CALCULATION OF THE VALUES OF HYPERGEOMETRIC FUNCTIONS

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Abstract

The algorithm of calculation of the values of hypergeometric functions with the arbitrary number of upper and lower parameters is proposed. In the work it is shown how to choose the parameters used in the algorithm in order to minimize the time of calculations.

Key words and phrases: Hypergeometric function; calculation; algorithm; optimization; minimum time of calculations; reduction formulas.

Introduction

Importance of special functions for modern science and technical disciplines is obvious. The quantity of special functions used in applied tasks has sharply increased for the last decades. And in a whole set of these functions, hypergeometric functions represent one of the major classes.

If, in a theoretical or an applied problem in which, for example, it is required to calculate an integral or to solve a differential equation, it was possible to present the sought function in an explicit form with the use of special functions, it is necessary that the obtained expression could be investigated qualitatively or, at least, quantitatively. Otherwise the value of the obtained result will be doubtful. When quantitative estimations of the result are required, it is necessary that calculation of the values of all used special mathematical functions could be carried out more quickly and easier than it would be required for realization of the algorithm of direct numerical solution of the initial problem.

One of possible algorithms of calculation of the values of hypergeometric functions is offered in this work. The described algorithm, apparently, has been applied for the first time by the author of this work to a special case in his old publication [1].

It is known that often the same problem of calculus mathematics can be solved by different methods, and each of such methods has both advantages

and imperfections in comparison with others. This, in particular, concerns the algorithms of calculation of special functions.

There are powerful modern software packages like MatLab, Maple, Mathematica and other annually updated packages in which effective methods of calculation of different special functions are realized. In particular, in the specified packages, any hypergeometric functions of complex parameters and the argument are calculated with the help of any personal computer very quickly and with high accuracy (100 true decimal significant digits and more). Nevertheless still there are some important problems connected with calculation of these functions which have not been resolved so far [2],[3]. To such problems, in particular, refer: the problem of analytic extension of the functions in the vicinity of boundary of the circle of convergence of expansion, in the vicinity of singular points $z = 1$ and $z = \infty$ in the cases of sets of parameters which are close to or coinciding with the logarithmic variant and the problem of calculation of entire hypergeometric functions in the vicinity of essentially singular point $z = \infty$ at big values z and $\mathrm{Re}\, z$.

In the later specified case, fundamental difficulty is connected with the presence of special effect – cancelation. This effect arises at summation of sign-changing terms when the sum of the series is many orders less than the initial terms of the series. Thus there is a reduction in true significant digits; the relative error for the sum proves to be much greater than the relative error for the terms.

In work [4], the algorithm of calculation of hypergeometric function of two variables is offered. The authors of some works (see, for example, [5],[6],[7]) offer original methods of calculation of special types of hypergeometric functions which appear in different applied problems.

The advantage of the method offered in this work is that, at certain assumptions, the realization of this method demands the minimum quantity of mathematical actions and, consequently, the minimum time of calculations.

In this work the following designations are used:

j, k, r, l, m, n, R, L, M, N
– integer variables and parameters;

x, y, t, a, b, c, d, h, p, q, s, A, B, C, D, H, ω, φ, ψ
– real variables and parameters;

w, z, ξ, η, α, β, γ, $\varkappa$, λ, μ, ν, ϱ, σ, τ, ζ
– complex variables and parameters;

$$\mathcal{F}_k(z) \equiv \prod_{j=0}^{k-1}(z+j); \qquad \widetilde{\mathcal{F}}_k(z) \equiv (-1)^k \cdot \mathcal{F}_k(-z) = \prod_{j=0}^{k-1}(z-j)$$

$\mathcal{F}_k(z)$ is the Pochhammer polynomial of degree k, which is usually designated by $(z)_k$ [8],[9].

For derivatives, the reduced designation is used everywhere: d_z instead of $\frac{d}{dz}$ and ∂_z instead of $\frac{\partial}{\partial z}$.

1 Essence of the Method

Let's consider the function:

$$\Phi(z) \equiv {}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1, ..., \gamma_n;\, z)$$

where $n > 0$. Let it be required to determine the value of this function for the given complex values of argument and parameters, and ε be the parameter describing the admissible relative deviation of approximate value of the function from its exact value.

Let's designate:

$$r \equiv \max\{m, n+1\}.$$

The considered function satisfies the r-order linear differential equation.

For calculation of values of the function $\Phi(z)$, it is expedient to use its Taylor series if the modules of upper parameters of this function are small, and the modules of its lower parameters are big. Thus the Taylor series of the considered function either represents a quickly convergent power series, or, even if this series is divergent, it is an asymptotic series giving a good approximation of the function $\Phi(z)$.

If the parameters of the function $\Phi(z)$ assume any values, by means of power series, it is possible to determine the values of r hypergeometric functions of the same order as $\Phi(z)$ the parameters of which differ from the parameters of initial function by integers, and then to express $\Phi(z)$ in terms of the values of calculated functions. For this purpose, it is necessary to use the reduction formulas which are also called the recurrence relations [8],[9],[11].

It is known [11] that there exist $2m + n$ linealy independent linear relations between fixed function ${}_mF_n$ and its $2\,(m + n)$ adjacent functions (i.e. such functions ${}_mF_n$ which arguments and parameters have the same values except for one parameter which values can differ by ± 1). Usually such relations are also called the formulas of reduction. In particular, for the function ${}_2F_1$, they are Gauss formulas of reduction [8]. However we shall present these formulas in a nonconventional form; Appendix 1 is devoted to them.

The offered process of calculation of the values of function $\Phi(z)$ together with its first $r - 1$ derivatives by z can be conventionally divided into 3 stages:

a) There is calculated the r-dimensional vector $\boldsymbol{u} \equiv [u_1, ..., u_r]$ with components

$$u_l = \mathrm{d}_z^{l-1}{}_mF_n(\alpha_1 + M_1, ..., \alpha_m + M_m;\, \gamma_1 + N_1, ..., \gamma_n + N_n;\, z),$$

where M_1, ..., M_m, N_1, ..., N_n are the integers chosen so that values $|\alpha_j + M_j|$ were small in an admissible degree, and values $\mathrm{Re}\,(\gamma_k + N_k)$ were big enough ($j = 1, ..., m$; $k = 1, ..., n$). Then each of functions u_l is the sum of quickly converging power series:

$$u_l = \sum_{j=0}^{L-1} U_l^{(j)} + \varrho_l(L); \qquad U_l^{(j)} = \frac{\mathcal{F}_{l+j-1}(\alpha_1)...\mathcal{F}_{l+j-1}(\alpha_m)}{\mathcal{F}_{l+j-1}(\gamma_1)...\mathcal{F}_{l+j-1}(\gamma_n)} \cdot \frac{z^j}{j!}.$$

Here L is the least integer for which the condition $|\varrho_l(L)| \le \varepsilon$ is fulfilled.

Values $U_l^{(L)}$ may be calculated consistently by means of simple recurrence formulas.

b) There is calculated the r-dimensional vector $\boldsymbol{v} \equiv [v_1, ..., v_r]$ with components $v_k \equiv \mathrm{d}_z^{k-1}{}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1 + N_1, ..., \gamma_n + N_n;\, z)$. For this purpose, by means of recurrence formulas, the vectors with components

$$v_k^{(R_1...R_m)} \equiv \mathrm{d}_z^{k-1}{}_mF_n(\alpha_1 + R_1, ..., \alpha_m + R_m;\, \gamma_1 + N_1, ..., \gamma_n + N_n;\, z)$$

for various integers R_j from ranges $0 \le R_j \le M_j$ are consistently calculated; $j = 1, ..., m$.

c) There is calculated the r-dimensional vector $\boldsymbol{w} \equiv [w_1, ..., w_r]$ with components $w_k \equiv \mathrm{d}_z^{k-1}{}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1, ..., \gamma_n;\, z)$. For this purpose, by means of recurrence formulas, the vectors with components

$$w_k^{(L_1...L_n)} \equiv \mathrm{d}_z^{k-1}{}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1 + L_1, ..., \gamma_n + L_n;\, z)$$

for various integers L_k from ranges $0 \le L_k \le N_k$ are consistently calculated; $k = 1, ..., n$.

2 Estimation of Error

The L-th term in the expansion of function

$${}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1 + N_1, ..., \gamma_n + N_n;\, z)$$

into a power series is equal to $w_L = \varphi(L, N_1, ..., N_n)$, where

$$\varphi(\lambda, \nu_1, ..., \nu_n) \equiv \frac{\prod\limits_{j=1}^{m} \Gamma(\lambda + \alpha_j) \cdot \prod\limits_{k=1}^{n} \Gamma(\nu_k + \gamma_k)}{\prod\limits_{j=1}^{m} \Gamma(\alpha_j) \cdot \prod\limits_{k=1}^{n} \Gamma(\lambda + \nu_k + \gamma_k)} \cdot \frac{z^\lambda}{\Gamma(\lambda + 1)}.$$

The module of function $\varphi(L, N_1, ..., N_n)$ can serve as a measure of accuracy of approximation of the function $\Phi(z)$ by the L-th partial sum of its Taylor series.

Let's consider the equation $|\varphi(\lambda, \nu_1, ..., \nu_n)| = \varepsilon$. This equation can also be written down in the following form:

$$F(\lambda, \nu_1, ..., \nu_n) = 0,$$

where

$$F(\lambda, \nu_1, ..., \nu_n) \equiv E + (1-m) \cdot \ln\sqrt{2\pi} + \ln\left|\varphi(\lambda, \nu_1, ..., \nu_n) \cdot \prod_{j=1}^{m} \Gamma(\alpha_j)\right|;$$

$$E \equiv \ln\left|\frac{(2\pi)^{(m-1)/2}}{\varepsilon \cdot \Gamma(\alpha_1)...\Gamma(\alpha_m)}\right|.$$

The function $F(\lambda, \nu_1, ..., \nu_n)$ is of practical interest at big values λ, $|\nu_1|$, ..., $|\nu_n|$. It is convenient for presenting in the form of

$$F(\lambda, \nu_1, ..., \nu_n) =$$
$$= E + \lambda \cdot \ln|z| + ns\lambda - ns\lambda \cdot \ln\lambda - A \cdot \ln\lambda -$$
$$- \sum_{k=1}^{n} \lambda \cdot \ln(1+\xi_k) - \sum_{k=1}^{n} \lambda\xi_k \cdot \ln(1+1/\xi_k) - \varrho(1, \lambda) + \sum_{j=1}^{m} \operatorname{Re} \varrho(\alpha_j, \lambda) +$$
$$+ \sum_{k=1}^{n} \operatorname{Re}\Big(\varrho(1/2 + i \operatorname{Im}\gamma_k, \lambda\xi_k) - \varrho(1/2 + i \operatorname{Im}\gamma_k, \lambda + \lambda\xi_k)\Big),$$

where

$$s \equiv 1 + (1-m)/n; \qquad A \equiv 1/2 + m/2 - \sum_{j=1}^{m} \operatorname{Re}\alpha_j,$$
$$\xi_k \equiv (\nu_k + \operatorname{Re}\gamma_k - 1/2)/\lambda \qquad (k = 1, ..., n);$$

$\varrho(\alpha, \lambda)$ is the remainder of asymptotical representation of the function $\ln\Gamma(\lambda + \alpha)$ at big values $|\lambda|$:

$$\ln\Gamma(\lambda + \alpha) = \ln\sqrt{2\pi} + (\lambda + \alpha - 1/2) \cdot \ln\lambda - \lambda + \varrho(\alpha, \lambda).$$

The parameter E can be estimated as follows: let's designate

$$h_j \equiv \frac{1}{2}\, 2\pi \cdot |\operatorname{Im}\alpha_j| \qquad (j = 1, ..., m);$$
$$E_0 \equiv (m-1) \cdot \ln\sqrt{2\pi} + \ln(1/\varepsilon).$$

Then

$$E = E_0 + \frac{1}{2} \cdot \sum_{j=1}^{m} \ln\left(\frac{\sinh h_j}{h_j}\right) + \Delta E' =$$

$$= E_0 + \frac{1}{2} \sum_{j=1}^{m} \big(h_j - \ln(1+2h_j)\big) + \Delta E' + \Delta E'',$$

where

$$\Delta E' \equiv \frac{1}{2} \sum_{j=1}^{m} \ln\left(\frac{h_j}{\sinh h_j \cdot |\Gamma(\alpha_j)|^2}\right);$$

$$\Delta E'' \equiv \frac{1}{2} \sum_{j=1}^{m} \ln\left(\left(1 - e^{-2h_j}\right)\left(1 + \frac{1}{2h_j}\right)\right).$$

The corrections $\Delta E'$ and $\Delta E''$ may be estimated by means of relations:

$$0 \le \Delta E' \le m\,\gamma_0 \qquad (\text{at } 1 \le \operatorname{Re}\alpha_j \le 2);$$
$$0 \le \Delta E'' \le m\,g_0/2,$$

where $\gamma_0 = 0.1214\,8629...$ is the maximum value of the function $-\ln\Gamma(x+1)$ at $x > -1$; $g_0 = 0.2611\,5245...$ is the maximum value of the function

$$g(x) = \ln\big((1 - e^{-x})(1 + 1/x)\big)$$

at $x \ge 0$.

Practically, usually $E_0 \ge 20$ and corrections $\Delta E'$ and $\Delta E''$ may be neglected at determination of the parameter E.

3 Estimation of the Time of Calculations

The time necessary for performance of all actions specified in 1 may be presented in the form of

$$T = L\,T_z + \sum_{j=1}^{m} M_j\,T_\alpha + \sum_{k=1}^{n} N_k\,T_\gamma,$$

where L, M_j, N_k are the numbers defined in 1;

T_z is the average time interval over all $k \in [0, ..., L-1]$ between consecutive calculations of partial sums

$$\sum_{j=1}^{k} \boldsymbol{U}^{(j)} \quad \text{and} \quad \sum_{j=1}^{k+1} \boldsymbol{U}^{(j)};$$

T_α is the average time interval over the set of parameters $R_1, ..., R_m$ $(0 \le R_j \le M_j; \quad j = 1, ..., m)$ between consecutive calculations of two vectors $\boldsymbol{v}^{(R_1 ... R_m)}$, all indexes R_j of which coincide except for one differing by unit;

T_γ is the average time interval over the set of parameters $L_1, ..., L_n$ $(0 \le L_k \le N_k; \quad k = 1, ..., n)$ between consecutive calculations of two vectors $\boldsymbol{w}^{(L_1 ... L_n)}$, all indexes L_k of which coincide except for one differing by unit.

The numbers N_1, ..., N_n should be chosen so that the value

$$\omega \equiv L + \sum_{k=1}^{n} N_k \varkappa / n,$$

where $\varkappa \equiv n\, T_\gamma / T_z$, was minimal. Thus the time T also proves to be minimal.

The parameter $\varkappa$ should be determined experimentally; it depends on the features of the computer, the programming language and the composed program. Usually it belongs to the interval [0.5, 2].

Practically, the numbers L, N_1, ..., N_n may be considered approximately equal to real numbers λ, ν_1, ..., ν_n rounded to the nearest integer values, which satisfy the equation $\varphi(\lambda, \nu_1, ..., \nu_n) = \varepsilon$, and for which the value

$$\Omega \equiv \lambda + \sum_{k=1}^{n} \nu_k \varkappa / n$$

takes on the minimal value. These numbers may be determined by the system of equations:

$$F(\lambda, \nu_1, ..., \nu_n) = 0;$$
$$\Big(\partial_\lambda - (n/\varkappa) \cdot \partial_{\nu_k}\Big) F(\lambda, \nu_1, ..., \nu_n) = 0 \qquad (k = 1, ..., n).$$

Here $F(\lambda, \nu_1, ..., \nu_n)$ and $\varphi(\lambda, \nu_1, ..., \nu_n)$ are the functions defined in 2; ε is the parameter describing the required accuracy of calculation of the function.

4 Estimation of Optimum Parameters

The parameters λ, ν_1, ..., ν_n defined in 3 may be estimated as follows:

let's designate

$$s \equiv 1 + (1 - m)/n; \qquad \tau \equiv |z|^{1/n} \cdot (n/E)^s;$$

$$A \equiv 1/2 + m/2 - \sum_{j=1}^{m} \mathrm{Re}\,\alpha_j;$$

let s and A be the parameters assigned in 2; let's designate:

$$\tau \equiv |z|^{1/n} \cdot (n/E)^s,$$

and let μ and ξ be the solutions of the following system of equations:

$$E + ns\mu - A \cdot \ln \mu - n\mu \cdot (\xi + 1/\varkappa) \cdot \ln(1 + 1/\xi) = 0;$$
$$\ln |z| - ns \cdot \ln \mu - n \cdot \ln(1 + \xi) + (n/\varkappa) \cdot \ln(1 + 1/\xi) = 0.$$

Then parameters λ, ν_1, ..., ν_n may be determined by means of the asymptotical series:

$$\lambda \sim \mu + \sum_{l=1}^{\infty} \lambda^{(l)} \cdot \mu^{-l};$$
$$\nu_k \sim 1/2 - \mathrm{Re}\,\gamma_k + \xi\mu + \sum_{l=1}^{\infty} \nu_k^{(l)} \cdot \mu^{-l}$$

($\mu \to \infty$; $k = 1, ..., n$). The expressions for the factors $\lambda^{(l)}$ and $\nu_k^{(l)}$ are not given here because of their bulkiness. Besides, of practical interest is the case when the value μ is big, and hence $\lambda \approx \mu$ and $\nu_k \sim 1/2 - \mathrm{Re}\,\gamma_k + \xi\mu$.

In particular, if $A = 0$, then the parameter μ is expressed explicitly by ξ:

$$\mu = (E/n) \cdot \Big(-s + (\xi + 1/\varkappa) \cdot \ln(1 + 1/\xi)\Big)^{-1}, \tag{1}$$

and the parameter ξ is the root of the equation:

$$\xi\,(1 + \xi)^{\varkappa - 1}\big(-s + (\xi + 1/\varkappa) \cdot \ln(1 + 1/\xi)\big)^{-\varkappa s} = \tau^{\varkappa}. \tag{2}$$

This equation can be also rewritten in the form of

$$\xi^{-1}\,(1 + 1/\xi)^{1/\varkappa - 1}\,\big(-s + (\xi + 1/\varkappa) \cdot \ln(1 + 1/\xi)\big)^{s} = \tau^{-1}. \tag{3}$$

Before analyzing the equations determining the parameters ξ and μ, it is necessary to make a number of remarks. First, at drawing up of the computer program for calculation of hypergeometric functions, it would be simply unreasonable to waste time for searching the optimum way of solving the problem comparable with the time of realization of this way. Therefore the procedure of determination of parameters ξ and μ should be extremely simplified; perforce it is necessary to confine to a rough approximations. Therefore, in particular, at determination of parameter ξ, it is necessary to

use equation (2), and at determination of μ – explicit expression (1) even if the condition $A = 0$ is not satisfied.

Secondly, if there is no opportunity to derive a simple explicit expression for approximate value ξ as a function of τ, then this function should be considered for big values of τ since at small values τ, so also at small values $|z|$, the process of calculation of hypergeometric function will be rapid in any case.

Thirdly, if a partial sum of some power series is used for approximation of the parameter ξ^{-1}, then the number of terms M of this series should be taken such that the first M terms of expansion were positive. This will provide a correct character of the dependence of approximate value ξ on τ and will exclude the occurrence of extremums "in the unauthorized place".

The character of dependence of parameter ξ determined by equation (2) on τ is as follows: at $\tau > 0$, this function increases, its plot has no inflections; at $\tau \to 0$, the asymptotical formula $\xi \sim \tau^{\varkappa}\big(\ln(1 + 1/\tau)\big)^{\varkappa s}$ is true, and, at $\tau \to \infty$, the asymptotical formula depends on values s and $\varkappa$ and looks like $\xi \sim \xi_B,\ \xi \sim (\omega\tau)^{1/3},\ \xi \sim (\omega\tau)^{1/2}$ or $\xi \sim \omega\tau$ $(\omega,\, \xi_B = \text{const})$.

5 Scheme of Calculation

Summing up all the above mentioned, it is possible to represent the scheme of calculation of the function $\Phi(z)$ as the sequence of actions:

a) the time intervals T_z and T_γ defined in 3 are estimated, and the value of the parameter $\varkappa = n\,T_\gamma/T_z$ is calculated;

b) the values of parameters

$$h_j = \frac{1}{2}\,2\pi \cdot |\mathrm{Im}\,\alpha_j| \qquad (j = 1, ..., m);$$
$$E_0 = (1 - m) \cdot \ln\sqrt{2\pi} + \ln(1/\varepsilon);$$
$$E \approx E_0 + \frac{1}{2}\sum_{j=1}^{m}\big(h_j - \ln(1 + 2\,h_j)\big);$$
$$s = 1 + (1 - m)/n; \qquad \tau = |z|^{1/n} \cdot (n/E)^s$$

are calculated;

c) the approximate value of the parameter ξ which is the root of equivalent equations (2) and (3) (see below) is determined;

d) by means of formula (1), there are determined the parameter μ and the numbers $L = \mathcal{R}(\mu)$ and $N_k = \mathcal{R}(\infty/\in - \mathrm{Re}\,\gamma_{\|} + \mu\xi)$, where $\mathcal{R}(\S)$ is the integer function the value of which is the value of the variable x rounded to the nearest integer;

e) the values of functions u_k, v_k and w_k are consistently determined according to the scheme described in 1.

6 Determination of Parameter ξ

At determination of parameter ξ, a few cases are possible:

• If $m = n + 1$, so $s = 0$ (in particular, it is satisfied for function ${}_2F_1(...)$), then

$$\xi \approx \begin{cases} \sum\limits_{k=1}^{M} P_k(\varkappa) \cdot \tau^{\varkappa k} & \text{at } \tau \le \tau_B \\ \left(\sum\limits_{k=1}^{M} P_k(1/\varkappa) \cdot \tau^{-k} \right)^{-1} & \text{at } \tau \ge \tau_B \end{cases},$$

where $M \ge 3$; $\tau_B = \frac{1}{\varkappa} |\varkappa - 1|^{1-1/\varkappa}$;

$$P_k(\varkappa) = \frac{(-1)^{k-1}}{k!} \cdot \widetilde{\mathcal{F}}_{k-1}(\varkappa k - 2) = \prod_{j=2}^{k} (1 - \varkappa k/j).$$

$$P_1(\varkappa) = 1; \quad P_2(\varkappa) = 1 - \varkappa; \quad P_3(\varkappa) = \frac{1}{2}(1 - \varkappa)(2 - 3\varkappa);$$

$$P_4(\varkappa) = \frac{1}{3}(1-\varkappa)(1-2\varkappa)(3-4\varkappa); \quad P_5(\varkappa) = \frac{1}{24}(1-\varkappa)(2-5\varkappa)(3-5\varkappa)(4-5\varkappa).$$

• If $m = n$ (i.e. $s = 1$) and $0 < \varkappa < 2$ then

$$\xi \approx \begin{cases} \frac{(\omega\tau)^{1/2}}{1+Q\,(\omega\tau)^{-1/2}} & \text{at } (6 - \sqrt{6})/5 < \varkappa < (6 + \sqrt{6})/5; \\ (\omega\tau)^{1/2} & \text{otherwise} \end{cases},$$

where $\omega = 1/\varkappa - 1/2$; $Q = (-12\,\omega^2 + 12\,\omega - 1)/(24\,\omega)$.

• If $m = n$ (i.e. $s = 1$) and $\varkappa = 2$ then

$$\xi \approx \left(\sum_{k=1}^{M} Q_k \cdot (12/\tau)^{k/3} \right)^{-1},$$

where $M \ge 3$; $Q_1 = 1$; $Q_2 = 1/2$; $Q_3 = 19/120$; $Q_4 = 1/30$; $Q_5 = 881/201600$.

• If either $m = 0$ (i.e. $s = 1 + 1/n$; in particular, it is satisfied for function ${}_0F_1(...)$) or $m = n$ (i.e. $s = 1$) and $\varkappa > 2$, then

$$\xi \approx \xi_B \cdot \left(1 + Q\,(\omega\tau)^{-1/s}\right)^{-1},$$

Table 0.1: Table of limiting values of ξ

$\varkappa$	ξ_B $(s=2)$
0.75	0.5106 5335
1.00	0.2550 0097
1.25	0.1326 8996
1.50	0.0712 5568
1.75	0.0393 1574

$\varkappa$	ξ_B $(s=1)$	ξ_B $(s=2)$
2.00	∞	0.0221 9102
2.25	1.0443 9706	0.0127 5570
2.50	0.4130 5229	0.0074 3669
2.75	0.2194 7387	0.0043 8252
3.00	0.1319 7214	0.0026 0350

where ξ_B is the root of equation

$$(\xi_B + 1/\varkappa) \cdot \ln(1 + 1/\xi_B) = s;$$

$$\omega = \xi_B^{-1/\varkappa}(1+\xi_B)^{1/\varkappa - 1}; \qquad Q = \left(\frac{1+\varkappa\xi_B}{\varkappa(1+\xi_B)} - \frac{s\,\varkappa\xi_B}{1+\varkappa\xi_B}\right)^{-1}.$$

The values of the parameter ξ_B for different values of $\varkappa$ are given at $s = 1$ and $s = 2$ in Table 0.1.

• If $n \geq 2$ and $1 < m < n+1$, so $0 < s < 1$, (in particular, it is satisfied for function ${}_2F_2(...)$), then

$$\xi \approx \begin{cases} \frac{\omega\tau}{1+Q\,(\omega\tau)^{-1}} & \text{at } \varkappa > 2/(2-s); \\ \omega\tau & \text{otherwise} \end{cases},$$

where $\omega = (1-s)^s$; $Q = 1/2 - (\varkappa^{-1} - 1/2)/(1-s)$.

7 Final Remarks

As follows from the description of algorithm, the realization of the offered method is possible under the following conditions. First, the range of the argument should be such that the phenomenon of cancelation was excluded. Secondly, the recurrence relations should be numerically steady. Unfortunately, the question on, at which values of the argument and the parameters, the latter condition proves to be fulfilled remains open. However for some practically important classes of hypergeometric functions (for example, for some functions of statistical distributions), it is possible to consider the condition of stability to be guaranteed.

In some cases, the recurrence relation in the direction backward may prove to be steady. Therefore, probably it is expedient to modify the offered algorithm so that the use of recurrence relations in both directions – forward and backward was supposed in it. But it is a subject of the further research.

1 Appendix. Reduction Formulas for Hypergeometric Function in the Matrix Form

The reduction formulas in the form in which they are presented in this work allow us to derive rather easily the linear relations like

$$\sum_{k=1}^{r} a_k\, u_k = 0$$

between any hypergeometric functions u_k of type ${}_mF_n$, the arguments of which have the same values and the corresponding parameters differ by integers; $r \equiv \max\{m, n+1\}$.

Main formulas of this section may be derived by means of formulas of differentiation [9] for the function $\Phi(z) = {}_mF_n(\alpha_1, ..., \alpha_m;\ \gamma_1, ..., \gamma_n;\ z)$:

$$\mathrm{d}_z^M \Phi(\beta z) = \beta^M \cdot \frac{\mathcal{F}_M(\alpha_1)\,\mathcal{F}_M(\alpha_2)...\mathcal{F}_M(\alpha_m)}{\mathcal{F}_M(\gamma_1)\,\mathcal{F}_M(\gamma_2)...\mathcal{F}_M(\gamma_n)} \cdot \\ \cdot {}_mF_n\big(\alpha_1 + M, ..., \alpha_m + M;\ \gamma_1 + M, ..., \gamma_n + M;\ \beta z\big);$$

$$\mathrm{d}_z^M \big(z^{\alpha_1+M-1} \cdot \Phi(\beta z)\big) = \\ = \mathcal{F}_M(\alpha_1) \cdot z^{\alpha_1-1} \cdot {}_mF_n\big(\alpha_1 + M, \alpha_2, ..., \alpha_m;\ \gamma_1, ..., \gamma_n;\ \beta z\big);$$

$$\mathrm{d}_z^M \big(z^{\gamma_1-1} \cdot \Phi(\beta z)\big) = \\ = \mathcal{F}_M(\gamma_1 - M) \cdot z^{\gamma_1-M-1} \cdot {}_mF_n\big(\alpha_1, ..., \alpha_m;\ \gamma_1 - M, \gamma_2, ..., \gamma_n;\ \beta z\big)$$

$(M = 0, 1, ...)$, in view of symmetry of the function $\Phi(z)$ concerning parameters and the differential equation to which this function satisfies.

Let

$$\boldsymbol{M} = [M_1, M_2, ..., M_m], \quad \boldsymbol{N} = [N_1, N_2, ..., N_n], \\ \boldsymbol{\mu} = [\mu_1, \mu_2, ..., \mu_m], \quad \boldsymbol{\nu} = [\nu_1, \nu_2, ..., \nu_n]$$

be any sequences of integers, and let's consider the functions:

$$\Phi(z) = {}_mF_n(\alpha_1, ..., \alpha_m;\ \gamma_1, ..., \gamma_n;\ \beta z) \quad \text{and} \\ f(z) = {}_mF_n(\alpha_1 + M_1, ..., \alpha_m + M_m;\ \gamma_1 + N_1, ..., \gamma_n + N_n;\ \beta z),$$

– two hypergeometric functions of the same order the corresponding parameters of which differ by integers. We shall designate the sequences of upper and lower parameters of the function $\Phi(z)$ by $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$, respectively:

$$\boldsymbol{\alpha} = [\alpha_1, \alpha_2, ..., \alpha_m]; \quad \boldsymbol{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_n].$$

The function $\Phi(z)$ satisfies the linear differential equation of the order $r = \max\{m, n+1\}$, which we shall present in the form of

$$\sum_{k=0}^{r} \eta_k(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z) \cdot \mathrm{d}_z^k \Phi(z) = 0,$$

and we assume $\eta_r(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z) = 1$.

Further, let's designate by $[\mathbf{g}_R(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)]$ and $[\mathbf{h}_L(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)]$ $(R = 1, 2, ..., m;\ L = 1, 2, ..., n)$ the sequences of the matrixes depending on variables $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$ and z, the elements of which are determined by the following formulas:

$$\langle j|\mathbf{g}_R(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)|k\rangle =$$
$$= \delta_{jk} + \alpha_R^{-1}\Big((j-1)\cdot\delta_{jk} + z\cdot\delta_{(j+1),k} - z\,\eta_{k-1}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)\cdot\delta_{jr}\Big);$$
$$\langle j|\mathbf{h}_L(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)|k\rangle =$$
$$= \delta_{jk} + (\gamma_L - 1)^{-1}\Big((j-1)\cdot\delta_{jk} + z\cdot\delta_{(j+1),k} - z\,\eta_{k-1}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, z)\cdot\delta_{jr}\Big),$$

and let $\mathbf{A}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z)$ and $\mathbf{B}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z)$ be two mutually inverse matrixes depending on variables $\boldsymbol{\alpha}$, $\boldsymbol{\gamma}$, $\boldsymbol{M}$, $\boldsymbol{N}$ and z and determined as follows:

$$\mathbf{B}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z) = \mathbf{G}_1 \cdot \mathbf{G}_2 \cdot ... \cdot \mathbf{G}_m \cdot \mathbf{H}_1 \cdot \mathbf{H}_2 \cdot ... \cdot \mathbf{H}_n,$$

where

$$\mathbf{G}_R = \begin{cases} \prod\limits_{k=0}^{M_R-1} \big(\mathbf{g}_R(\alpha_1 + M_1, ..., \alpha_{R-1} + M_{R-1}, \alpha_R + k, \alpha_{R+1}, ..., \alpha_m, \boldsymbol{\gamma}, z)\big)^{-1} \\ \qquad\qquad \text{at } M_R \geq 0 \\ \prod\limits_{k=1}^{-M_R} \mathbf{g}_R(\alpha_1 + M_1, ..., \alpha_{R-1} + M_{R-1}, \alpha_R - k, \alpha_{R+1}, ..., \alpha_m, \boldsymbol{\gamma}, z) \\ \qquad\qquad \text{at } M_R \leq 0 \end{cases}$$

$(R = 1, ..., m)$;

$$\mathbf{H}_L = \begin{cases} \prod\limits_{k=1}^{N_L} \mathbf{h}_L(\boldsymbol{\alpha}+\boldsymbol{M}, \gamma_1+N_1, ..., \gamma_{L-1}+N_{L-1}, \gamma_L+k, \gamma_{L+1}, ..., \gamma_n, z) \\ \qquad\qquad \text{at } N_L \geq 0 \\ \prod\limits_{k=0}^{-N_L-1} \big(\mathbf{h}_L(\boldsymbol{\alpha}+\boldsymbol{M}, \gamma_1+N_1, ..., \gamma_{L-1}+N_{L-1}, \gamma_L-k, \gamma_{L+1}, ..., \gamma_n, z)\big)^{-1} \\ \qquad\qquad \text{at } N_L \leq 0 \end{cases}$$

$(L = 1, ..., n)$.

The determining formulas for matrixes $\mathbf{A}$ and $\mathbf{B}$ allow us to derive a number of formulas describing the properties of these matrixes: different explicit expressions, recurrence relations, the values of matrixes at special values of parameters, etc.

The function $\Phi(z)$ and its derivatives are expressed in terms of function $f(z)$ and its derivatives – as follows:

$$\mathrm{d}_z^{j-1} f(z) = \sum_{k=1}^{r} A_{jk}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z) \cdot \mathrm{d}_z^{k-1} \Phi(z);$$
$$\mathrm{d}_z^{j-1} \Phi(z) = \sum_{k=1}^{r} B_{jk}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z) \cdot \mathrm{d}_z^{k-1} f(z)$$

$(j = 1, ..., r)$.

From these relations, it follows that the function $f(z)$ is expressed in terms of function $\Phi(z)$ and its first $r-1$ derivatives – by the formula:

$$f(z) = \sum_{k=1}^{r} A_k \cdot \mathrm{d}_z^{k-1} \Phi(z),$$

where the sequence of factors $[A_1, ..., A_r]$ represents the first row of the matrix $\mathbf{A}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z)$ (i.e. $A_k = A_{1k}(\boldsymbol{\alpha}, \boldsymbol{\gamma}, \boldsymbol{M}, \boldsymbol{N}, z)$). These factors may be consistently determined by special recurrence formulas.

As an example, for the Kummer hypergeometric function for some special values of M and N, the values of factors A_1 and A_2 are given in Table .2. Similar tables can be drawn up for any other hypergeometric function.

Table .2: Values of Factors A_1 and A_2 Defined by Conditions:

$$u(z) \equiv {}_1F_1(\alpha,\gamma,z); \qquad {}_1F_1(\alpha+M,\gamma+N,z) = A_1\cdot u(z) + A_2\cdot \mathrm{d}_z u(z).$$

$_N\backslash^M$	-1	0	1
-1	$A_1 = 1 - z(\gamma-1)^{-1}$ $A_2 = z(\gamma-1)^{-1}$	$A_1 = 1$ $A_2 = z(\gamma-1)^{-1}$	$A_1 = 1 + z(\gamma-1)^{-1}$ $A_2 = z(1+z\alpha^{-1})(\gamma-1)^{-1}$
0	$A_1 = 1 - z(\gamma-\alpha)^{-1}$ $A_2 = z(\gamma-\alpha)^{-1}$	$A_1 = 1$ $A_2 = 0$	$A_1 = 1$ $A_2 = z\alpha^{-1}$
1	$A_1 = \gamma(\gamma-2\alpha+1-z)\cdot$ $\cdot(\gamma-\alpha)^{-1}(\gamma-\alpha+1)^{-1}$ $A_2 = \gamma(z+\alpha-1)\cdot$ $\cdot(\gamma-\alpha)^{-1}(\gamma-\alpha+1)^{-1}$	$A_1 = \gamma(\gamma-\alpha)^{-1}$ $A_2 = -\gamma(\gamma-\alpha)^{-1}$	$A_1 = 0$ $A_2 = \gamma\alpha^{-1}$

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