ON VARIATIONAL FORMULATION OF SOME NONLOCAL BOUNDARY VALUE PROBLEMS BY SYMMETRIC CONTINUATION OPERATION OF A FUNCTION

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Abstract

Some nonlocal boundary value problems are considered. The variational formulation using the introduction of the scalar product by symmetric continuation operation of a function is studied.

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It is well known that symmetry of the operators related to many classical boundary value problems can be realized by means of the Green formula [1]. In the case of nonlocal boundary value problems [2], [3], direct application of the Green's formula does not allow us to get an operator symmetry and, all the more, a positive definiteness [1]. Such a circumstance poses problems for the variational formulation of the problems of the mentioned type (minimization problem of the quadratic functional in the energetic space). Therefore, any approach which permits a variational formulation of nonlocal boundary value problems is of current importance. Our approach is connected with the introduction of the scalar product by means of symmetric continuation operation, and with the proof of the positive definiteness of operators of some nonlocal boundary value problems on the special lineals [4], [5], [6].

In the present article the following nonlocal boundary value problem is considered. Let us find a function $u(x) \in C^{(2)}\ ]-a,\, 0[\, \cap C[-a,\, 0]$ for which

$$-(k(x)u'(x))' + (Bu)(x) = f(x), \quad x \in]-a,\, 0[, \tag{1}$$

$$u(-a) = 0, \tag{2}$$

$$\int_{-\xi}^{0} k(x)u'(x)dx = 0, \tag{3}$$

where $\xi\in]0,a[$ is a fixed point, $f(x)\in C[-a,0]$, $k(x)\in C^{(1)}[-a,0]$, $k(x)\geq k_0>0$, B is a linear operator which acts from $C[-a,0]$ into $C[-a,0]$ and satisfies the condition

$$(Bv)(0)=0,\ \ \forall v(x)\in C[-a,0]. \tag{4}$$

Note, that when the function $k(x)$ is constant, expression (3) presents Bitsadze-Samarskii nonlocal boundary condition [2].

The aim of the paper is to realize variational formulation of problem (1)-(3) concerning an operator B in the case of satisfaction of certain conditions.

We fix some notations and definitions which will be used throughout the paper. We denote by $D[-a,0]$ a lineal of all real functions such that each of its functions $v(x)$ be defined a.e. on $[-a,0]$, $|v(0)|<+\infty$ and $v(x)\in L_2[-a,0]$.

Note that to give the function $u(x)\in D[-a,0]$ actually means to designate a couple $(v(x),v(0))$ ($x\in[-a,0[$). Functions $v_1(x)$ and $v_2(x)$ are the same elements of the lineal $D[-a,0]$ if $v_1(x)=v_2(x)$ a.e. on $[-a,0[$ and $v_1(0)=v_2(0)$.

On the lineal $D[-a,0]$ define symmetric continuous operator τ by the following way:

$$\tau v(x)=\begin{cases} v(x), & \text{when} \quad x\in[-a,0], \\ -v(-x)+2v(0), & \text{when} \quad x\in]0,\xi]. \end{cases} \tag{5}$$

By the operator τ function $\tilde{v}(x)=\tau v(x)$ defined a.e. on the $[-a,\xi]$, is corresponded to each function $v(x)$ and function $\tilde{v}(x)-v(0)$ is odd a.e. on the $[-\xi,\xi]$.

Let us define a scalar product on lineal $D[-a,0]$.

$$[u,v]=\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{u}(s)\tilde{v}(s)dsdx. \tag{6}$$

By the scalar product (6) the lineal $D[-a,0]$ becomes pre-Hilbert space which we denote via $H[-a,0]$. For the norm produced by scalar product (6) we use the notation $||\cdot||_H$:

$$||v||_H^2=\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{v}^2(s)dsdx.$$

Theorem 1. The norm defined on the lineal $D[-a,0]$ by the equality

$$||v||^2=||v||_{L_2}^2+v^2(0), \quad ||v||_{L_2}^2=\int_{-a}^{0}v^2(x)dx), \tag{7}$$

is equivalent to the norm $||\cdot||$.

Proof. By simple transformations we get

$$\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{v}^2(s)dsdx = 2\xi\int_{-a}^{0}v^2(x)dx + 2\xi^2v^2(0) - 4v(0)\int_{-\xi}^{0}(\xi+x)v(x)dx. \quad (8)$$

For any $\varepsilon > 0$

$$|2v(0)(\xi+x)v(x)| \le \frac{v^2(0)(\xi+x)^2}{\varepsilon\xi} + \varepsilon\xi v^2(x).$$

Integrating the last inequality we get

$$\left|2v(0)\int_{-\xi}^{0}(\xi+x)v(x)dx\right| \le \frac{\xi^2}{3\varepsilon}v^2(0) + \varepsilon\xi\int_{-\xi}^{0}v^2(x)dx. \quad (9)$$

If take into account (8) and (9) we have

$$\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{v}^2(s)dsdx \ge 2\xi(1-\varepsilon)\int_{-a}^{0}v^2(x)dx + 2\xi^2\left(1-\frac{1}{3\varepsilon}\right)v^2(0). \quad (10)$$

Let $\varepsilon \in \left]\frac{1}{3},\ 1\right]$, then, it is evident from (10) that there exists a constant $m > 0$ such that

$$\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{v}^2(s)dsdx \ge m||v||^2\ . \quad (11)$$

From (8) and (9) it is easy to see that there exists a number $M > 0$ such that

$$\int_{-\xi}^{\xi}\int_{-a}^{x}\tilde{v}^2(s)dsdx \le M||v||^2. \quad (12)$$

Inequalities (11) and (12) prove the validity of the Theorem 1.

Corollary. $H[-a,0]$ is a Hilbert space.

Below the operator B besides condition (4) satisfies the positiveness conditions:

$$[Bu,\ u] \ge 0, \quad [Bu,v] = [u,Bv], \quad \forall\ u,v \in C[-a,\ 0]. \quad (13)$$

Let $D_A[-a,\ 0]$ lineal of the functions of the space $H[-a,\ 0]$ be a domain of definition of the operator $Au = -(ku')' + Bu$. For each function $v(x)$ of the lineal $D_A[-a,\ 0]$ the following conditions are fulfilled.

$$v(x) \in C^{(2)}[-a,\ 0],\ v(-a) = 0,\ v^{(i)}(0) = 0,\ \ i = 1,\ 2, \quad (14)$$

$$\int_{-\xi}^{0} k(x)u'(x)dx = 0. \tag{15}$$

It is easy to show that the following is valid.

Theorem 2. The lineal $D_A[-a,\,0]$ is dense in $H[-a,\,0]$.

Thus, an operator A acts from the lineal $D_A[-a,\,0]$ into the $H[-a,\,0]$.

Theorem 3. The operator A is symmetric on the lineal $D_A[-a,\,0]$.

Proof. Suppose that

$$\tau\left(k(x)u'(x)\right)' = \overline{k}(x)(\tilde{u}'(x))', \tag{16}$$

where

$$\overline{k}(x) = \begin{cases} k(x), & \text{when } x \in [-a,\,0], \\ k(-x), & \text{when } x \in [0,\,\xi]\,. \end{cases} \tag{17}$$

Indeed, when $x \in]0,\,\xi]$ we have

$$\tau(k(x)u'(x))' = -(ku')'(-x) + 2(ku')'(0) = -(ku')'(-x) = ((ku')'(-x))' =$$

$$= (k(-x)u'(-x))' = (k(-x)(-u(-x) + 2u(0))')' = \overline{k}(x)\tilde{u}'((x))'.$$

Taking into account (16) we have

$$\int_{-\xi}^{\xi}\int_{-a}^{x} \tau(k(s)u'(s))'\;\tilde{v}(s)dsdx = \int_{-\xi}^{\xi}\int_{-a}^{x} (\overline{k}(s)\tilde{u}'(s))'\tilde{v}(s)dsdx =$$

$$= \int_{-\xi}^{\xi} [\overline{k}\tilde{u}'\tilde{v}|_{-a}^{x} - \int_{-a}^{x} \overline{k}(s)\,\tilde{u}'(s)\,\tilde{v}'(s)ds]dx = \int_{-\xi}^{\xi} \overline{k}(x)\,\tilde{u}'(x)\,\tilde{v}'(x)dx-$$

$$-\int_{-\xi}^{\xi}\int_{-a}^{x} \overline{k}(s)\tilde{u}'(s)\tilde{v}'(s)dsdx = v(0)\int_{-\xi}^{\xi} \overline{k}(x)\,\tilde{u}'(x)\,dx - \int_{-\xi}^{\xi}\int_{-a}^{x} \overline{k}(s)\tilde{u}'(s)\tilde{v}'(s)dsdx =$$

$$= 2v(0)\int_{-\xi}^{0} k(x)u'(x)dx - \int_{-\xi}^{\xi}\int_{-a}^{x} \overline{k}(s)\tilde{u}'(s)\tilde{v}'(s)dsdx = -\int_{-\xi}^{\xi}\int_{-a}^{x} \overline{k}(s)\tilde{u}'(s)\tilde{v}'(s)dsdx,$$

which, considering the symmetry of the operator B, proves the symmetry of the operator A on the lineal D_A.

Theorem 4. The operator A is positively defined on the $D_A[-a,\,0]$.

Proof. Preliminarily, let us prove Poincare-type inequality. Let $x \in [-\xi,\,\xi]$ and $s \in [-a, x]$.

We have

$$\tilde{u}^2(s) = (\int_{-a}^{s} \tilde{u}'(t)dt)^2 \le \int_{-a}^{s} dt \int_{-a}^{s} \tilde{u}'^2(t)dt \le (s+a) \int_{-a}^{x} \tilde{u}'^2(t)dt,$$

from which

$$\int_{-a}^{x} \tilde{u}^2(s)ds \le \frac{(s+a)^2}{2}\Big|_{-a}^{x} \int_{-a}^{x} \tilde{u}'^2(s)ds \le \frac{(\xi+a)^2}{2} \int_{-a}^{x} \tilde{u}'^2(s)ds.$$

By integrating we get

$$\int_{-\xi}^{\xi} \int_{-a}^{x} \tilde{u}^2(s)dsdx \le \frac{(\xi+a)^2}{2} \int_{-\xi}^{\xi} \int_{-a}^{x} \tilde{u}'^2(s)dsdx. \tag{18}$$

Considering (18) we obtain

$$[Au,u] = \int_{-\xi}^{\xi} \int_{-a}^{x} \overline{k}(s)(\tilde{u}')^2(s)dsdx + [Bu,u] \ge \frac{2k_0}{(\xi+a)^2}[u,u], \tag{19}$$

that proves positively definiteness of the symmetric operator A.

Thus, A is an operator defined positively on the dense lineal $D_A[-a,\, 0]$ in the Hilbert space $H[-a,0]$. Follow the standard way [1]. Onto the lineal $D_A[-a,\, 0]$, let us introduce a new scalar product

$$[u,v]_A = [Au,v] = \int_{-\xi}^{\xi} \int_{-a}^{x} \overline{k}(s)\tilde{u}'^2(s)\tilde{v}'(s)dsdx + [Bu,v]. \tag{20}$$

For the corresponding norm we use notation $||\cdot||_A$.

$$||u||_A^2 = \int_{-\xi}^{\xi} \int_{-a}^{x} \overline{k}(s)(\tilde{u}')^2(s)dsdx + [Bu,u]. \tag{21}$$

By scalar product (20) the Lineal $D_A[-a,\, 0]$ is transformed into the pre-Hilbert space. Denote it via $S_A[-a,\, 0]$. Complete this space with the norm (21) which, as is easy to show, is equivalent to the norm defined by the equality

$$|||u|||^2 = ||u||^2_{W_2^1} + u^2(0). \tag{22}$$

Denote with $H_A[-a,\,0]$ a Hilbert space obtained as a result of completing. The space consists in those functions of $W_2^1[-a,\,0]$ space which satisfy the conditions

$$u(-a)=0,\quad \int\limits_{-\xi}^{0} k(x)u'(x)dx=0. \tag{23}$$

Let $\alpha\in R$. Consider a pair $(f(x),\,\alpha)$. It defines the unique function $f_\alpha(x)$ of the space $H[-a,\,0]$. For each such function, a functional

$$F_\alpha(v)=[v,v]_A-2[f_\alpha,v] \tag{24}$$

has unique minimizing function $u_\alpha(x)\in H_A[-a,\,0]$ which satisfies the relation

$$[u_\alpha,v]_A=[f_\alpha,v] \tag{25}$$

for $\forall v(x)\in H_A[-a,0]$.

As is easy to see,

$$u_\alpha(x)=\;u_0(x)+\alpha\omega(x), \tag{26}$$

where $\omega(x)$ is a minimizing function of the functional (24) in that case when the first term of the pair $(f(x),\,\alpha)$ is identical to zero function on the $[-a,\,0[$, and $\alpha=1$.

Theorem 5. Let $u(x)$ be a solution of problem (1)-(3). Then it is a minimizing function of the functional $F_0(v)$ in the space $H[-a,\,0]$.

Proof. Making simple transformations we get

$$\begin{aligned}F_0(v)=2\xi\int\limits_{-a}^{0}((k(x)v')^2(x)-2f(x)v(x))dx+\\ +4v(0)\int\limits_{-\xi}^{0}(\xi+x)f(x)dx+\int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x}(\widetilde{Bv})(s)\tilde{v}(s)dsdx.\end{aligned} \tag{27}$$

If we consider that

$$\int\limits_{-\xi}^{\xi}\int\limits_{-a}^{x}\widetilde{Bu}\,\tilde{v}dsdx=\int\limits_{-\xi}^{\xi}x'\int\limits_{-a}^{x}\widetilde{Bu}\,\tilde{v}dsdx=x\int\limits_{-a}^{x}\widetilde{Bu}\,\tilde{v}ds|_{-\xi}^{\xi}-\int\limits_{-\xi}^{\xi}x\widetilde{Bu}\,\tilde{v}dx=\xi\int\limits_{-a}^{\xi}\widetilde{Bu}\,\tilde{v}dx+$$

$$+\xi\int\limits_{-a}^{-\xi}\widetilde{Bu}\,\tilde{v}dx-\int\limits_{-\xi}^{\xi}x\widetilde{Bu}\,\tilde{v}dx=2\xi\int\limits_{-a}^{-\xi}Buvdx+\xi\int\limits_{-\xi}^{\xi}\widetilde{Bu}\,\tilde{v}dx-\int\limits_{-\xi}^{\xi}x\widetilde{Bu}\,\tilde{v}dx=$$

$$= 2\xi \int_{-a}^{-\xi} Buvdx + \xi \int_{-\xi}^{\xi} \widetilde{Bu}\,\tilde{v}dx - \int_{-\xi}^{\xi} x\widetilde{Bu}\,\tilde{v}dx = 2\xi \int_{-a}^{-\xi} Buvdx + 2\xi \int_{-\xi}^{0} (Bu)(x)(v(x) - v(0))dx -$$

$$-2v(0) \int_{-\xi}^{0} xBudx = 2\xi \int_{-a}^{-\xi} Buvdx + 2\xi \int_{-\xi}^{0} Buvdx - 2\xi v(0) \int_{-\xi}^{0} Budx - 2v(0) \int_{-\xi}^{0} xBudx =$$

$$= 2\xi \int_{-a}^{0} Buvdx - 2v(0) \int_{-\xi}^{0} (\xi + x)Budx,$$

then a variation of a functional $F_0(v)$ on the solution of problem (1)-(3) has the form:

$$\delta F_0(u) = \frac{d}{d\varepsilon} F_0(u + \varepsilon v)|_{\varepsilon=0} = 2\xi \int_{-a}^{0} (2k(x)u'(x)v'(x) - 2f(x)v(x))dx +$$

$$+4v(0) \int_{-\xi}^{0} (\xi + x)f(x)dx + 2 \int_{-\xi}^{\xi} \int_{-a}^{x} (\widetilde{Bu})(s)\tilde{v}(s)dsdx = 4\xi \Big(k(x)u'(x)v(x)|_{-a}^{0} -$$

$$- \int_{-a}^{0} ((ku')' + f)vdx \Big) + 4v(0) \int_{-\xi}^{0} (\xi + x)fdx + 4\xi \int_{-a}^{0} Buvdx - 4v(0) \int_{-\xi}^{0} (\xi + x)Budx =$$

$$= 4\xi k(0)u'(0)v(0) + 4\xi \int_{-a}^{0} [-(ku')' + Bu - f]vdx + 4v(0) \int_{-\xi}^{0} (\xi + x)(f - Bu)dx =$$

$$= 4\xi k(0)u'(0)v(0) + 4v(0) \int_{-\xi}^{0} (\xi + x)(-ku')'dx = 4\xi k(0)u'(0)v(0) +$$

$$+4v(0)[-(\xi + x)ku'|_{-\xi}^{0} + \int_{-\xi}^{0} ku'dx] = 4\xi k(0)u'(0)v(0) - 4v(0)\xi k(0)u'(0) +$$

$$+4v(0) \int_{-\xi}^{0} ku'dx = 0.$$

Which proves the validity of the Theorem 5.

Note that conditions (4) and (13) are fulfilled, for example, when $(Bv)(x) = q(x)v(x)$, where $q(x) \in C[-a, 0]$, $q(x) \geq 0$, when $x \in [-a, -\xi]$ and $q(x) \equiv 0$, when $x \in [-\xi, 0]$.

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