ON STABILITY AND CONVERGENCE OF SYMMETRIC THREE-LAYER SEMIDISCRETE SCHEME FOR ABSTRACT ANALOGUE OF NONLINEAR KIRCHHOFF WAVE EQUATION

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Abstract

In the present work Cauchy problem for abstract analogue of nonlinear Kirchhoff wave equation is considered. For approximate solution of this problem symmetric three-layer semi-discrete scheme is constructed. Stability and convergence for the offered scheme is shown.

Key words and phrases: nonlinear Kirchhoff wave equation, three-layer semidiscrete scheme.

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Let us consider the Cauchy problem for abstract hyperbolic equation in the Hilbert space H:

$$\frac{d^2u(t)}{dt^2} + A^2u\left(t\right) + a\left(\left\|A^{1/2}u\right\|^2\right)Au\left(t\right) = 0, \quad t \in [0,T]\,, \tag{1}$$

$$u\left(0\right) = \varphi_0, \quad \frac{du\left(0\right)}{dt} = \varphi_1. \tag{2}$$

where A is a self-adjoint (A does not depend on t), positively defined (generally unbounded) operator with the definition domain $D\left(A\right)$, which is everywhere dense in H, i.e. $\overline{D\left(A\right)} = H, \quad A = A^*$ and

$$(Au,u) \geq \nu \left\|u\right\|^2, \quad \forall u \in D\left(A\right), \quad \nu = const > 0,$$

where by $\|\cdot\|$ and $(\cdot,\cdot)$ are defined correspondingly the norm and scalar product in H; $a\left(\left\|A^{1/2}u\right\|^2\right) = \lambda + \left\|A^{1/2}u\right\|^2$, $\lambda > 0$; φ_0 and φ_1 are given vectors from H; $u\left(t\right)$ is a continuous, twice continuously differentiable, searched function with values in H.

As in the linear case (see [1], T. 1.5 p. 301) $u\left(t\right)$ vector function with values in H, defined on the interval $[0,T]$ is called a solution of the problem

(1)-(2) if it satisfies the following conditions: (a) $u(t)$ is twice continuously differentiable in the interval $[0,T]$; (b) $u(t) \in D\left(A^2\right)$ for any t from $[0,T]$ and the function $A^2 u(t)$ is continuous; (c) $u(t)$ satisfies equation (1) on the $[0,T]$ interval and the initial condition (2). Here continuity and differentiability is meant by metric H. Existence and uniqueness of the solution of the problem (1)-(2) is shown in [2].

Equation (1) is an abstract analogue of nonlinear Kirchhoff wave equation. Nonlinear Kirchhoff wave equation for stick has the following form:

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}-\left(\lambda+\int_0^L u_\xi^2(\xi,t)\,d\xi\right)\frac{\partial^2 u}{\partial x^2}=0.$$

We are searching solution of the problem (1)-(2) by the following semidiscrete scheme:

$$\begin{aligned}\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2} &+ A^2\frac{u_{k+1}+u_{k-1}}{2}\\ &+ a\left(\left\|A^{1/2}u_k\right\|^2\right)\frac{Au_{k+1}+Au_{k-1}}{2}=0, \qquad (3)\end{aligned}$$

where $k=1,...,n-1,\ \tau=T/n\ (n>1)$.

As an approximate solution $u(t)$ of problem (1)-(2) at point $t_k=k\tau$ we declare u_k-s, $u(t_k)\approx u_k$.

Theorem 1. *Vectors $(u_{k+1}-u_k)/\tau$, $A^{1/2}u_k$ and Au_k are equally bounded, i.e there exist constants M_1, M_2 and M_3 (independent of n) such that*

$$\begin{aligned}\left\|\frac{u_k-u_{k-1}}{\tau}\right\| &\le M_1,\\ \|Au_k\| &\le M_2, \quad \left\|A^{1/2}u_k\right\|\le M_3, \quad k=1,...,n.\end{aligned}$$

Proof. If we multiply scalarly both sides of equality (3) on vector $u_{k+1}-u_{k-1}=(u_{k+1}-u_k)+(u_k-u_{k-1})$, we obtain:

$$\begin{aligned}&\left\|\frac{u_{k+1}-u_k}{\tau}\right\|^2+\frac12\|Au_{k+1}\|^2+\frac12 a\left(\left\|A^{1/2}u_k\right\|^2\right)\left\|A^{1/2}u_{k+1}\right\|^2\\ =&\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2+\frac12\|Au_{k-1}\|^2+\frac12 a\left(\left\|A^{1/2}u_k\right\|^2\right)\left\|A^{1/2}u_{k-1}\right\|^2. \qquad (4)\end{aligned}$$

Let us introduce denotations:

$$\alpha_k=\left\|\frac{u_k-u_{k-1}}{\tau}\right\|^2, \quad \beta_k=\|Au_k\|^2, \quad \gamma_k=\left\|A^{1/2}u_k\right\|^2.$$

Then (4) will have the following form:

$$\begin{aligned} & \alpha_{k+1}+\frac{1}{2}\left(\beta_{k+1}+\beta_k\right)+\frac{1}{2}\left(\lambda+\gamma_k\right)\gamma_{k+1} \\ = & \alpha_k+\frac{1}{2}\left(\beta_k+\beta_{k-1}\right)+\frac{1}{2}\left(\lambda+\gamma_k\right)\gamma_{k-1}. \end{aligned}$$

Whence we have:

$$\lambda_{k+1}=\lambda_k+\varepsilon_k,$$

where

$$\begin{aligned} \lambda_k &= \alpha_k+\frac{1}{2}\left(\beta_k+\beta_{k-1}\right)+\frac{1}{2}\left(\lambda+\gamma_{k-1}\right)\gamma_k, \\ \varepsilon_k &= \frac{1}{2}\lambda\left(\gamma_{k-1}-\gamma_k\right). \end{aligned}$$

Obviously from (4) we obtain:

$$\begin{aligned} \lambda_{k+1} &= \lambda_1+\left(\varepsilon_1+\varepsilon_2+\ldots+\varepsilon_k\right) \\ &= \lambda_1+\frac{1}{2}\lambda\left(\left(\gamma_0-\gamma_1\right)+\left(\gamma_1-\gamma_2\right)+\ldots+\left(\gamma_{k-1}-\gamma_k\right)\right) \\ &= \lambda_1+\frac{1}{2}\lambda\left(\gamma_0-\gamma_k\right). \end{aligned}$$

Therefore we have:

$$\begin{aligned} & \alpha_{k+1}+\frac{1}{2}\left(\beta_{k+1}+\beta_k\right)+\frac{1}{2}\left(\lambda+\gamma_k\right)\gamma_{k+1}+\frac{1}{2}\lambda\gamma_k \\ = & \alpha_1+\frac{1}{2}\left(\beta_1+\beta_0\right)+\frac{1}{2}\left(\lambda+\gamma_1\right)\gamma_0+\frac{1}{2}\lambda\gamma_0. \end{aligned}$$

From here it follow that α_k, β_k and γ_k are equally bounded. ■

The following theorem takes place (below everywhere c denotes positive constant):

Theorem 2. *Let u_k and $\overline{u}_k$ be solutions of difference equation (3) corresponding to initial vectors (u_0,u_1) and $(\overline{u}_0,\overline{u}_1)$. Then for $z_k=u_k-\overline{u}_k$ the following estimates are true:*

$$\begin{aligned} \left\|A^{1/2}z_{k+1}\right\| &\le e^{ct_k}\left(\sqrt{2}\left(\left\|A^{1/2}z_0\right\|+\left\|A^{-1/2}\frac{\Delta z_0}{\tau}\right\|\right)\right. \\ &\quad\left.+\tau\left\|A^{1/2}\frac{\Delta z_0}{\tau}\right\|+c\tau\left\|A^{1/2}z_1\right\|\right), \qquad (5) \\ \left\|Az_{k+1}\right\| &\le e^{ct_k}\left(\sqrt{2}\left(\left\|Az_0\right\|+\left\|\frac{\Delta z_0}{\tau}\right\|\right)\right. \\ &\quad\left.+\tau\left\|A\frac{\Delta z_0}{\tau}\right\|+c\tau\left\|Az_1\right\|\right), \qquad (6) \end{aligned}$$

$$\begin{aligned}\left\|\frac{\Delta z_k}{\tau}\right\| &\leq \|Az_0\|+\sqrt{2}\left\|\frac{\Delta z_0}{\tau}\right\| \\ &+ct_k e^{ct_k}\left(\sqrt{2}\left(\|Az_0\|+\left\|\frac{\Delta z_0}{\tau}\right\|\right)\right. \\ &\left.+\tau\left\|A\frac{\Delta z_0}{\tau}\right\|+c\tau\|Az_1\|\right), \qquad (7)\end{aligned}$$

where $k=1,...,n-1$, $\Delta z_k = z_{k+1}-z_k$.

Proof of **Theorem 2** is based on lemma, which we state below.

Let us consider in Hilbert space H the following difference equation:

$$\frac{u_{k+1}-2u_k+u_{k-1}}{\tau^2}+A^2\frac{u_{k+1}+u_{k-1}}{2}=f_k, \qquad (8)$$

where $k=1,...,n-1$, u_0,u_1 and f_k are the given vectors of H.

The following lemma takes place.

Lemma 3. (see [3]) *For difference problem (8) the following estimates are true:*

$$\begin{aligned}\left\|A^{2s}u_{k+1}\right\| &\leq \sqrt{2}\left(\left\|A^{2s}u_0\right\|+\left\|A^{2s-1}\frac{\Delta u_0}{\tau}\right\|\right)+\tau\left\|A^{2s}\frac{\Delta u_0}{\tau}\right\| \\ &+ \tau\sum_{i=1}^{k}\left\|A^{2s-1}f_i\right\|, \quad 0\leq s\leq 1/2, \qquad (9)\end{aligned}$$

$$\left\|\frac{\Delta u_k}{\tau}\right\| \leq \|Au_0\|+\sqrt{2}\left\|\frac{\Delta u_0}{\tau}\right\|+\tau\sum_{i=1}^{k}\|f_i\|, \qquad (10)$$

where $k=1,...,n-1$, $\Delta u_k=u_{k+1}-u_k$.

Let us return to proof of Theorem 2.

Proof of Theorem 2. From (3) it follows:

$$u_{k+1}=\widetilde{L}_k u_k-u_{k-1}, \quad k=1,...,n-1, \qquad (11)$$

where

$$\widetilde{L}_k=2\left(I+\frac{\tau^2}{2}A^2+\frac{\tau^2}{2}a\left(\left\|A^{1/2}u_k\right\|^2\right)A\right)^{-1}.$$

Let us rewrite (11) as follows:

$$u_{k+1}=Lu_k-u_{k-1}+\left(\widetilde{L}_k-L\right)u_k, \quad k=1,...,n-1, \qquad (12)$$

where

$$L=2\left(I+\frac{\tau^2}{2}A^2\right)^{-1}.$$

Obviously we have

$$\widetilde{L}_k - L = \left(L^{-1} - \widetilde{L}_k^{-1}\right) L\widetilde{L}_k = -\frac{\tau^2}{4} a \left(\left\|A^{1/2}u_k\right\|^2\right) AL\widetilde{L}_k.$$

Taking into account this transformation, from (11) it follows:

$$\frac{u_{k+1} - 2u_k + u_{k-1}}{\tau^2} + A^2 \frac{u_{k+1} + u_{k-1}}{2} = -\frac{1}{2} a \left(\left\|A^{1/2}u_k\right\|^2\right) A\widetilde{L}_k u_k. \quad (13)$$

It is obvious that according to (13) $z_k = u_k - \overline{u}_k$ will satisfy the following equation:

$$\frac{z_{k+1} - 2z_k + z_{k-1}}{\tau^2} + A^2 \frac{z_{k+1} + z_{k-1}}{2} = -\frac{1}{2} g_k, \quad (14)$$

where

$$\begin{aligned} g_k &= a \left(\left\|A^{1/2}u_k\right\|^2\right) A\widetilde{L}_k u_k - a \left(\left\|A^{1/2}\overline{u}\right\|^2\right) A\overline{L}_k \overline{u}_k, \\ \overline{L}_k &= 2\left(I + \frac{\tau^2}{2}A^2 + \frac{\tau^2}{2} a \left(\left\|A^{1/2}\overline{u}_k\right\|^2\right) A\right)^{-1}. \end{aligned}$$

From (14) according to (9) it follows (correspondingly to $s = 1/4$ and $s = 1/2$):

$$\begin{aligned} \left\|A^{1/2}z_{k+1}\right\| &\le \sqrt{2}\left(\left\|A^{1/2}z_0\right\| + \left\|A^{-1/2}\frac{\Delta z_0}{\tau}\right\|\right) + \tau \left\|A^{1/2}\frac{\Delta u_0}{\tau}\right\| \\ &\quad + \frac{\tau}{2}\sum_{i=1}^{k}\left\|A^{-1/2}g_i\right\|, \qquad (15) \\ \left\|Az_{k+1}\right\| &\le \sqrt{2}\left(\left\|Az_0\right\| + \left\|\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|A\frac{\Delta u_0}{\tau}\right\| \\ &\quad + \frac{\tau}{2}\sum_{i=1}^{k}\left\|g_i\right\|. \qquad (16) \end{aligned}$$

The following representation is obvious

$$\begin{aligned} g_k = {} & \left(\left\|A^{1/2}u_k\right\| - \left\|A^{1/2}\overline{u}_k\right\|\right)\left(\left\|A^{1/2}u_k\right\| + \left\|A^{1/2}\overline{u}_k\right\|\right)\widetilde{L}_k A u_k \\ & -\frac{1}{4}\tau^2 a \left(\left\|A^{1/2}\overline{u}_k\right\|^2\right)\left(\left\|A^{1/2}u_k\right\| - \left\|A^{1/2}\overline{u}_k\right\|\right) \\ & \times \left(\left\|A^{1/2}u_k\right\| + \left\|A^{1/2}\overline{u}_k\right\|\right) A\widetilde{L}_k\overline{L}_k A u_k \\ & + a\left(\left\|A^{1/2}\overline{u}_k\right\|^2\right)\overline{L}_k\left(Au_k - A\overline{u}_k\right). \qquad (17) \end{aligned}$$

From (15) taking into account (17) it follows:

$$\begin{aligned}\left\|A^{1/2}z_{k+1}\right\| &\leq \sqrt{2}\left(\left\|A^{1/2}z_0\right\| + \left\|A^{-1/2}\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|A^{1/2}\frac{\Delta u_0}{\tau}\right\| \\ &+\frac{\tau}{2}\sum_{i=1}^{k}\left(\left\|A^{1/2}z_i\right\|\left(\left\|A^{1/2}u_i\right\| + \left\|A^{1/2}\overline{u}_i\right\|\right)\left\|\widetilde{L}_i\right\|\left\|A^{1/2}u_i\right\|\right. \\ &+\frac{1}{4}\tau^2 a\left(\left\|A^{1/2}\overline{u}_i\right\|^2\right)\left\|A^{1/2}z_i\right\| \\ &\times\left(\left\|A^{1/2}u_i\right\| + \left\|A^{1/2}\overline{u}_i\right\|\right)\left\|\widetilde{L}_i\right\|\left\|A\overline{L}_i\right\|\left\|A^{1/2}u_i\right\| \\ &\left.+\, a\left(\left\|A^{1/2}\overline{u}_i\right\|^2\right)\left\|\overline{L}_i\right\|\left\|A^{1/2}z_i\right\|\right). \end{aligned} \tag{18}$$

If we take into account that $\left\|A^{1/2}u_k\right\|$ and $\left\|A^{1/2}\overline{u}_k\right\|$ are equally bounded and, in addition, $\left\|\widetilde{L}_k\right\| \leq 2$, $\left\|\overline{L}_k\right\| \leq 2$ and

$$\frac{1}{4}\tau^2 a\left(\left\|A^{1/2}\overline{u}_k\right\|^2\right)\left\|A\overline{L}_k\right\| \leq 1,$$

then from (15) we obtain:

$$\begin{aligned}\left\|A^{1/2}z_{k+1}\right\| &\leq \sqrt{2}\left(\left\|A^{1/2}z_0\right\| + \left\|A^{-1/2}\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|A^{1/2}\frac{\Delta u_0}{\tau}\right\| \\ &+c\tau\sum_{i=1}^{k}\left\|A^{1/2}z_i\right\|. \end{aligned} \tag{19}$$

Let us introduce the following denotations

$$\begin{aligned}\delta_0 &= \sqrt{2}\left(\left\|A^{1/2}z_0\right\| + \left\|A^{-1/2}\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|A^{1/2}\frac{\Delta u_0}{\tau}\right\|, \\ \varepsilon_k &= \left\|A^{1/2}z_k\right\|. \end{aligned}$$

then the inequality (19) can be rewritten as:

$$\varepsilon_{k+1} \leq \delta_0 + c\tau\sum_{i=1}^{k}\varepsilon_k.$$

From here by the induction can be obtained (discrete analog of Gronwell's lemma):

$$\begin{aligned}\varepsilon_{k+1} &\leq (1+c\tau)^{k-1}\delta_0 + c\tau(1+c\tau)^{k-1}\varepsilon_1 \\ &= (1+c\tau)^{k-1}(\delta_0 + c\tau\varepsilon_1). \end{aligned} \tag{20}$$

If we take into consideration that

$$(1+c\tau)^k \leq e^{c(k\tau)} = e^{ct_k},$$

then from (20) we obtain (5).

From (16) with account of (17) analogously to (19) we obtain

$$\begin{aligned}\|Az_{k+1}\| &\leq \sqrt{2}\left(\|Az_0\| + \left\|\frac{\Delta z_0}{\tau}\right\|\right) + \tau\left\|A\frac{\Delta z_0}{\tau}\right\| \\ &+ c\tau\sum_{i=1}^{k}\left(\left\|A^{1/2}z_i\right\| + \|Az_i\|\right).\end{aligned}$$

From here, if we take into account that

$$\left\|A^{1/2}z_k\right\| = \left\|A^{-1/2}\left(Az_k\right)\right\| \leq \frac{1}{\sqrt{\nu}}\|Az_k\|, \tag{21}$$

then analogously to (5) we obtain (6).

Let us show the estimate (7). From (14) according to (10) it follows:

$$\left\|\frac{\Delta z_k}{\tau}\right\| \leq \|Az_0\| + \sqrt{2}\left\|\frac{\Delta z_0}{\tau}\right\| + c\tau\sum_{i=1}^{k}\left(\left\|A^{1/2}z_i\right\| + \|Az_i\|\right).$$

Obviously from here with account of (21) we obtain:

$$\left\|\frac{\Delta z_k}{\tau}\right\| \leq \|Az_0\| + \sqrt{2}\left\|\frac{\Delta z_0}{\tau}\right\| + c\tau\sum_{i=1}^{k}\|Az_i\|.$$

From here with account of (6) follows (7). ■

The following theorem takes place.

Theorem 4. *Let the following conditions be fulfilled: (a)* $u_0 = \varphi_0$ *and* $\varphi_0 \in D\left(A^2\right)$; *(b)* $u_1 = \varphi_0 + \tau\varphi_1 + \frac{\tau^2}{2}\varphi_2$, $\varphi_2 = -\left(A^2\varphi_0 + a\left(\left\|A^{1/2}\varphi_0\right\|^2\right)A\varphi_0\right)$ *and* $\varphi_2 \in D\left(A\right)$; *(c) Solution* $u\left(t\right)$ *of problem (1)-(2) is continuously differentiable to third degree including and* $u'''\left(t\right)$ *satisfies Holder's inequality with index* λ $(0 < \lambda \leq 1)$*; (d)* $u'\left(t\right) \in D\left(A^2\right)$ *for every* $t-$ *from* $[0,T]$ *and function* $A^2u'\left(t\right)$ *satisfy Holder inequality with parameter* λ $(0 < \lambda \leq 1)$*. Then for error* $\widetilde{z}_k = u\left(t_k\right) - u_k$ *the following estimate is true:*

$$\max_{1\geq k\leq n-1}\|A\widetilde{z}_k\| \leq c\tau^{1+\lambda}, \tag{22}$$

$$\max_{1\geq k\leq n-1}\left\|\frac{\Delta\widetilde{z}_k}{\tau}\right\| \leq c\tau^{1+\lambda}. \tag{23}$$

Proof.

Let us write down the equation (1) at point $t=t_k$ in the following form:

$$\begin{aligned} &\frac{\Delta^2 u\left(t_{k-1}\right)}{\tau^2}+A^2\frac{u\left(t_{k+1}\right)+u\left(t_{k-1}\right)}{2}\\ +\quad &a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)\frac{Au\left(t_{k+1}\right)+Au\left(t_{k-1}\right)}{2}=r_\tau\left(t_k\right), \qquad (24)\end{aligned}$$

where

$$\begin{aligned} r_\tau\left(t_k\right) &= r_{0,\tau}\left(t_k\right)+r_{1,\tau}\left(t_k\right)+r_{2,\tau}\left(t_k\right),\\ r_{0,\tau}\left(t_k\right) &= \frac{\Delta^2 u\left(t_{k-1}\right)}{\tau^2}-u''\left(t_k\right),\\ r_{1,\tau}\left(t_k\right) &= \frac{1}{2}A^2\left(\Delta^2 u\left(t_{k-1}\right)\right),\\ r_{2,\tau}\left(t_k\right) &= \frac{1}{2}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)A\left(\Delta^2 u\left(t_{k-1}\right)\right).\end{aligned}$$

From (24) we have

$$u\left(t_{k+1}\right)=L_k u\left(t_k\right)-u\left(t_{k-1}\right)+\frac{\tau^2}{2}L_k r_\tau\left(t_k\right),\quad k=1,...,n-1, \qquad (25)$$

where

$$L_k=2\left(I+\frac{\tau^2}{2}A^2+\frac{\tau^2}{2}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)A\right)^{-1}.$$

Let us rewrite (25) in the form:

$$\begin{aligned} u\left(t_{k+1}\right) &= Lu\left(t_k\right)-u\left(t_{k-1}\right)+\frac{\tau^2}{2}Lr_\tau\left(t_k\right)+\left(L_k-L\right)u\left(t_k\right)\\ &+\frac{\tau^2}{2}\left(L_k-L\right)r_\tau\left(t_k\right),\quad k=1,...,n-1. \qquad (26)\end{aligned}$$

Obviously we have

$$L_k-L=\left(L^{-1}-L_k^{-1}\right)LL_k=-\frac{\tau^2}{4}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)ALL_k.$$

Taking into account this transformation, from (26) it follows:

$$\begin{aligned} &\frac{u\left(t_{k+1}\right)-2u\left(t_k\right)+u\left(t_{k-1}\right)}{\tau^2}+A^2\frac{u\left(t_{k+1}\right)+u\left(t_{k-1}\right)}{2}\\ = \quad &-\frac{1}{2}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)AL_k u\left(t_k\right)\\ &+r_\tau\left(t_k\right)-\frac{\tau^2}{4}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)AL_k r_\tau\left(t_k\right). \qquad (27)\end{aligned}$$

By taking (13) from (27) we obtain:

$$\frac{\widetilde{z}_{k+1}-2\widetilde{z}_k+\widetilde{z}_{k-1}}{\tau^2}+A^2\frac{\widetilde{z}_{k+1}+\widetilde{z}_{k-1}}{2}=-\frac{1}{2}\widetilde{g}_k+\widetilde{r}_\tau\left(t_k\right), \tag{28}$$

where

$$\begin{aligned}\widetilde{g}_k &= a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)AL_ku\left(t_k\right)-a\left(\left\|A^{1/2}u_k\right\|^2\right)A\widetilde{L}_ku_k\\ \widetilde{r}_\tau\left(t_k\right) &= r_\tau\left(t_k\right)-\frac{\tau^2}{4}a\left(\left\|A^{1/2}u\left(t_k\right)\right\|^2\right)AL_kr_\tau\left(t_k\right).\end{aligned}$$

Analogously to (17) we have:

$$\begin{aligned}\widetilde{g}_k &= \left(\left\|A^{1/2}u\left(t_k\right)\right\|-\left\|A^{1/2}u_k\right\|\right)\\ &\times\left(\left\|A^{1/2}u\left(t_k\right)\right\|+\left\|A^{1/2}u_k\right\|\right)L_kAu\left(t_k\right)\\ &-\frac{1}{4}\tau^2a\left(\left\|A^{1/2}u_k\right\|^2\right)\left(\left\|A^{1/2}u\left(t_k\right)\right\|-\left\|A^{1/2}u_k\right\|\right)\\ &\times\left(\left\|A^{1/2}u\left(t_k\right)\right\|+\left\|A^{1/2}u_k\right\|\right)AL_k\widetilde{L}_kAu\left(t_k\right)\\ &+a\left(\left\|A^{1/2}u_k\right\|^2\right)\widetilde{L}_k\left(Au\left(t_k\right)-Au_k\right).\end{aligned} \tag{29}$$

From (28) according to (9) with account of (29) we obtain:

$$\begin{aligned}\left\|A\widetilde{z}_{k+1}\right\| &\leq \sqrt{2}\left(\left\|A\widetilde{z}_0\right\|+\left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\|\right)+\tau\left\|A\frac{\Delta\widetilde{z}_0}{\tau}\right\|\\ &+\frac{\tau}{2}\sum_{i=1}^{k}\left(\left\|A^{1/2}\widetilde{z}_i\right\|\left(\left\|A^{1/2}u\left(t_i\right)\right\|+\left\|A^{1/2}u_i\right\|\right)\right.\\ &\times\left\|L_i\right\|\left\|Au\left(t_i\right)\right\|+\frac{1}{4}\tau^2a\left(\left\|A^{1/2}u_i\right\|^2\right)\left\|A^{1/2}\widetilde{z}_i\right\|\\ &\times\left(\left\|A^{1/2}u\left(t_i\right)\right\|+\left\|A^{1/2}u_i\right\|\right)\left\|L_i\right\|\left\|A\widetilde{L}_i\right\|\left\|Au\left(t_i\right)\right\|\\ &\left.+\,a\left(\left\|A^{1/2}u_i\right\|^2\right)\left\|\widetilde{L}_i\right\|\left\|A\widetilde{z}_i\right\|\right)+\tau\sum_{i=1}^{k}\left\|\widetilde{r}_\tau\left(t_i\right)\right\|.\end{aligned} \tag{30}$$

If we take into account that $\left\|A^{1/2}u\left(t_k\right)\right\|, \left\|Au\left(t_k\right)\right\|$ and $\left\|A^{1/2}u_k\right\|$ are equally bounded and, in addition, $\left\|\widetilde{L}_k\right\|\leq 2$, $\left\|L_k\right\|\leq 2$, $\left\|A^{1/2}\widetilde{z}_k\right\|\leq\frac{1}{\sqrt{\nu}}\left\|A\widetilde{z}_k\right\|$ and

$$\frac{1}{4}\tau^2a\left(\left\|A^{1/2}u_k\right\|^2\right)\left\|A\widetilde{L}_k\right\|\leq 1,$$

then from (30) we obtain:

$$\begin{aligned}\|A\widetilde{z}_{k+1}\| &\leq \sqrt{2}\left(\|A\widetilde{z}_0\| + \left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\|\right) + \tau\left\|A\frac{\Delta\widetilde{z}_0}{\tau}\right\| \\ &\quad + c\tau\sum_{i=1}^{k}\|A\widetilde{z}_i\| + \tau\sum_{i=1}^{k}\|\widetilde{r}_\tau(t_i)\|. \qquad (31)\end{aligned}$$

From here, according to discrete analog of Gronwell's lemma, we obtain:

$$\begin{aligned}\|A\widetilde{z}_{k+1}\| &\leq e^{c_0 t_k}\left(\sqrt{2}\left(\|A\widetilde{z}_0\| + \left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\|\right) + \tau\left\|A\frac{\Delta\widetilde{z}_0}{\tau}\right\|\right. \\ &\quad \left. + c\tau\|A\widetilde{z}_1\| + \tau\sum_{i=1}^{k}\|\widetilde{r}_\tau(t_i)\|\right). \qquad (32)\end{aligned}$$

It is obvious that the following inequality is true:

$$\begin{aligned}\|\widetilde{r}_\tau(t_k)\| &= \left\|r_\tau(t_k) - \frac{\tau^2}{4}a\left(\left\|A^{1/2}u_n(t_k)\right\|^2\right)AL_k r_\tau(t_k)\right\| \\ &\leq \|r_\tau(t_k)\| + \frac{\tau^2}{4}a\left(\left\|A^{1/2}u_n(t_k)\right\|^2\right)\|AL_k\|\,\|r_\tau(t_k)\| \\ &\leq 2\|r_\tau(t_k)\| \leq 2\left(\|r_{0,\tau}(t_k)\| + \|r_{1,\tau}(t_k)\| + \|r_{2,\tau}(t_k)\|\right). \qquad (33)\end{aligned}$$

According to how smooth is function $u(t)$, the following formulas are true:

$$\begin{aligned}\frac{\Delta^2 u(t_{k-1})}{\tau^2} - u''(t_k) &= \frac{1}{\tau^2}\int_{t_k}^{t_{k+1}}\int_{t_k}^{t}\int_{t_k}^{s}\left(u'''(\xi) - u'''(t_k)\right)d\xi ds dt + \\ &\quad \frac{1}{\tau^2}\int_{t_{k-1}}^{t_k}\int_{t_{k-2}}^{t}\int_{t_kt}^{s}\left(u'''(t_k) - u'''(\xi)\right)d\xi ds dt, \qquad (34)\end{aligned}$$

$$\begin{aligned}\Delta^2 u(t_{k-1}) &= \int_{t_k}^{t_{k+1}}\left(u'(t) - u'(t_k)\right)dt \qquad (35) \\ &\quad + \int_{t_{k-1}}^{t_k}\left(u'(t_k) - u'(t)\right)dt,\end{aligned}$$

$$u(t_1) = u_0 + \tau u'(0) + \int_0^{\tau}\left(u'(t) - u'(0)\right)dt, \qquad (36)$$

$$u(t_1) = u_0 + \tau u'(0) + \frac{\tau^2}{2}u''(0) \qquad (37)$$

$$+\int_0^{\tau}\int_0^{t}\int_0^{s} u'''(\xi)\,d\xi ds dt.$$

According to conditions (a), (b) and (d) of Theorem 4, from (36) it follows:

$$\begin{aligned}\|A(\Delta\widetilde{z}_0)\| &= \|A(\widetilde{z}_1-\widetilde{z}_0)\| = \|A\widetilde{z}_1\| = \|A(u(t_1)-u_1)\| \\ &= \left\|-\frac{\tau^2}{2}A\varphi_2+\int_0^{\tau} A\left(u'(t)-u'(0)\right)dt\right\| \le c\tau^{1+\lambda}. \quad (38)\end{aligned}$$

According to conditions (a), (b) and (c) of Theorem 4, from (37) it follows:

$$\left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\| = \frac{1}{\tau}\|u(t_1)-u_1\| = \frac{1}{\tau}\int_0^{\tau}\int_0^{t}\int_0^{s}\left\|u'''(\xi)\right\| d\xi ds dt \le c\tau^2. \quad (39)$$

According to condition (c) of Theorem 4, from (34) it follows:

$$\|r_{0,\tau}(t_k)\| = \left\|\frac{\Delta^2 u(t_{k-1})}{\tau^2} - u''(t_k)\right\| \le c\tau^{1+\lambda}. \quad (40)$$

According to condition (c) of Theorem 4, from (35) it follows:

$$\|r_{j,\tau}(t_k)\| \le c\tau^{1+\lambda}, \quad j=1,2. \quad (41)$$

From (33), with account of inequalities (40)-(41), it follows:

$$\|\widetilde{r}_\tau(t_k)\| \le\le c\tau^{1+\lambda}. \quad (42)$$

From (32), with account of inequalities (38),(39) and (42), follows (22).
Now let us show the estimate (23). From (28) according to (10), analogously to (31), it is obtained:

$$\left\|\frac{\Delta\widetilde{z}_k}{\tau}\right\| \le \|A\widetilde{z}_0\| + \sqrt{2}\left\|\frac{\Delta\widetilde{z}_0}{\tau}\right\| + c\tau\sum_{i=1}^{k}\|A\widetilde{z}_i\| + \tau\sum_{i=1}^{k}\|\widetilde{r}_\tau(t_i)\|.$$

From here with account of estimates (38),(39), (42) and (22), we obtain (23) ■

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