ASYMPTOTICAL EXPANSION OF HYPERGEOMETRIC FUNCTIONS IN TERMS OF PARAMETERS

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Abstract

The formulas which allow approximating the hypergeometric functions with the arbitrary number of upper and lower parameters by more simple functions by means of their expansion into asymptotical series are derived. The specified auxiliary functions are investigated in detail. The obtained results are tested in concrete examples.

Key words and phrases: Hypergeometric function; parameters; asymptotical expansion; limit formula; formula of degeneration.

Introduction

In the enormous list of special mathematical functions, hypergeometric functions are distinguished by special practical importance. First, a lot of other special functions, in particular, integral functions and functions of statistical distribution can be expressed in terms of them. Secondly, a rather extensive class of linear differential equations supposes solutions with the use of hypergeometric functions, which makes these functions irreplaceable in many applications (see, for example, publications of the author of this work [1],[2]; a full list of similar works would amount to tens or even hundreds pages).

If, in any theoretical or applied problem, it is required to determine an unknown function, always when there is such an opportunity, they try to present this function in the simplest form, supposing both the numerical and the theoretical analysis of the obtained result. In relation to the hypergeometric function, an opportunity for analysis and research of qualitative characteristics usually proves to be more considerable, the less is the number of parameters of this function. Therefore the cases when the hypergeometric function can be approximated by the expression containing the hypergeometric functions of lower order are of interest.

In this work, for achievement of this purpose, the asymptotical expansion of hypergeometric functions is used. The formulas given below, where such expansions are used, generalize and render more precisely the well-known *formulas of degeneration* [3].

In work [4], the formulas of asymptotical expansion of Gaussian hypergeometric function were considered; they represent special cases of formulas considered in this work. In [4] main formulas are received with the use of integrated representations for the Gaussian function and the method of saddle points; thus in this work the general expressions for the coefficient of expansion are not given.

The authors of some works (see for example [5]-[9]) offer the formulas of asymptotical expansion for different special types of hypergeometric functions. In this work, unlike the specified publications, hypergeometric functions with any number of upper and lower parameters in most general form are investigated.

In this work, the following designations are used:

j, k, r, l, m, n, R, L, M, N – integer variables;

x, y, t, a, b, c, d, h, p, q, s, A, B, C, D, H, ω, φ, ψ – real variables;

α, β, γ, $\varkappa$, λ, μ, ν, ϱ, σ, τ, ζ – complex variables;

C_j^k are the binomial coefficients; $(-1)^{r-k} \cdot s_r^{(k)}$ are the Stirling numbers of the first kind (the multiplier $(-1)^{r-k}$ here is used so that all numbers $s_r^{(k)}$ were nonnegative); σ_r^k are the Stirling numbers of the second kind; $B_k(z)$ are the Bernoulli polynomials;

$$\mathcal{F}_k(z) \equiv \prod_{j=0}^{k-1} (z+j); \qquad \widetilde{\mathcal{F}}_k(z) \equiv (-1)^k \cdot \mathcal{F}_k(-z) = \prod_{j=0}^{k-1} (z-j)$$

$\mathcal{F}_k(z)$ is the Pochhammer polynomial of degree k, which is usually designated by $(z)_k$ [3],[10].

For the derivative, the reduced designation is used everywhere: d_z instead of $\frac{d}{dz}$.

Besides, the following auxiliary functions are used:

$$h_r(z) \equiv (z-r) \sum_{k=0}^{r} \frac{(-1)^{r-k}\, \sigma_{r+k}^k}{(r+k)!\,(r-k)!} \cdot \frac{\mathcal{F}_r(z+1)}{(z+k)};$$

$$\chi_r(\lambda, z) \equiv \widetilde{\mathcal{F}}_r(z) \sum_{k=0}^{r} \frac{1}{k!}\, h_{r-k}(z) \cdot \lambda^k; \qquad W_{kj}(\mu) = \frac{1}{j!}\, \chi_{k-j}(1-\mu,\, -j),$$

and also $U_{jk}(\lambda)$, $V_{jk}(\mu)$, $\Upsilon_{jk}(\alpha, \gamma, \lambda, \mu)$ – the functions which are the coefficients of expansion of some expressions including $\chi_k(\lambda, z)$ in series by

$\widetilde{\mathcal{F}}_j(z)$:

$$\chi_k(\lambda, z) = \sum_{j=k}^{2k} U_{kj}(\lambda) \cdot \widetilde{\mathcal{F}}_j(z); \qquad \chi_k(1-\mu, -z) = \sum_{j=0}^{2k} V_{kj}(\mu) \cdot \widetilde{\mathcal{F}}_j(z);$$

$$\sum_{l=0}^{r} \frac{(-1)^l}{\alpha^{r-l}\,\gamma^l} \cdot \chi_{r-l}(\lambda, z)\,\chi_l(1-\mu,\,-z) = \sum_{k=0}^{2r} \Upsilon_{rk}(\alpha, \gamma, \lambda, \mu) \cdot \widetilde{\mathcal{F}}_k(z).$$

1 Asymptotical Expansions

Let's designate:

$$\Phi_0(z) = {}_mF_n(\alpha_1, ..., \alpha_m;\, \gamma_1, ..., \gamma_n;\, \beta z).$$

This function will be used for approximation of hypergeometric functions of a higher order.

At deriving different formulas for hypergeometric functions there often are useful the following relations:

$$(z\mathrm{d}_z + A)^m \cdot (z^\lambda\, I) = (z^\lambda\, I) \cdot (z\mathrm{d}_z + A + \lambda)^m; \qquad (1)$$

$$z^m \mathrm{d}_z^m = \widetilde{\mathcal{F}}_m(z\mathrm{d}_z) \qquad (2)$$

(I is the unit operator).

In particular, at proving the validity of formulas (3), (4) and (5) given below, the differential equations which approximating and approximated functions $\Phi_0(z)$ and $\Phi(\eta, z)$ satisfy, the values of these functions and their derivatives at $z = 0$, and also formulas (1), (2) and some relations for functions $\chi_k(\lambda, \xi)$ given in Section 2 can be used.

The case of small values of argument and big values of one of the upper parameters. Let's consider the function:

$$\Phi(\eta, z) = {}_{m+1}F_n(\lambda + \alpha_0/\eta,\, \alpha_1, ..., \alpha_m;\, \gamma_1, ..., \gamma_n;\, \eta\beta z).$$

The following formula of degeneration takes place for it:

$$\lim_{\eta\to 0} \Phi(\eta, z) = \Phi_0(z\alpha_0) \qquad (n \geq m).$$

A more exact approximation is given by the following *formula of asymptotical expansion*:

$$\Phi(\eta, z) \sim \sum_{k=0}^{\infty} \Phi_k(z\alpha_0) \cdot (\eta/\alpha_0)^k \qquad (\eta \to 0), \qquad (3)$$

where the coefficients of expansion are determined by the relations:

$$\Phi_k(z) = \chi_k(\lambda,\, z\mathrm{d}_z) \cdot \Phi_0(z) = \sum_{j=k}^{2k} U_{kj}(\lambda) \cdot z^j\, \mathrm{d}_z^j\, \Phi_0(z).$$

In particular,

$$\Phi_1(z) = \left(\lambda\, z\mathrm{d}_z + \frac{1}{2}\, z^2 \mathrm{d}_z^2 \right)\, \Phi_0(z);$$

$$\Phi_2(z) = \left(\frac{1}{2}\, \lambda(\lambda+1) \cdot z^2 \mathrm{d}_z^2 + \frac{1}{6}\,(2+3\lambda) \cdot z^3 \mathrm{d}_z^3 + \frac{1}{8}\, z^4 \mathrm{d}_z^4 \right)\, \Phi_0(z).$$

A symbolic notation of the formula of asymptotical expansion:

$$\Phi(\eta, z) \sim \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \Big(B_k(\lambda + z\mathrm{d}_z) - B_k(\lambda) \Big) \cdot (\eta/\alpha_0)^{k-1} \right) \cdot \Phi_0(z\alpha_0)$$

$$(\eta \to 0).$$

The case of big values of argument and one of the lower parameters. Let's consider the function:

$$\Phi(\eta, z) = {}_mF_{n+1}(\alpha_1, ..., \alpha_m;\; \mu + \gamma_0/\eta, \gamma_1, ..., \gamma_n;\; \beta z/\eta).$$

The following formula of degeneration takes place for it:

$$\lim_{\eta \to 0} \Phi(\eta, z) = \Phi_0(z/\gamma_0)$$

$(n \ge m)$ or $(m = n+1;\; |z| < 1;\; \mathrm{Re}\,(\gamma_0/\eta) \ge 0)$.

A more exact approximation is given by the following *formula of asymptotical expansion*:

$$\Phi(\eta, z) \sim \sum_{k=0}^{\infty} \Phi_k(z/\gamma_0) \cdot (\eta/\gamma_0)^k \qquad (\eta \to 0), \tag{4}$$

where the coefficients of expansion are determined by the relations:

$$\Phi_k(z) = (-1)^k \cdot \chi_k(1-\mu,\, -z\mathrm{d}_z) \cdot \Phi_0(z) = (-1)^k \sum_{j=0}^{2k} V_{kj}(\mu) \cdot z^j\, \mathrm{d}_z^j\, \Phi_0(z).$$

In particular,

$$\Phi_1(z) = -\left(\mu\, z\mathrm{d}_z + \frac{1}{2}\, z^2 \mathrm{d}_z^2 \right)\, \Phi_0(z);$$

$$\Phi_2(z) = \left(\mu^2\, z\mathrm{d}_z + \frac{1}{2}\,(1+3\mu+\mu^2)\, z^2 \mathrm{d}_z^2 + \frac{1}{6}\,(4+3\mu)\, z^3 \mathrm{d}_z^3 + \frac{1}{8}\, z^4 \mathrm{d}_z^4 \right)\, \Phi_0(z);$$

A symbolic notation of the formula of asymptotical expansion:

$$\Phi(\eta, z) \sim \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)}\Big(B_k(\mu) - B_k(\mu + z\mathrm{d}_z)\Big) \cdot (\eta/\gamma_0)^{k-1}\right) \cdot \Phi_0(z/\gamma_0)$$
$$(\eta \to 0).$$

The case of big values of one of the upper and one of the lower parameters. Let's consider the function:

$$\Phi(\eta, z) = {}_{m+1}F_{n+1}(\lambda + \alpha_0/\eta, \alpha_1, ..., \alpha_m;\ \mu + \gamma_0/\eta, \gamma_1, ..., \gamma_n;\ \beta z).$$

The following formula of degeneration takes place for it:

$$\lim_{\eta \to 0} \Phi(\eta, z) = \Phi_0(z\alpha_0/\gamma_0)$$

$(n \geq m)$ or $(m = n + 1;\ |z\alpha_0/\gamma_0| < 1;\ \mathrm{Re}\,(\gamma_0/\eta) \geq 0)$.

A more exact approximation is given by the following *formula of asymptotical expansion*:

$$\Phi(\eta, z) \sim \sum_{k=0}^{\infty} \Phi_k(z\alpha_0/\gamma_0) \cdot \eta^k \qquad (\eta \to 0), \tag{5}$$

where the coefficients of expansion are determined by the relations:

$$\Phi_r(z) = \sum_{l=0}^{r} \frac{(-1)^l}{\alpha_0^{r-l}\,\gamma_0^l} \cdot \chi_{r-l}(\lambda,\ z\mathrm{d}_z)\,\chi_l(1-\mu,\ -z\mathrm{d}_z) \cdot \Phi_0(z) =$$
$$= \sum_{j=0}^{2r} \Upsilon_{rj}(\alpha_0, \gamma_0, \lambda, \mu) \cdot z^j\,\mathrm{d}_z^j\,\Phi_0(z).$$

In particular,

$$\Phi_1(z) = \left(\big(\lambda/\alpha_0 - \mu/\gamma_0\big)\,z\mathrm{d}_z + \frac{1}{2}\big(1/\alpha_0 - 1/\gamma_0\big)\,z^2\mathrm{d}_z^2\right)\,\Phi_0(z).$$

A symbolic notation of the formula of asymptotical expansion:

$$\Phi(\eta, z) \sim \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)}\Big(B_k(\lambda + z\mathrm{d}_z) - B_k(\lambda)\Big) \cdot (\eta/\alpha_0)^{k-1} - \right.$$
$$\left. - \frac{(-1)^k}{k(k-1)}\Big(B_k(\mu - z\mathrm{d}_z) - B_k(\mu)\Big) \cdot (\eta/\gamma_0)^{k-1}\right) \cdot \Phi_0(z)$$
$$(\eta \to 0).$$

A special case. The formulas given in the previous item become essentially simpler in the case when $\alpha_0 = \gamma_0$.

Let's consider the function:

$$\Phi(\eta, z) = {}_{m+1}F_{n+1}(\mu - \lambda + 1/\eta, \alpha_1, ..., \alpha_m;\ \mu + 1/\eta, \gamma_1, ..., \gamma_n;\ \beta z).$$

The following formula of asymptotical expansion takes place for it:

$$\Phi(\eta, z) \sim \sum_{k=0}^{\infty} \Phi_k(z) \cdot \eta^k \qquad (\eta \to 0), \tag{6}$$

where the coefficients of expansion are determined by the relations:

$$\Phi_r(z) = (-1)^r \sum_{j=0}^{r} W_{rj}(\mu) \cdot \mathcal{F}_j(\lambda) \cdot z^j \, \mathrm{d}_z^j \, \Phi_0(z).$$

In particular,

$$\Phi_1(z) = -\lambda \, z \mathrm{d}_z \, \Phi_0(z).$$

$$\Phi_2(z) = \left(\mu \, \lambda \, z \mathrm{d}_z + \frac{1}{2} \lambda \, (\lambda + 1) \, z^2 \mathrm{d}_z^2 \right) \Phi_0(z);$$

$$\Phi_3(z) = -\left(\mu^2 \, \lambda \, z \mathrm{d}_z + (\mu + 1/2) \, \lambda \, (\lambda + 1) \, z^2 \mathrm{d}_z^2 + \frac{1}{6} \lambda \, (\lambda + 1) \, (\lambda + 2) \, z^3 \mathrm{d}_z^3 \right) \Phi_0(z).$$

The case of big values of one of the lower parameters. Let's consider the function:

$$\Phi(\eta, z) = {}_{m}F_{n+1}(\alpha_1, ..., \alpha_m;\ \mu + 1/\eta, \gamma_1, ..., \gamma_n;\ z).$$

The following limit formula takes place for it:

$$\lim_{\eta \to 0} \Phi(\eta, z) = 1,$$

if the function $\Phi(\eta, z)$ is represented as a convergent series of powers z and $\arg(1/\eta) < 2\pi/2$.

A more exact approximation is given by the following *formula of asymptotical expansion*:

$$\Phi(\eta, z) \sim \sum_{k=0}^{\infty} \sum_{j=0}^{k} W_{kj}(\mu) \cdot \frac{\mathcal{F}_j(\alpha_1) ... \mathcal{F}_j(\alpha_m)}{\mathcal{F}_j(\gamma_1) ... \mathcal{F}_j(\gamma_n)} \cdot (-z)^j \, (-\eta)^k; \tag{7}$$

i.e.

$$\Phi(\eta, z) \sim 1 +$$

$$+ \frac{\alpha_1 ... \alpha_m}{\gamma_1 ... \gamma_n} \cdot z \, \eta + \left(\frac{\alpha_1 ... \alpha_m}{\gamma_1 ... \gamma_n} \cdot \mu \, z + \frac{\alpha_1(\alpha_1 + 1) ... \alpha_m(\alpha_m + 1)}{\gamma_1(\gamma_1 + 1) ... \gamma_n(\gamma_n + 1)} \cdot \frac{z^2}{2} \right) \cdot \eta^2 + ... \,.$$

Asymptotical expansion of quotient of two gamma functions. Let's consider the function:

$$\varphi(\eta, z) = \eta^z \cdot \frac{\Gamma(1/\eta + z + \lambda)}{\Gamma(1/\eta + \lambda)}.$$

Using a well-known formula of asymptotic expansion of the gamma function [11], it is possible to obtain:

$$\ln \varphi(\eta, z) \sim \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \big(B_k(\lambda + z) - B_k(\lambda)\big) \cdot \eta^{k-1}\right)$$

$(\eta \to 0;\ \arg(1/\eta) < 2\pi/2)$.

It turned out that, for asymptotical representation of $\varphi(\eta, z)$, the function $\chi_k(\lambda, z)$ introduced in this work may be also used:

$$\varphi(\eta, z) \sim \sum_{k=0}^{\infty} \chi_k(\lambda, z) \cdot \eta^k \qquad (\eta \to 0);$$

$$1/\varphi(\eta, z) \sim \sum_{k=0}^{\infty} (-1)^k \cdot \chi_k(1 - \lambda, -z) \cdot \eta^k \qquad (\eta \to 0).$$

The symbolic asymptotical relations

$$\sum_{k=0}^{\infty} \chi_k(\lambda, z) \cdot \eta^k = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \Big(B_k(\lambda + z) - B_k(\lambda)\Big) \cdot \eta^{k-1}\right) \quad (\eta \to 0);$$

$$\sum_{k=0}^{\infty} (-1)^k \cdot \chi_k(1 - \mu,\ -z) \cdot \eta^k = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k}{k(k-1)} \Big(B_k(\mu) - B_k(\mu + z)\Big) \cdot \eta^{k-1}\right) \quad (\eta \to 0)$$

may be used as alternative definitions of the functions $\chi_k(\lambda, z)$.

2 Main Properties of Functions $\chi_r(\lambda, z)$

Elementary properties:

The defining relations for the functions $\chi_r(\lambda, z)$ are given in the introduction.

From these relations, it follows that, at $r < 0$, $\chi_r(\lambda, z) = 0$, and at $r \geq 0$, $\chi_r(\lambda, z)$ is a polynomial of degree $2r$.

Functional relations:

$$\chi_r(\lambda,z)=(-1)^r\cdot\chi_r(1-\lambda-z,z);$$

$$\chi_r(\lambda,\xi+z)=\sum_{k=0}^{r}\chi_{r-k}(\lambda,\xi)\cdot\chi_k(\lambda+\xi,z).$$

$$\chi_r(\lambda+\mu,\ z)=\sum_{k=0}^{r}\frac{\mu^k}{k!}\cdot\chi_{r-k}(\lambda,z)\cdot\mathcal{F}_k(z-r+1);$$

Shift of argument:

$$\chi_r(\lambda,\ z+1)=\chi_r(\lambda,z)+(\lambda+z)\cdot\chi_{r-1}(\lambda,z);$$

Recurrence relations:

$$\chi_r(\lambda,z)=r^{-1}\cdot\sum_{k=1}^{r}(-1)^{k+1}\cdot(k+1)^{-1}\cdot\chi_{r-k}(\lambda,z)\cdot\big(B_{k+1}(\lambda+z)-B_{k+1}(\lambda)\big);$$

$$\chi'_r(\lambda,z)=r^{-1}\cdot\sum_{k=1}^{r}(k+1)^{-1}\cdot\chi'_{r-k}(\lambda,z)\cdot\big(B_{k+1}(\lambda+z)-B_{k+1}(\lambda)\big),$$

where

$$\chi'_r(\mu,z)\equiv\chi_r(1-\mu,-z).$$

Values of the polynomials $\chi_r(\lambda,z)$ at integer values z:

$$\chi_r(\lambda,\ N)=\sum_{k=0}^{\min\{r,N\}}C_N^k\cdot s_{N-k}^{N-r}\cdot\mathcal{F}_k(\lambda)\qquad(N\ge r);$$

$$\chi_r(\lambda,\ N)=\sum_{k=0}^{\min\{r,N\}}C_{N-k}^{r-k}\cdot s_N^{N-k}\cdot\lambda^{r-k}\qquad(N\ge 0);$$

$$\chi_r(\lambda,\ N)=0\qquad 0\le N<r;$$

$$\chi_r(\lambda,r)=\mathcal{F}_r(\lambda);\qquad \chi_{r-1}(\lambda,r)=\mathrm{d}_\lambda\mathcal{F}_r(\lambda);$$

$$\chi_r(0,N)=s_N^{N-m};\qquad \chi_r(0,r+1)=r!.$$

Other closed forms: these relations directly follow from the formulas given in the previous item:

$$\chi_r(\lambda,\ z)=\sum_{k=r}^{2r}\sum_{L=0}^{r}\sum_{j=0}^{k}\frac{(-1)^{k-j}\,C_j^L}{j!(k-j)!}\cdot s_{j-L}^{j-r}\cdot\mathcal{F}_L(\lambda)\cdot\widetilde{\mathcal{F}}_k(z);$$

$$\chi_r(\lambda,\ z)=\sum_{k=r}^{2r}\sum_{L=0}^{r}\sum_{j=0}^{k}\frac{(-1)^{k-j}\,C_{j-L}^{r-L}}{j!(k-j)!}\cdot s_j^{j-L}\cdot\lambda^{r-L}\cdot\widetilde{\mathcal{F}}_k(z);$$

$$\chi_r(0,z) = \sum_{k=0}^{r}\sum_{j=0}^{k} \frac{(-1)^{k-j}}{(r+j)!(k-j)!} \cdot s_{r+j}^{j} \cdot \widetilde{\mathcal{F}}_{r+k}(z).$$

3 Main Properties of Functions $U_{jk}(\lambda)$, $V_{jk}(\lambda)$, $W_{jk}(\lambda)$ and $\Upsilon_{jk}(...)$

Closed forms for the functions $U_{rk}(\lambda)$, $V_{jk}(\lambda)$ and $\Upsilon_{jk}(...)$:

$$U_{rk}(\lambda) = \sum_{L=0}^{r}\sum_{j=0}^{k} \frac{(-1)^{k-j}\, C_j^L}{j!\,(k-j)!} \cdot s_{j-L}^{j-r} \cdot \mathcal{F}_L(\lambda);$$

$$U_{rk}(\lambda) = \sum_{L=0}^{r}\sum_{j=0}^{k} \frac{(-1)^{k-j}\, C_{j-L}^{r-L}}{j!\,(k-j)!} \cdot s_j^{j-L} \cdot \lambda^{r-L};$$

$$V_{rk}(\mu) = \sum_{j=\max\{1,k\}}^{2r} U_{rj}(1-\mu) \cdot (-1)^j C_{j-1}^{k-1} \cdot j!/k!.$$

$$\Upsilon_{rm}(\alpha,\gamma,\lambda,\mu) = \sum_{l=0}^{r} \frac{(-1)^l}{\alpha^{r-l}\,\gamma^l} \cdot \Upsilon'_{rlm}(\lambda,\mu),$$

where

$$\Upsilon'_{rlm}(\lambda,\mu) = \sum_{k=\max\{0,\,l-1\}}^{2l} \sum_{j=0}^{k} (-1)^k\, C_{m+j}^{j}\, \frac{k!}{(k-j)!}\, U_{r-l,m+j-k}(\lambda) \cdot \\ \cdot\Big(U_{lk}(1-\mu) + (k+1)\, U_{l,k+1}(1-\mu)\Big).$$

Closed forms for the functions $W_{kj}(\mu)$: The defining relation for these functions is given in the Introduction. Besides, at $1 \le k \le r$,

$$W_{rk}(\mu) = \frac{1}{k!} \sum_{l=0}^{r-k} \sum_{j=0}^{r-k} (-1)^{r+k+j} \, C_{2r-k}^{r-k-j} \, C_{r+j-1}^{k-1} \, C_{r-k+j}^{l} \, s_{r-k-l+j}^{j} \, \mathcal{F}_l(1-\mu);$$

$$W_{rk}(\mu) = \frac{1}{k!} \sum_{l=0}^{r-k} \sum_{j=0}^{r-k} (-1)^{r+k+j} \, C_{2r-k}^{r-k-j} \, C_{r+j-1}^{k-1} \, C_{l+j}^{j} \, s_{r-k+j}^{l+j} \, (1-\mu)^l;$$

$$W_{rk}(\mu) = \frac{1}{(k-1)!} \sum_{l=0}^{r-k} \sum_{j=0}^{r-k-l} \frac{(-1)^{r+k+j}}{r-l+j} \, C_{2r-2k-2l}^{r-k-l+j} \, C_{2r-2l-k}^{k} \, C_{r-1}^{l} \, s_{r-k-l+j}^{j} \, (1-\mu)^l;$$

$$W_{rk}(\mu) = \frac{1}{k!} \Big((-1)^{r-k} \, C_{r-1}^{k-1} \, (1-\mu)^{r-k} + \sum_{l=0}^{r-2k} (-1)^l \, \sigma_{r-l}^{k} \, C_{r-1}^{l} \, (1-\mu)^l;$$

$$+ \sum_{l=1}^{\min\{r-k,\,k-1\}} \sum_{j=1}^{l} (-1)^{r-k-j} \, C_{r-1}^{r-k-l} \, C_{k+l}^{k-j} \, C_{k-j-1}^{l-j} \, \sigma_{l+j}^{j} \, (1-\mu)^{r-k-l} \Big).$$

Recurrence relations:

$$U_{rk}(\lambda) = \frac{1}{k} \, (\lambda + k - 1) \, U_{r-1,k-1}(\lambda) + \frac{1}{k} \, U_{r-1,k-2}(\lambda);$$

$$V_{rk}(\mu) = (\mu + k - 1) \, V_{r-1,k}(\mu) + \frac{1}{k} \, (\lambda + 2k - 2) \, V_{r-1,k-1}(\mu) + \frac{1}{k} \, V_{r-1,k-2}(\mu);$$

$$W_{rk}(\mu) = (\mu + k - 1) \cdot W_{r-1,k}(\mu) + \frac{1}{k} \cdot W_{r-1,k-1}(\mu);$$

For the functions $\upsilon_{rk} = \Upsilon_{rk}(\alpha, \gamma, \lambda, \mu)$, at fixed values of the parameters α, γ, λ and μ, the following formula takes place:

$$\upsilon_{rk} = -\gamma^{-1} \, (\mu + k - 1) \, \upsilon_{r-1,k} +$$

$$+ \frac{1}{k} \Big(\alpha^{-1} \, (\lambda + k - 1) - \gamma^{-1} \, (\mu + 2k - 2) \Big) \cdot \upsilon_{r-1,k-1} + \frac{1}{k} \left(\alpha^{-1} - \gamma^{-1} \right) \cdot \upsilon_{r-1,k-2}.$$

Functional relations:

$$\Upsilon_{rk}(1, 1, \mu - \lambda, \mu) = (-1)^r \, W_{rk}(\mu) \, \mathcal{F}_k(\lambda); \tag{8}$$

$$\Upsilon_{r,2k}(1, 2, \ \lambda, \ 2\lambda) = (-1)^{r+k} \, 4^{-k} \, W_{rk}(\lambda + 1/2); \tag{9}$$

$$\Upsilon_{r,2k+1}(1, 2, \ \lambda, \ 2\lambda) = 0; \tag{10}$$

$$V_{rl}(\mu) = \sum_{j=0}^{l} \frac{(-1)^{l-j}}{(l-j)!} \, W_{r+j,j}(\mu); \qquad W_{rl}(\mu) = \sum_{j=0}^{\min\{l,2r-2l\}} \frac{1}{(l-j)!} \, V_{r-l,j}(\mu).$$

Table of the first functions $u_{kj} = U_{kj}(\lambda)$:

$u_{00} = 1;$

$u_{11} = \lambda;$ $\quad u_{12} = \frac{1}{2};$

$u_{22} = \frac{1}{2}\,\lambda(\lambda + 1);$ $\quad u_{23} = \frac{1}{6}\,(2 + 3\lambda);$ $\quad u_{24} = \frac{1}{8};$

$u_{33} = \frac{1}{6}\,\lambda(\lambda+1)(\lambda+2);$ $\quad u_{34} = \frac{2}{24}\,(3 + 7\lambda + 3\lambda^2);$ $\quad u_{35} = \frac{1}{120}\,(20 + 15\lambda);$ $\quad u_{36} = \frac{1}{48};$

Table of the first functions $v_{kj} = V_{kj}(\mu)$:

$v_{00} = 1;$

$v_{10} = 0;$ $\quad v_{11} = \mu;$ $\quad v_{12} = \frac{1}{2};$

$v_{20} = 0;$ $\quad v_{21} = \mu^2;$ $\quad v_{22} = \frac{1}{2}\,(1 + 3\mu + \mu^2);$ $\quad v_{23} = \frac{1}{6}\,(4 + 3\mu);$ $\quad v_{24} = \frac{1}{8};$

Table of the first functions $w_{kj} = W_{kj}(\mu)$:

$w_{00} = 1;$

$w_{10} = 0;$ $\quad w_{11} = 1;$

$w_{20} = 0;$ $\quad w_{21} = \mu;$ $\quad w_{22} = 1/2;$

$w_{30} = 0;$ $\quad w_{31} = \mu^2;$ $\quad w_{32} = \mu + 1/2;$ $\quad w_{33} = 1/6;$

$w_{40} = 0;$ $\quad w_{41} = \mu^3;$ $\quad w_{42} = \frac{1}{2}\,(3\mu^2 + 3\mu + 1);$ $\quad w_{43} = \frac{1}{2}\,(\mu + 1);$ $\quad w_{44} = \frac{1}{24}.$

Table of the first functions $\upsilon_{kj} = \Upsilon_{kj}(\mu)$:

$\upsilon_{00} = 1;$

$\upsilon_{10} = 0;$ $\quad \upsilon_{11} = \alpha^{-1}\,\lambda + (-\gamma)^{-1}\,\mu;$ $\quad \upsilon_{12} = \frac{1}{2}\,(\alpha^{-1} - \gamma^{-1});$

4 Asymptotical Expansion of Hypergeometric Functions Satisfying the Second-Order Differential Equations

In view of practical importance, in this item the above-stated formulas of asymptotical expansion for hypergeometric functions satisfying the second-order differential equations are specially considered. Most of the formulas given below allow approximating the hypergeometric functions by elementary functions.

Expansion of the function ${}_0F_1(...)$*:*

$$ {}_0F_1(\mu+\gamma/\eta,\, z/\eta) \sim e^{z/\gamma}\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{2k} V_{kj}(\mu)\cdot\left(\frac{z}{\gamma}\right)^j\cdot\left(\frac{-\eta}{\gamma}\right)^k ; $$

$$ {}_0F_1(\mu+1/\eta,\, z) \sim \sum_{k=0}^{\infty}\sum_{j=0}^{k} W_{kj}(\mu)\cdot(-z)^j\cdot(-\eta)^k . $$

Note: the last of the given relations is a particular case of formula (7); but it can also be derived from the formula for asymptotical expansion of the function ${}_1F_1(\lambda+\alpha/\eta,\, \mu+\gamma/\eta,\, z)$ with consideration for formulas (9) and (10) and the formula of transformation:

$$ {}_0F_1\big(\gamma+1/2,\, z^2/4\big) = e^{-z}\cdot{}_1F_1(\gamma, 2\gamma, 2z) \qquad (2\gamma\neq 0,-1,-2,...). $$

Expansion of the function ${}_2F_0(...)$*:*

$$ {}_2F_0(\tau,\, \lambda+\alpha/\eta,\, \eta z) \sim (1-\alpha z)^{-\tau}\cdot\sum_{k=0}^{\infty}\sum_{j=k}^{2k} U_{kj}(\lambda)\cdot\mathcal{F}_j(\tau)\cdot\left(\frac{\alpha z}{1-\alpha z}\right)^j\cdot\left(\frac{\eta}{\alpha}\right)^k . $$

Expansion of the function ${}_1F_1(...)$*:*

$$ {}_1F_1(\lambda+\alpha/\eta,\, \gamma,\, \eta z) \sim \sum_{k=0}^{\infty}\sum_{j=k}^{2k} U_{kj}(\lambda)\cdot z^j\, \mathrm{d}_z^j\, {}_0F_1(\gamma,\, \alpha z)\cdot\left(\frac{\eta}{\alpha}\right)^k ; $$

$$ {}_1F_1(\alpha,\, \mu+\gamma/\eta,\, z/\eta) = $$
$$ \sim (1-z/\gamma)^{-\alpha}\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{2k} V_{kj}(\mu)\cdot\mathcal{F}_j(\alpha)\cdot\left(\frac{z/\gamma}{1-z/\gamma}\right)^j\cdot\left(\frac{-\eta}{\gamma}\right)^k ; $$

$$ {}_1F_1(\lambda+\alpha/\eta,\, \mu+\gamma/\eta,\, z) \sim e^{z\alpha/\gamma}\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{2k} \Upsilon_{kj}(\alpha,\gamma,\lambda,\mu)\cdot\left(\frac{z\alpha}{\gamma}\right)^j\cdot\eta^k ; $$

$$ {}_1F_1(\mu-\lambda+1/\eta,\, \mu+1/\eta,\, z) = e^z\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{k} W_{kj}(\mu)\cdot\mathcal{F}_j(\lambda)\cdot z^j\cdot(-\eta)^k ; $$

$$ {}_1F_1(\alpha,\, \mu+1/\eta,\, z) \sim \sum_{k=0}^{\infty}\sum_{j=0}^{k} W_{kj}(\mu)\cdot\mathcal{F}_j(\alpha)\cdot(-z)^j\cdot(-\eta)^k . $$

Note: the last of the given relations is a particular case of formula (7); but it can also be derived from the formula for asymptotical expansion of the function ${}_1F_1(\mu - \lambda + 1/\eta,\ \mu + 1/\eta,\ z)$ with consideration for formula (8) and the formula of transformation:

$$ {}_1F_1(\alpha, \gamma, z) = e^z \cdot {}_1F_1(\gamma - \alpha, \gamma, -z). $$

Expansion of the function ${}_2F_1(...)$:

$$ {}_2F_1(\lambda + \alpha/\eta,\ \tau,\ \gamma,\ \eta z) \sim \sum_{k=0}^{\infty} \sum_{j=k}^{2k} U_{kj}(\lambda) \cdot z^j \, \mathrm{d}_z^j \, {}_1F_1(\tau,\ \gamma,\ \alpha z) \cdot \left(\frac{\eta}{\alpha}\right)^k; $$

$$ {}_2F_1(\alpha_1,\ \alpha_2,\ \mu + \gamma/\eta,\ z/\eta) \sim \sum_{k=0}^{\infty} \sum_{j=0}^{2k} V_{kj}(\mu) \cdot z^j \, \mathrm{d}_z^j \, {}_2F_0(\alpha_1,\ \alpha_2,\ z/\gamma) \cdot \left(\frac{-\eta}{\gamma}\right)^k; $$

$$ {}_2F_1(\tau,\ \lambda + \alpha/\eta,\ \mu + \gamma/\eta,\ z) \sim $$

$$ \sim (1 - z\alpha/\gamma)^{-\tau} \cdot \sum_{k=0}^{\infty} \sum_{j=0}^{2k} \Upsilon_{kj}(\alpha, \gamma, \lambda, \mu) \cdot \mathcal{F}_j(\tau) \cdot \left(\frac{z\alpha/\gamma}{1 - z\alpha/\gamma}\right)^j \cdot \eta^k; $$

$$ {}_2F_1(\tau,\ \mu - \lambda + 1/\eta,\ \mu + 1/\eta,\ z) \sim $$

$$ \sim (1 - z)^{-\tau} \cdot \sum_{k=0}^{\infty} \sum_{j=0}^{k} W_{kj}(\mu) \cdot \mathcal{F}_j(\lambda) \cdot \mathcal{F}_j(\tau) \cdot \left(\frac{z}{1 - z}\right)^j \cdot (-\eta)^k; $$

$$ {}_2F_1(\alpha_1,\ \alpha_2,\ \mu + 1/\eta,\ z) \sim \sum_{k=0}^{\infty} \sum_{j=0}^{k} W_{kj}(\mu) \cdot \mathcal{F}_j(\alpha_1)\, \mathcal{F}_j(\alpha_2) \cdot (-z)^j \cdot (-\eta)^k. $$

Note: the last of the given relations is a particular case of formula (7); but it can also be derived from the formula for asymptotical expansion of the function ${}_2F_1(\tau,\ \mu - \lambda + 1/\eta,\ \mu + 1/\eta,\ z)$ with consideration for formula (8) and the formula of transformation:

$$ {}_2F_1(\alpha_1, \alpha_2;\ \gamma;\ z) = (1 - z)^{-\alpha_1} \cdot {}_2F_1\big(\alpha_1, \gamma - \alpha_2;\ \gamma;\ z/(z - 1)\big). $$

5 Control Examples

At drawing up the control examples intended for checking the correctness of the obtained results, it is expedient to consider such values of parameters of hypergeometric functions at which these functions are expressed in terms of elementary functions. For example, the function ${}_0F_1(\gamma, z)$ where parameter γ is equal to a semi-integer number is represented in the form of combination of trigonometrical or hyperbolic functions; the functions ${}_1F_1(...)$, ${}_2F_0(...)$ and ${}_2F_1(...)$ where the upper parameter (or one of the upper parameters) is equal to a negative integer are represented in the form of polynomials.

The examples, in each of which the formula of asymptotical expansion for some function $\Phi(\varepsilon, x)$ is checked, have been considered. In the table below, the expressions for tested functions $\Phi(\varepsilon, x)$, the values of the parameters for these functions and the intervals of change in argument $[x_{min}, x_{max}]$ are presented. The values of functions were calculated for 20 equidistant values $x = x_j$ from the interval $[x_{min}, x_{max}]$.

N	$\Phi(\varepsilon, x)$	$[x_{min}, x_{max}]$	Values of parameters
1	${}_0F_1(\mu + \gamma/\varepsilon,\ x/\varepsilon)$	[1.5, 2.5]	$\gamma = 1.0$; $\mu = 0.5$; $\varepsilon = 0.06667$
2	${}_0F_1(\mu + \gamma/\varepsilon,\ x)$	[0.4, 3.4]	$\gamma = 1.0$; $\mu = 2.8$; $\varepsilon = 0.07874$
3	${}_2F_0(\lambda + \alpha/\varepsilon,\ \tau,\ \varepsilon x)$	[−13.7, −1.3]	$\tau = -7$; $\alpha = -6.7$; $\lambda = 2.3$; $\varepsilon = 0.05$
4	——”——	[0.3, 2.96]	$\tau = 3.7$; $\lambda = 0$; $\alpha = -1$; $\varepsilon = 1/15$
5	${}_1F_1(\alpha,\ \mu + \gamma/\varepsilon,\ x/\varepsilon)$	[1.3, 13.7]	$\alpha = -7$; $\gamma = -6.7$; $\mu = 2.3$; $\varepsilon = 0.05$
6	${}_1F_1(\lambda + \alpha/\varepsilon,\ \mu + \gamma/\varepsilon,\ x)$	[1.3, 13.7]	$\alpha = -1$; $\gamma = -2.9$; $\lambda = 0$; $\mu = 3.7$; $\varepsilon = 0.06667$
7	${}_1F_1(\alpha,\ \mu + \gamma/\varepsilon,\ x)$	[1.3, 13.7]	$\alpha = -7$; $\gamma = -6.7$; $\mu = 2.3$; $\varepsilon = 0.05$
8	${}_2F_1(\lambda + \alpha/\varepsilon,\ \tau,\ \mu + \gamma/\varepsilon,\ x)$	[0.3, 3.7]	$\tau = -7$; $\alpha = -6.7$; $\gamma = 5.4$; $\lambda = 2.3$; $\mu = -3.1$; $\varepsilon = 0.05$
9	${}_2F_1(\alpha_1,\ \alpha_2,\ \mu + \gamma/\varepsilon,\ x)$	[0.3, 3.7]	$\alpha_1 = -7$; $\alpha_2 = -6.7$; $\gamma = 5.4$; $\mu = -3.1$; $\varepsilon = 0.05$

Let's designate by $M + 1$ the number of terms considered in the used asymptotical expansion, i.e. let us suppose that, in this expansion, there are considered the terms of the order not greater than ε^M. For each of tested functions, for first several values M beginning from $M = 2$, the tables describing the accuracy of approximation of the function $\Phi(\varepsilon, x)$ by asymptotical expansion were drawn up. Columns of each such table contain the following data: the number of row j, the value of argument x_j, the corresponding exact value of function y_j, the absolute deviation Δy_j of the approximate value of the function calculated by the asymptotical formula from its exact value and the relative deviation $\Delta y_j / y_j$.

As a whole, the results of calculation can be characterized as follows. In each example, except for example 4, quite satisfactory results were obtained. Values $|\Delta y_j / y_j|$ are less than unit and contain a several zeros after the decimal point before significant digits, and the accuracy improves by an order or two at each increase in number M by unit. In example 4 the asymptotical expansion gives satisfactory results for small values $|x|$; at increasing x, from some value (equal to about 1.6), the calculated values of the function start to differ sharply from the exact values.

6 Final Remarks

The purpose of this work was the derivation of asymptotical formulas for hypergeometric functions which generalize and render more precisely the known limit formulas. But most attention is given to studying the properties of auxiliary functions used in the expansion. In this connection, the estimations of remainders of asymptotical expansions which would establish the conditions and the limits of applicability of the derived formulas have not been given in this work; it is a subject of the further research.

It is obvious that the conditions of applicability of formulas of degeneration for hypergeometric functions are the necessary conditions for applicability of the asymptotical expansions generalizing the given formulas. At the same time, it should be noted that conditions $n \geq m$ and $m = n + 1$; $|z| < 1$; $\mathrm{Re}\,(1/\eta) \geq 0$ given in [3] which should guarantee the applicability of formulas of degeneration for functions ${}_{m+1}F_n(\lambda + \alpha_0/\eta,\, \alpha_1, ..., \alpha_m;\, \gamma_1, ..., \gamma_n;\, \eta z)$ and ${}_mF_{n+1}(\alpha_1, ..., \alpha_m;\, \mu{+}\gamma_0/\eta, \gamma_1, ..., \gamma_n;\, z/\eta)$, sometimes prove to be excessive, which some of the control examples considered in this work point to.

In work [4], it is shown that the formulas of asymptotical expansions of the functions ${}_2F_1(\lambda{+}1/\eta,\, \tau,\, \mu{-}1/\eta,\, z)$ and ${}_2F_1(\alpha_1,\, \alpha_2,\, \mu{+}1/\eta,\, z)$ can be generalized on the cases when z belongs to some subregions of the exterior of the circle $|z| > 1$ if $\arg(1/\eta) < 2\pi/4$. Apparently, similar generalizations can be also obtained for other hypergeometric functions.

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