ONE METHOD OF CONSTRUCTING A FORMAL SYSTEM[1]

Kh. Rukhaia, L. Tibua, G. Chankvetadze, B. Dundua

I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
0143 University Street 2, Tbilisi, Georgia

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Abstract

Research in the field of automated theorem proving mainly has been conducted in two directions: (a) Simple representation of the input problem through the improvement if the logical language, and (b) Search for effective proof methods and their implementation. Results of this research are essentially based on the first-order theory. In this theory the τ operator sign of Bourbaki does not occur in the basic symbols, nor is it possible to introduce it through Pkhakadze's rational system of rules for defining contracting symbols. The absence of the τ operator sign in a theory in some sense restricts its expressive power. In this paper the $\mathcal{TSR}$ logic is constructed, whose language, as its basic symbols, includes the τ operator sign and S and R operator signs of substitution. In this theory the existential and universal quantifiers are defined by the rational system of the defining rules. The same system is used to deductively extend and develop the language of $\mathcal{TSR}$ theory and, therefore, it has sufficient expressive power.

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1 Introduction

Languages of modern formal mathematical theories use quite poor alphabets, where sets of variables are usually countable and the number of other symbols is rather small. For example, Bourbaki's set theory contains seven such symbols: $\square$, τ, $\vee$, $\neg$, $=$, $\in$, $\rightarrow$. Using them, one can define any symbol needed in formal mathematics as a *contracting symbol*. Forms, formulas, and terms over an alphabet extended by contracting symbols are called *contracted forms*, *contracted formulas*, and *contracted terms*, respectively. While considering various subjects of mathematical theory, we are essentially dealing with contracted forms that denote forms of the given theory.

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One should consider contracted forms as the forms that they denote (abbreviate). But the principal difficulty is that the forms denoted by contracted forms are usually very long. For example, as it is shown in Bourbaki's set theory [1], a simple contracted term like the constant 1 denotes a term that consisting of tens of thousands of symbols.

Hence, it is necessary to operate directly on contracted forms, instead of the forms that they denote. It requires to identify certain general rules for operating on contracted forms. For example, it is desirable to have a rule that states that for substitutions on contracted forms there exist substitutions on the corresponding main forms. In general, it is desirable to have rules for operating on contracted forms that are natural counterparts of similar rules that operate on main forms. However, often it is not possible to find a precise relationship between operations on contracted forms and operations on main forms. One can establish such relationships modulo congruence, if the class of contracting symbols satisfy certain conditions.

This naturally leads us to an important problem of finding a rational system of definition rules for contracted symbols. Such a system should be general enough to satisfy the following two conditions:

1. It should satisfy requirements of all main formal mathematical theories in the sense that the rules of the system make possible to introduce all the contracting symbols used in those theories.

2. The contracted forms obtained by the rules of the system should have "nice properties". Namely, they should satisfy the above mentioned natural rules for operating on them.

However, these two conditions contradict each other. Therefore it is very difficult to find a rational system of definition rules for contracted symbols. Church showed that Hilbert-Bernays rational system of definition rules for contracted symbols [3] is more strict then some other systems, but they are not general enough. In [2] Church himself essentially gave a rather rational system for propositional calculus and predicate logic, but this system is not enough for first order theories, in particular for set theory. Normally, definition rules for contracting symbols in first order theories are not restricted, which makes it impossible to define general rules for operating on contracted forms. Often, instead of this, when introducing contracting symbols, one extends the theory with new symbols, hence moving to a new theory. However, this approach has also its difficulties.

Pkhakadze in [4] approached the problem of designing a rational system of rules for defining contracting symbols by classifying the types of contracting symbols. In his classification, contracting symbols have types I–VII, II', IV', and VI', where the system of definition rules for symbols of types I–IV, II', and IV' is rational. Moreover, the system, on the one hand, is general enough to make it possible to introduce all the contracting symbols used in classical mathematical theories and, on the other hand, has rich properties enough to guarantee freedom of operating on contracted forms.

For definition rules of contracting symbols to be general, it was necessary to introduce such types of contracting symbols that their corresponding contracted forms define, up to a congruence, the forms denoted by these contracted forms.

2 Definitions

We first very briefly recall some basic definitions and results from Notation Theory [4].

We denote a formal mathematical theory by $\mathcal{T}$. The theory obtained from $\mathcal{T}$ by adding contracting symbols in the alphabet as symbols of corresponding types is denoted by $\widetilde{\mathcal{T}}$. New symbols—operators and operator signs—are introduced in some order, and each definition has the form

$$\sigma x_1 x_2 \ldots x_n A_1 A_2 \ldots A_n \text{—} B \qquad (*)$$

where σ is the contracting symbol and $x_1, x_2, \ldots, x_n$ are metavariables such that each x ranges over a nonempty class of all quantifier symbols of the given theory that satisfies the following properties:

- For each element in it, the class contains all the variables of the same type as the element.
- For each constant in it, the class contains all the quantifier constants of the same type as the given constant.

As for $A_1, A_2, \ldots, A_n$, they are metavariables such that one of following conditions holds:

1. Each of the A's range over the class of all formulas.
2. Each of the A's ranges over the class of all terms.
3. Some of the A's (but not all of them) range over the class of formulas and the remained ones range over the class of all terms.

In the expression $(*)$ the natural numbers m and n be 0 under the following conditions: If $m \neq 0$ then σ is an operator sign with the weight (m, n), otherwise σ is a simple operator with weight n. Moreover, if $n = 0$ then σ

is a derived propositional constant when B is formula, and a derived object constant when B is term.

Moreover, the derived symbol σ is in case (1) (i.e., when all A's are formulas) *logical*, in (2) *special*, and in (3) *logico-special.* It is either *relational* or *substantive* depending whether in the right hand side of the definition $(*)$ one has a formula or a term for each system of metavariables.

Even after such a specialization of the left hand side of definitions the class of contracting symbols remains quite big. The right hand side has to be specialized as well. Based on such a specialization, Pkhakadze in [4] introduced the definition types I–IV, II', and IV'. Although the definition of these types is quite complicated, in practice it is not hard to establish the type of a definition. Moreover, most definitions have the type I that can be identified relatively easily.

One has significant freedom when operates on contracted forms that correspond to contacting symbols introduced by the definitions of the types I–IV, II', da IV'. It is guaranteed by the important properties of contracted forms given in [5]. Moreover, the class of contracting symbols that correspond to these types is quite big. For the contracting symbols of the other four types, the restrictions used in the definitions of the types I–IV, II', and IV' are indeed necessary.

Pkhakadze also described algorithmic processes for reconstructing main forms from contracted ones.

3 The Language

We will rely on the principles given in the pervious section to build the $\mathcal{TSR}$ logic. The language of $\mathcal{TSR}$ logic consists of the following symbols:

1. Fundamental symbols:
   (a) Logical connectives: $\neg$ (of the weight 1), $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ (each of the weight 2).
   (b) Logical operational sign τ of the weight $(1, 1)$.
   (c) Substantive special substitution operator S of the weight $(1, 2)$.
   (d) Relational logico-special substitution operator R of the weight $(1, 2)$ and with the logicality indicator 2.
   (e) Object letters: $X_0, X_1, \ldots$.
   (f) Predicate symbols $=$ and $\in$, each of the weight 2, and predicate letters: $P_0^n, Q_0^n, P_1^n, Q_1^n, \ldots$.
   (g) Functional symbol $\supset$ that has the weight 2, and functional letters: $f_0^n, g_0^n, f_1^n, g_1^n, \ldots$.

(h) [and] (left and right brackets)

2. Signs, introduced by the definitions of the types I, II and II'.

Finite sequence of signs of $\mathcal{TSR}$ are called a *word* of $\mathcal{TSR}$ logic. The words $\tau X_0, \tau X_1, \ldots$ are the $\mathcal{TSR}$ logic operators with the weight 1. The words $SX_0, SX_1, \ldots$ and $RX_0, RX_1, \ldots$ are the $\mathcal{TSR}$ logic operators with the weight 2. Besides, the operators $SX_0, SX_1, \ldots$ are substantive partial quantifiers with the binding indicator 2, and the operators $RX_0, RX_1, \ldots$ are logico-special partial quantifiers with logicality and binding indicator 2. Note that the metaletters $A, B, A_1, B_1, \ldots$, resp. $T, U, T_1, U_1, \ldots$, resp. $x, y, x_1, y_1, \ldots$, are metavariables that range on the class of all formulas, resp. terms, resp. subject letters, of the $\mathcal{TSR}$ logic.

Formulas and *terms* of $\mathcal{TSR}$ logic are defined in following way:

1. Subject letters are simple terms.

2. If σ is an n-ary relational logical (resp. special) operator, then $\sigma A_1 \ldots A_n$ (resp. $\sigma T_1 \ldots T_n$) is either a formula or a term depending whether σ is relational or substantive.

3. Let $c_1, \ldots, c_n$ be a sequence of formulas. If σ is an n-ary logico-special operator whose logicality indicator is $(n_1, \ldots, n_k)$, and $c_{n_1}, \ldots, c_{n_k}$ is the maximal subsequence of the sequence $c_1, \ldots, c_n$ consisting of formulas only, then $\sigma c_1, \ldots, c_n$ is either a formula or a term depending whether σ is relational or substantive.

4. C is formula or term if and only if it is derived by the three rules above.

The expression $\tau x A$ is a term that denotes a privileged subject that has the property A, if such a subject exists. Otherwise it is a term that denotes an arbitrary element in the interpretation domain. For example, $\tau x(x^2 = 1)$ is a term that denotes a privileged number whose squaring gives 1. For instance, "1" can be such a term. The expression $\tau x(x^2 = -1)$ is a term that denotes some number from the interpretation domain of real numbers, e.g., "0".

A form of $\mathcal{TSR}$ logic is called a *fundamental forms*, or a form of level 0, if it is constructed from fundamental symbols.

The symbols of level n of the $\mathcal{TSR}$ logic are defined as follows:

1. The fundamental symbols of the $\mathcal{TSR}$ logic are symbols of level 0.

2. A symbol of level n, $n > 1$, is a contracting symbol of $\mathcal{TSR}$ logic introduced by a definition such that each symbol in its right hand side has the level less than n, and there is at least one symbol with the level $n-1$.

Below we label definitions of the types I, II, and II' with the $D_k[i,j]$ where k is the number of the definition, i denotes the type of the definition, and j denotes the level of the operator obtained by the definition.

$$D_1[I,I] \qquad \exists xA\text{—}Rx\tau xAA.$$

Reads: "There exists x such that A". The operator $\exists x$ is logical relational.

$$D_2[I,II] \qquad \forall xA\text{—}\neg\exists x\neg A.$$

Reads: "For all x A". The operator $\forall x$ is logical relational.

$$D_3[II',III] \qquad \langle set\rangle xA\text{—}\tau y[\forall x[x \in y \leftrightarrow a]].$$

where the variable y is different from x and does not occur in A. The operator $\langle set\rangle$ is logical substantive.

$$D_4[II',IV] \qquad \langle complement\rangle UT\text{—}\langle set\rangle x[x \in T \wedge x \notin U].$$

where the variable x does not occur in the terms T and U. Reads: "The complements of the set U with respect of the set T. The operator $\langle complement\rangle$ is special substantive.

$$D_5[II',I] \qquad \langle represent\rangle xUT\text{—}\exists y[U = SxyT]].$$

where the variable y is different from x and does not occur in the terms T and U. Reads: "U can be represented as T with respect of the set x. The operator $\langle represent\rangle$ is special relational partial quantifier with the binding indicator (2).

$$D_6[I,I] \qquad \langle root\rangle xTA\text{—}RxTA.$$

Reads: "T is a solution of the formula A with respect to x". The operator $\langle root\rangle x$ is a logico-special relational partial quantifier with the logical and binding indicator (2).

$$D_7[II',III] \qquad \langle subset\rangle xTA\text{—}\tau y[\forall x[x \in y \leftrightarrow [x \in T \wedge A]]].$$

where x and y are distinct variables and y do not occur in T and A. Reads: "The set of all the elements of T with the property A". The operator

$\langle subset \rangle x$ is a logico-special substantial partial quantifier with the logical and binding indicator (2).

The notions of explicit and implicit axioms and axiom schema of $\mathcal{TSR}$ logic are as in [1]. The inference rule in $\mathcal{TSR}$ logic is Modus Ponens and the axiom schemes are the following:

HA1. $[A \vee A] \rightarrow A$

HA2. $A \rightarrow [A \vee B]$

HA3. $[A \vee B] \rightarrow [B \vee A]$

HA4. $[A \rightarrow B] \rightarrow [[A_1 \vee A] \rightarrow [A_1 \vee B]]$

HA5. $RxTA \rightarrow \exists xA$

HA6. $RxTA \leftrightarrow (T/x)A$

HA7. $SxTU = (T/x)U$

Note that in the axiom schemas HA6 and HA7 the substitution $(T/x)A$ does not bind free variables in T.

HA8. $\forall x[A \leftrightarrow B] \rightarrow [\tau xA = \tau xB]$

HA9. $\forall x[A \leftrightarrow B] \wedge [T = U]] \rightarrow [RxTA \leftrightarrow RxUB]$

HA10. $[\forall x[T = U] \wedge [T_1 = U_1]] \rightarrow [SxT_1T = SxU_1U]$

Last, assume C—C_1 is a D_m ($m = 1, 2, \ldots$) definition, then $C \leftrightarrow C_1$ (respectively $C = C_1$) is an axiom schema if C is a formula (resp. C is a term). This axiom schema is denoted by HAD_m.

Examples of the axiom schema HAD_m are:

HAD_1. $\exists x$—$Rx\tau xAA$.

HAD_3. $\langle set \rangle xA = \tau y \forall x[x \in y \leftrightarrow A]$ where y does not occur in A.

The axioms and inference rules of $\mathcal{TSR}$ logic imply validity of the counterparts of all the deductive criteria from [1] and the following theorems:

Theorem 3.1 *If A is a main formula of $\mathcal{TSR}$ and B is a contracted form of A, then $\vdash B$ if and only if $\vdash A$.*

Theorem 3.2 *Let the type of the operator $\sigma x_1 \ldots x_m$, $m \geq 0$, be I, II, or II'. If all the operators in the right side of the definition of $\sigma x_1 \ldots x_m$ are invariant then $\sigma x_1, \ldots x_m$ is invariant.*

Theorem 3.3 *All operators of $\mathcal{TSR}$ are invariant.*

Note that $\mathcal{TSR}$ logic is more expressive than predicate logic. Indeed:

1. As it was shown above, one can introduce the $\exists$ operational sign in the $\mathcal{TSR}$ logic by a definition of type I. However, it is not possible to define the τ operational sign in predicate logic by the definitions of type I,II, and II' of Pkhakadze.

2. τxA is a term of $\mathcal{TSR}$ logic, which is a name of an object with the property A. Hence, $\mathcal{TSR}$ logic is a Henkin-type theory.

Another advantage of the $\mathcal{TSR}$ logic over the other logical theories is that in $\mathcal{TSR}$ it is possible to introduce by definitions of type I, II, and II' all contracting symbols that are rational in the sense defined above.

4 Open Problem

We have a similar situation in artificial languages. Here computer (machine) languages play the rôle of the main language, and programming languages can be considered as languages extended by contracting symbols. Each programming language is obtained by introducing a new operator (these operators can be called contracting symbols). Programming languages are closer to natural languages, than computer languages. When writing programs, a programmer uses general, yet unproved, laws that connect computer languages and programming languages. The use of these laws is based on intuition. This is why program testing becomes necessary, but testing can not guarantee program correctness. If these laws can be proved for programming languages, then program testing will not be necessary. To prove them, it will be necessary to find a mathematical notion of contracting symbol (i.e., an operator of a programming language). Hence, a general theory of programming should be created in the same method that was used for creating the Notation Theory. To achieve this goal, operators of modern programming languages should be studied, a rational notion of contracting symbol should be introduced, and the theory of contracting symbols for artificial languages should be constructed. Creating such a theory essentially means to create a general theory of programming. It would imply reliability of programs written in the language of this theory. Moreover, such a theory would help to come up with recommendations on how machine languages should be, to which direction computers should develop, what kind of devices should be created for processing mathematical texts, etc.

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