\tau)}\big]\,d\sigma_\tau\\
&+ \sum_{k=1}^{N^\pm} c^k_\pm\, W^k_\pm(z), \quad z \in D^\pm,
\end{aligned}
\tag{26}
$$

where the resolvents Γ^1, Γ^2 and the vector $W_k(z)$ are as introduced above. $c^k_\pm$ $(k = 1, \dots, N^\pm)$ unknown real constants, $\Phi_\pm(z)$ are unknown vectors of the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, \rho)$, satisfying additional conditions

$$
\operatorname{Im} \int_\Gamma \Phi_\pm(t)\, d_{Q^\pm} t\, \Psi^j_\pm(t) = 0, \quad j = 1, \dots, N^\pm, \tag{27}
$$

where $\{\Psi^j_\pm\}$ form a complete system of linearly independent solutions of conjugate equations, they are continuous in whole plane and vanish at infinity.

The formula (16) allows us to reduce the problem (19) to the case of Q-holomorphic vectors. Note that the vectors $W^k_\pm(z)$, $k = 1, \dots, N^\pm$ have continuous derivatives up to the required order because of smoothness of the coefficients of the equations (17) and (18).

Finally we obtain the following result

Theorem. On the inequality

$$
\inf_{t \in \Gamma} \big|\det \Omega(t)\big| > 0 \tag{28}
$$

holds, then the problem (19) is Noetherian in the class $E^{\pm}_{l,m,p}(\Gamma, Q_\pm, A_\pm, B_\pm, \rho)$ if and only if

$$
\frac{1 + \nu_k}{p} \neq \mu_{jk}, \tag{29}
$$

where $\mu_{jk} = 1/2\pi \arg \lambda_{jk}$, $0 \leq \arg \lambda_{jk} < 2\pi$, $k = 1, \ldots, r$, $j = 1, \ldots, n$, λ_{jk} are the roots of the equation

$$\det \left[\Omega^{-1}(t_k + 0)\, \Omega(t_k - 0) - \lambda I\right] = 0 \tag{30}$$

and $\Omega(t)$ is the block-matrix

$$\Omega(t) = \begin{pmatrix} \frac{c_m(t)}{d_l(t)} & \frac{b_m(t)}{a_l(t)} \end{pmatrix}. \tag{31}$$

Using I. Vekua representations we obtain necessary and sufficient solvability conditions and index formulae for Problem (V) in case when the plane is cut along several regular arcs for analytic functions so-called cut plane in various functional classes. These problems are important in applications. We have considered the general differential boundary value problems for analytic vectors as well as boundary value problems with shift complex conjugation on a cut plane [8-11].

3 Degenerate Elliptic Systems

As was mentioned above I. Vekua's scientific interest was concentrated on construction of the theory of generalized analytic functions and its applications in geometry and in the theory of elastic shells. I. Vekua systematically indicated the necessity of investigation of irregular equations. Let now consider the following equation

$$\frac{\partial w}{\partial \overline{z}} + \frac{a(z)}{f(z)}\, w + \frac{b(z)}{g(z)}\, \overline{w} = 0, \tag{32}$$

in some domain G of z-plane; $a, b \in L_p(G)$, $p > 2$; f and g are analytic functions on G, they may have zeros of arbitrary order and essential singularities. I. Vekua called these functions as analytic regularizators of the coefficients of the equation (32).

One of the fundamental results (and important tool of investigation of this equation) of the theory of generalized analytic functions is the general representation of solution by the analytic functions. Precisely for any $w(z)$ there exists a function $\Phi(z)$ analytic in G, such that

$$w(z) = \Phi(z) \exp\{\Omega(z)\}, \tag{33}$$

where

$$\Omega(z) = \frac{\pi}{f(z)} \iint_G \frac{a(\zeta)}{\zeta - z}\, dG(\zeta) + \frac{\pi}{g(z)} \iint_G \frac{b(\zeta)}{\zeta - z}\, \frac{\overline{\omega(\zeta)}}{\omega(\zeta)}\, dG(\zeta). \tag{34}$$

For regular coefficients the converse of this relation is given in I. Vekua's famous monograph [13], by the given analytic function $\Phi(z)$ the solution $w(z)$ is constructed. For general case this important result was also generalized by himself.

In regular case this relation completely reveals the properties of generalized analytic functions however even if one of the functions f and g has essentially singular point then nothing containing is known on behavior of the solution of the equation (32) in the neighborhood of this point. It is unknown how to use the relation in this case too.

Incomparably more is known in case when f and g have zeros but do not have essential singularities. This type of equations are called Carleman-Vekua equations with polar singularities.

Consider typical and important in applications the following Carleman-Vekua equation with polar singularities

$$|z|^{\nu}\frac{\partial w}{\partial \overline{z}} + a(z)\,w + b(z)\,\overline{w} = 0, \tag{35}$$

where the real number $\nu > 0$, $a, b \in L_p(G)$, $p > 2$ and G contains some neighborhood of $z = 0$ except this point (perforated neighborhood of $z = 0$). For these equations (differing form the regular case $\nu = 0$) it case take place very unexpected phenomenas.

It is very meaningful I. Vekua's emotional attitude to their problematic, which he expressed as follows. "Some simple examples show the complicate character of these problems" [14].

To make it clear let's consider the following examples:

$$|z|^{\nu}\frac{\partial}{\partial \overline{z}} + \varepsilon(\cos\varphi + i\sin\varphi)w = 0, \tag{36}$$

where $\nu > 1$, $\varepsilon = \pm 1$.

It is easy to show that the solutions of this equation in the neighborhood of $z = 0$ have essentially different behavior for $\varepsilon = 1$ and $\varepsilon = -1$. It follows that the problem of construction of general theory of such singular equations is very different and indeterminable however the validity of the following proposition about the structure of solutions of these equations under general assumptions for given ν, a, b is proved: every solution $w(z)$ of the equation (35) satisfying the equation

$$w(z) = O(\Psi(z)), \quad z \to 0 \tag{37}$$

for some analytic in domain G function $\Psi(z)$ is identically zero; every function $\Psi(z)$ satisfying condition

$$\Psi(z) = O(w(z)), \quad z \to 0 \tag{38}$$

in domain G is identically zero for some solution $w(z)$.

From above the following conclusion holds: the structure of solutions of Carleman-Vekua equations with polar singularities is principally nonanalytic.

We have obtained correct statement and complete analysis of boundary value problems for sufficiently wide class of equations of such type. They are first order singular equations. The equations of higher order undoubtly are of much theoretical and practical interest. In this connection let's consider the following system

$$\sum_{k=0}^{m} z^{\nu_k} A_k \frac{\partial^k w}{\partial \overline{z}^k} = 0, \tag{39}$$

where m, ν are given natural numbers, A_k $(0 \le k \le m)$ are given complex square $n \times n$-matrices. Under the solution of this system we mean a vector-function $w = (w_1, \ldots, w_n)$ of the class $C^m(G)$ satisfying the system (39) in every point of G. Note, that G is as above perforated neighborhood of $z = 0$. Assume that

$$\det A_0 \ne 0, \quad \det A_m \ne 0, \;\; A_k \cdot A_j = A_j \cdot A_k, \;\; 0 \le j, k \le m. \tag{40}$$

Construct all possible polynomials of the form

$$\tau_m \zeta^m + \tau_{m-1}\zeta^{m-1} + \cdots + \tau_1 \zeta + \tau_0 = 0, \tag{41}$$

where the coefficients τ_k is some eigen-value of the matrix A_k, $(0 \le k \le m)$. Denote by Δ the set of all complex roots of these polynomials and introduce a number $\delta_0 = \min\limits_{\zeta \in \Delta} |\zeta|$, obviously $\delta_0 > 0$.

Along with the solution $w(z)$ of the system (39) construct its characteristic function

$$T_w(\rho) = \max_{0 \le \varphi \le 2\pi} \sum_{k=1}^{n} \sum_{p=0}^{m-1} \left| \frac{\partial^p}{\partial \overline{z}^p} (\rho\, e^{i\varphi}) \right|, \quad \rho > 0. \tag{42}$$

The following theorem holds:

Theorem. Let $\nu \ge 2$ and $\Psi(z)$ be some analytic function in G. Let the solution $w(z)$ of the system satisfy the condition

$$T_w(z) = O\left(|\Psi(z)| \exp\left\{ \frac{\delta}{|z|^\delta} \right\} \right), \quad z \to 0. \tag{43}$$

where δ is some number and $\sigma < \nu - 1$.

Then the solution $w(z)$ is identically zero vector-function. Moreover when the equation (43) is fulfilled then $w(z)$ is also trivial if

$$\sigma = \nu - 1, \quad \delta < \delta_0 \cos \pi\beta, \quad \beta = \max\left\{ \nu, \frac{\nu - 3}{2\nu - 2} \right\}. \tag{44}$$

Note that in particular, where $\nu = 2$ for this system we succeeded to state correct boundary value problem to make its complete analysis. These results are particularly published in [5–7].

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