SAINT-VENANT'S PROBLEMS FOR THREE-LAYERED CONCENTRIC ISOTROPIC CIRCULAR BEAMS

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Abstract

It is considered circular tube, composed by three different isotropic elastic materials. It is proposed that circular tubes inserted one of another and glued together along the length of borders surfaces. Of its kind three-layered concentric circular composed tube from a different isotropic elastic materials are considered Saint-Venant's problems.

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1 Basic Equations

Let us consider three-layered circular tube, composed by different isotropic elastic materials (obey Hook's low) occupying a composed circular cylindrical domain $\Omega = \Omega_1 + \Omega_2 + \Omega_3$. Consider a plane Ox_1x_2 of the cartesian coordinates $Ox_1x_2x_3$ of the "under" end of the cylindrical body, each domains Ω_1, Ω_2 and Ω_3 of an indicated tube are bounded by two planes $x_3 = 0, x_3 = l(l > 0)$ and by the the concentric circular cylindrical surfaces Γ_1, Γ_2, Γ_2, Γ_3 and Γ_3, Γ_4 respectively. It is proposed that the domains Ω_1,Ω_2 composed by the different elastic materials are glued together along the length of the borders of circular surfaces Γ_2 and Γ_3 (along the interfaces). By $\gamma_1, \gamma_2, \gamma_3$ and γ_4 we denote the lines obtained by cross-sections of the surfaces Γ_j by the plane parallel to the plane Ox_1x_2 .The radiuses of the circles (γ_j) are the circles R_1, R_2, R_3 and R_4 respectively. The equations of (γ_j) could be presented in the following form

$$(x_1)_{\gamma_j} = R_j \cos(\vartheta), (x_2)_{\gamma_j} = R_j \sin(\vartheta), \ (j = 1, 2, 3, 4; 0 \le \vartheta \le 2\pi)$$

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It is well-known that by N.Muskhelishvili [1] was stated three auxiliary problems for the plane deformation parallel to Ox_1x_2 for the continual components of displacement when the components of tension are discontinuous passing via of Ω_j to Ω_{j+1} from interfaces Γ_{j+1}.

In this paper these problems are stated for the three-layered domain.

Let us denote by $u(x_1,x_2)_{k(m)_j}$ and $\tau(x_1,x_2)_{k^{(m)})_j}$, $(k,l=1,2)$ the components of displacement and stresses of indicated auxiliary problems in the domains ω_j and consider the following three auxiliary planes deformation parallel to Ox_1x_2.

We consider the components of displacements and stresses independent from a variable x_3, the equations of static of elastic body and the following boundary conditions: on the inner and exterior surfaces $\Gamma_1(\gamma_1)$ and $\Gamma_4(\gamma_4)$ the following boundary conditions are satisfied

$$\tau_{nk}^{(m)} \equiv \tau_{n_1k}^{(m)} + \tau_{n_2k}^{(m)} = 0,\ (k=1,2; m=1,2,3), \tag{1.1}$$

and on the interfaces $\Gamma_2(\gamma_2)$ and $\Gamma_3(\gamma_3)$ are satisfied the following boundary-contact conditions:

$$[\tau_{nk}^{(m)}]_j - [\tau_{nk}^{(m)}]_{j+1} = 0, \tag{1.2}$$

$$[u_k^{(m)}]_j - [u_k^{(m)}]_{j+1} = [g_k^{(m)}]_j - [g^{(m)}]_{j+1}, \tag{1.3}$$

$$(k,j=1,2; m=1,2,3)$$

where $n(n_1,n_2)$ is exterior normal to $\Gamma_j(\gamma_j)$ and functions $g_k^{(m)}$ are given by the following equalities:

$$2g_1^{(1)} = -2g_2^{(2)} = \nu(x_1^2 - x_2^2),\ \ g_2^{(1)} = g_1^{(2)} = \nu x_1x_2,\ \ g_1^{(3)} = \nu x_1,\ g_2^{(3)} = \nu x_2,$$

ν is the Poison's ratio and the symbols $[\ ]_j$ and $[\ ]_{j+1}$ denote a limiting values on the interface $\Gamma_2 j(\gamma_j)$ of the expressions enclosed in the brackets taken from domains $\Omega_j(\omega_j)$ and $\Omega_{j+1}(\omega_{j+1})$ respectively.

Consider the generalized center of inertia and generalized principal axis of inertia. If the origin of cartesian coordinates $Ox_1x_2x_3$ coincide with the generalized center of inertia and axis Ox_1 and Ox_2 coincide with the generalized principal axis of inertia, then will be carried out the following relationship

$$J_{jk} + K_{jk} = 0,\ \ (j \neq k; j,k=1,2,3)$$

$$K_{jk} = K_{kj},\ \ J_{jj} + K_{jj} > 0,\ \ (j,k=1,2,3), \tag{1.4}$$

where

$$\tau_{33}^{(k)} = \nu(\tau_{11}^{(k)} + \tau_{22}^{(k)}),$$

$$J_{jk} = \int\int_\omega (Ex^{(j)}x^{(k)})d\omega,$$

$$K_{jk} = \int\int_{\omega} (x^{(j)}\tau_{33}^{(k)})d\omega, \tag{1.5}$$

$x^{(1)} = x_1, x^{(2)} = x_2, x^{(3)} = 1$, E is modulus of elasticity and $\omega = \omega_1 + \omega_2 + \omega_3$.

The equations of the elastic equilibrium for the components of displacements in each of domains Ω_j will be

$$\mu\Delta u_j + (\lambda + \mu)D_j\Theta = 0, \quad (j = 1, 2, 3), \tag{1.6}$$

where λ and μ are Lame's constants ,

$$\Delta = D_1^2 + D_2^2 + D_3^2. \ (D_j = \partial/\partial x_j)$$

We note that these constants and the components of displacements and stresses in every domain Ω_j will be different.

We admit that for the Saint-Venan's problems will be satisfied the following boundary-contact conditions

$$\tau_{nJ} \equiv \tau_{1j}n_1 + \tau_{2j}n_2 = 0$$

on Γ_1 and Γ_4,

$$[\tau_{nj}]_1 = [\tau_{nj}]_2,$$

$$[\tau_{nj}]_2 = [\tau_{nj}]_3, [u_j]_1 = [u_j]_2, \quad [u_j]_2 = [u_j]_3, \quad (j = 1, 2, 3). \tag{1.7}$$

Also the external forces applied to the "upper" base $z = l$ of composed body are statically equivalent to the bending transverse forces P_1 and P_2 and to external force P_3, bending couple-forces m_1 and m_2 and a torsion couple-forces m_3. The following conditions

$$\int\int_{\omega} \tau_{j3}d\omega = P_j, \quad (j = 1, 2, 3); \int\int_{\omega} (x_2\tau_{33} - x_3\tau_{23}) = m_1,$$

$$\int\int_{\omega} (x_3\tau_{13} - x_1\tau_{33}) = -m_2, \ \int\int_{\omega} (x_1\tau_{23} - x_2\tau_{13}) = m_3, \tag{1.8}$$

must be fulfilled (also in every cross-section $z = const$,where $0 < const \leq l$).

In the sequel will be used the well known formula [1]

$$\int\int_{\omega} \tau_{j3}d\omega = \int_{\gamma_1} x_j\tau_{n3}ds + \int_{\gamma_4} x_j\tau_{n3}ds +$$

$$+ \sum_{k=2,3} \int_{\gamma_j x_j} ([\tau_{n3}]_{k-1} - [\tau_{n3}]_k)ds + \int\int_{\omega} x_j D_3\tau_{33}d\omega, \quad (j = 1, 2),$$

where $\omega = \omega_1 + \omega_2 + \omega_3$.

By means of complex potentials $\varphi(z)$ and $\psi(z)$ we can express the components of displacements and stresses in the following form

$$2\mu(u_1+iu_2)=\kappa\varphi(z)-z\overline{\varphi'(z)}-\overline{\psi(z)},$$

$$\tau_{11}+\tau_{22}=2[\varphi'(z)+\overline{\varphi'(z)},$$

$$\tau_{22}-\tau_{11}+2i\tau_{12}=2[\overline{z}\varphi''(z)+\psi'(z)], \tag{1.9}$$

where $\varphi(z)$ and $\psi(z)$ are holomorphic functions of a complex variable $z=x_1+ix_2$, $(i^2=-1)$, the function $\overline{F(z)}$ is the conjugate complex to the function $F(z)$.

So three auxiliary problems of plane deformation (1)–(3) could be reduced to define in the domains ω_j complex potentials $\varphi_j^{(k)}(z)$ and $\psi_j^{(k)}(z)$ satisfying the following boundary conditions

$$[\varphi^{(k)}(t)_1+t\overline{\varphi'^{(k)}}(t)_1)+\overline{\psi^{(k)}(t)}_1]_1=C_1^{(k)}, \tag{1.10}$$

on γ_1,

$$[\varphi^{(k)}(t)_3+t\overline{\varphi'^{(k)}}(t)_3+\overline{\psi^{(k)}(t)_3}]_3=C_4^{(k)}, \tag{1.11}$$

on γ_4,

$$[\varphi^{(k)}(t)_1+t\overline{\varphi'^{(k)}(t)_1}+\overline{\psi^{(k)}}(t)_1]_1-[\varphi^{(k)}(t)_2+t\overline{\varphi'^{(k)}(t)_2}+\overline{\psi^{(k)}}(t)_2]_2=C_2^{(k)}, \tag{1.12}$$

on a circle-interface γ_2,

$$[\varphi^{(k)}(t)_2+t\overline{\varphi}'(k)(t)_2+\overline{\psi^{(k)}(t)_2}]_2-[\varphi^{(k)}(t)_3+t\overline{\varphi'^{(k)}(t)_3}+\overline{\psi^{(k)}(t)_3}_3]_3=C_3^{(k)}, \tag{1.13}$$

on a circle-interface γ_3,

$$[\alpha_1\varphi^{(k)}(t)_1-\beta_1 t\overline{\varphi'^{(k)}(t)_1}-\beta_1\overline{\psi^{(k)}(t)_1}]_1$$
$$-[\alpha_2\varphi^{(k)}(t)_2-\beta_2 t\overline{\varphi'^{(k)}(t)_2}-\beta_2\overline{\psi^{(k)}(t)_2}]_2=f_1^{(k)}(t), \tag{1.14}$$

on a circle-interface γ_2 ,

$$[\alpha_2\varphi^{(k)}(t)_2-\beta_2\overline{\varphi'^{(k)}(t)_2}-\beta_2\overline{\psi^{(k)}(t)_2}]_2$$
$$-[\alpha_3\varphi^{(k)}(t)_3-\beta_3\overline{\varphi'^{(k)}(t)_3}-\beta_3\overline{\psi^{(k)}}(t)_3]_3=f_2^{(k)}(t), \tag{1.15}$$

on a circle-interference γ_3, where $k=1,2,3$; $C_j^{(k)}$ are complex constants to be determined,

$$\alpha_j=\kappa_J(2\mu_j)^{-1}>0,\ \beta_j=(2\mu_j)^{-1}>0,$$

$$f_1^{(1)}(t) = (1/2)(\nu_1 - \nu_2)t^2, f_1^{(2)}(t) = (1/2)i(\nu_1 - \nu_2)t^2, f_1^{(3)}(t) = (\nu_1 - \nu_2)t,$$

$$f_2^{(1)}(t) = (1/2)(\nu_2 - \nu_3)t^2, f_2^{(2)}(t) = (1/2)i(\nu_2 - \nu_3)t^2, f_2^{(3)}(t) = (\nu_2 - \nu_3)t,$$

the variable t is an affix of point z on a circle $\gamma_j,\ (j = 1, 2, 3)$.

2 The solutions of an auxiliary problems

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a) *The solution of a first problem .*

In the first auxiliary problems the functions $f_j^{(2)}$ take the following values

$$f^{(2)}(t)_1 = 2^{-1}(\nu_1 - \nu_2)t^2,\ f^{(2)}(t)_2 = 2^{-1}(\nu_2 - \nu_3)t^2. \tag{2.1}$$

According to N.Muskhelishvili [1] the solution of the boundary-contact problem we seek in the form

$$\varphi_1^{(2)} = A_1^{(2)}z^2,\ \psi_1^{(2)} = A_{-1}^{(2)}z^{-2} + b_1^{(2)},$$

$$\varphi_2^{(2)} = A_2^{(2)}z^2, \psi_2^{(2)} = A_{-2}^{(2)}z^{-2} + b_2^{(2)},$$

$$\varphi_3^{(2)} = A_3^{(2)}z^2,\ \psi_3^{(2)} = A_{-3}^{(2)}z^{-2} + b_3^{(2)}, \tag{2.2}$$

where the constants $A_j^{(2)}, A_{-j}^{(2)}$ and $b_j^{(2)}$ will be determined.

Putting the expressions (2.2) into the equations (1.8)-(1.15) and take into the account the expression (2.1), we get

$$C_1^{(2)} = 2A_1^{(2)} + b_1^{(2)},\ C_2^{(2)} = 2(A_1^{(2)} - A_2^{(2)}) + b_1^{(2)} - b_2^{(2)},$$

$$C_3^{(2)} = 2(A_2^{(2)} - A_3^{(2)}) + b_2^{(2)} - b_3^{(2)},\ b_1^{(2)} = (\beta_1)^{-1}\beta_2 b^{(2)},\ b_2^{(2)} = (\beta_2)^{-1}\beta_3 b_3^{(2)}, \tag{2.3}$$

where $C_2^{(2)},\ C_3^{(2)} \neq 0$, (all constants has no effect on a stress state of the body)

$$A_3^{(2)} = -(R_4^4 - R_3^4)^{-1}[A_1^{(2)}(R_2^4 - R_1^4) + A_2^{(2)}(R_4^4 - R_3^4)],\ A_{-1}^{(2)} = -R_1^4 A_1^{(2)},$$

$$A_{-2}^{(2)} = A_1^{(2)}(R_2^4 - R_1^4) - R_2^4 A_2^{(2)},\ A_{-3}^{(2)} = R_4^4 A_3^{(2)},$$

$$A_3^{(2)} = -(R_4^4 - R_3^4)[A_1^{(2)}(R_2^4 - R_1^4) + A_2^{(2)}(R_3^4 - R_2^4)], \tag{2.4}$$

Substitute the expressions (2.2) and (2.3) in the boundary-contact conditions (1.14)-(1.15) and taking into the account the expressions (2.4), for $A_1^{(2)}$ and $A_2^{(2)}$ we get the following system

$$[\alpha_1 R_2^4 + \beta_1 R_1^4 + \beta_2(R_2^4 - R_1^4)]A_1^{(2)} - (\alpha_2 + \beta_2)R_2^4 A_2^{(2)} = R_2^4(\nu_1 - \nu_2),$$

$$[(\alpha_3 R_3^4 + \beta_3 R_4^4)(R_2^4 - R_1^4) - \beta_2(R_2^4 - R_1^4)(R_4^4 - R_3^4)]A_1^{(2)}$$
$$+[(\alpha_3 R_3^4 + \beta_3 R_4^4)(R_3^4 - R_2^4) + (\alpha_2 R_3^4 + \beta_2 R_2^4)(R_4^4 - R_3^4)]A_2^{(2)}$$
$$= (\nu_2 - \nu_3)R_3^4(R_4^4 - R_3^4), \tag{2.5}$$

From this system we obtain

$$A_1^{(2)} = (\nabla_3)^{-1}\{R_2^4 R_3^4(\alpha_2 + \beta_2)(R_4^4 - R_3^4)(\nu_2 - \nu_3)$$
$$+R_2^4(\nu_1 - \nu_2)[(\alpha_2 R_3^4 + \beta_2 R_2^4)(R_4^4 - R_3^4)(\alpha_3 R_3^4 + \beta_3 R_4^4)(R_3^4 - R_2^4)]\},$$
$$A_2^{(2)} = [R_2^4(\alpha_2 + \beta_2)]^{-1}\{A_1^{(2)}[\alpha_1 R_2^4 + \beta_1 R_1^4 + \beta_2(R_2^4 - R_1^4)] - R_2^4(\nu_1 - \nu_2)\},$$

where

$$\nabla_3 = \alpha_2\beta_2((R_2^4 - R_1^4)(R_3^4 - R_2^4)(R_4^4 - R_3^4) + (\alpha_3 R_3^4 + \beta_3 R_4^4)\{(R_2^4 - R_1^4)[(\alpha_2 + \beta_2)R_2^4 +$$
$$\beta_2(R_3^4 - R_2^4) + (\alpha_1 R_2^4 + \beta_1 R_1^4)(R_3^4 - R_2^4)\} > 0.$$

The other constants will be determined directly. Thus, the first auxiliary problem is solved completely.

b) *The solution of a second problem.*

We consider the case when in a boundary-contact conditions (1.12)-(1.17) the functions $f_1^{(3)}$ and $f_2^{(3)}$ have the following values

$$f_1^{(3)} = (2)^{-1} i(\nu_1 - \nu_2)t^2, \; f_2^{(3)} = (2)^{-1} i(\nu_2 - \nu_3)t^2; \tag{2.6}$$

In this case we seek the solutions in the form

$$\varphi^{(3)}(t)_1 = -iA_1^{(3)} z^2, \; \psi^{(3)}(t)_1 = -i(A_{-1}^{(3)} z^{-2} + b_1^{(3)}), \; \varphi^{(3)}(t)_2 = -iA_2^{(3)} z^2,$$

$$\psi^{(3)}(t)_2 = -i(A_{-2}^{(3)} z^{-2} + b_2^{(3)}), \;\; \varphi^{(3)}(t)_3 = -iA_3^{(3)} z^2,$$
$$\psi^{(3)}(t)_3 = -i(A_{-3}^{(3)} z^{-2} + b_3^{(3)}). \tag{2.7}$$

Putting these expressions in the conditions (1.12)-(1.17) we get the values of all coefficients $C_j^{(3)}$ and $b_j^{(3)}$, where $C_2^{(3)}$ and $C_3^{(3)}$ are an arbitrary constants differ from zero. The indicated constants has no effect on a stress-state of a body.

Take into the account (2.9) for the constants $A_j^{(3)}$, $A_{-j}^{(3)}$, $A_1^{(3)}$ and $A_2^{(3)}$ we get

$$A_{-1}^{(3)} = R_1^4 A_1^{(3)},\ \ \beta_2 A_{-2}^{(3)} = A_1^{(3)}(\alpha_1 R_2^4 - \beta_1 R_1^4) - A_2^{(3)}\alpha_2 R_2^4 - (2)^{-1}(\nu_1 - \nu_2),$$

$$A_{-3}^{(3)} = R_4^4 A_3^{(3)},\ \ A_3^{(3)} = -(R_4^4 - R_3^4)^{-1}[A_1^{(3)}(R_2^4 - R_1^4) + A_2^{(3)}(R_3^4 - R_2^4)],$$

$$A_1^{(3)}[\beta_2(R_2^4 - R_1^4) + \alpha_1 R_2^4 + \beta_1 R_1^4] - (\alpha_2 + \beta_2) R_2^4 A_2^{(3)} = (2)^{-1} R_2^4(\nu_1 - \nu_2),$$

$$A_1^{(3)}(R_2^4 - R_1^4)[(\alpha_3 R_3^4 + \beta_3 R_4^4) - \beta_2(R_4^4 - R_3^4)] + A_2^{(3)}[(\alpha_2 R_3^4 + \beta_2 R_2^4)(R_4^4 - R_3^4) + (\alpha_3 R_3^4 + \beta_3 R_4^4)(R_3^4 - R_2^4)] = (2)^{-1} R_3^4(R_4^4 - R_3^4)(\nu_2 - \nu_3).$$

From the above we get

$$2A_1^{(3)} = (\nabla_3)^{-1}[\delta_1 R_2^4(\nu_1 - \nu_2) + + R_3^4 R_2^4(R_4^4 - R_3^4)(\alpha_2 + \beta_2)(\nu_2 - \nu_3)],$$

where

$$\nabla_3 = \alpha_2(R_2^4 - R_1^4)(R_3^4 - R_2^4)(R_4^4 - R_3^4) + \delta_1(\alpha_1 R_2^4 + \beta_1 R_1^4) + R_2^4(R_2^4 - R_1^4)(\alpha_2 + \beta_2)(\alpha_3 R_3^4 + \beta_3 R_4^4) > 0.$$

And other coefficients will be determined directly.

c)*A solution of a third problem.*

For the third auxiliary problem in the boundary-contact conditions the functions $f_1^{(3}(t)$ and $f_2^{(3)}(t)$ have the following values

$$f_1^{(3)}(t) = (\nu_1 - \nu_2)t,\ \ f_2^{(3)}(t) = (\nu_2 - \nu_3)t. \qquad (2.8)$$

Taking into the account the results of N.Muskihelshvili [1], we seek the solutions in the form

$$\varphi_j^{(3)} = A_j^{(3)} z,\ \ \psi_j^{(3)} = A_{-j}^{(3)} z^{-1} + b_j^{(1)}, (j = 1, 2, 3), \qquad (2.9)$$

where the constants $A_j^{(3)}, A_{-j}^{(3)}$ and $b_j^{(3)}$ will be determined. Putting these expressions into the boundary conditions we get that two coefficients $C_3^{(3)}$ and $C_4^{(3}$ are an arbitrary constants differ from zero and this constants has no influence on the stress-state of the body. For the other constants from (1.12)-(1.17) we get

$$[(\alpha_1 - \beta_1)R_2^2 + 2\beta_{R_1}^2 + 2\beta_2(R_2^2 - R_1^2)]A_1^{(3)} - [(\alpha_2 - \beta_2)R_1^2 + 2\beta_2 R_2^2]A_2^{(3)} = (\nu_1 - \nu_2)R_2^2,$$

$$[(\alpha_2-\beta_2)R_3^2+2\beta_2R_2^2]A_2(3)-2\beta_2A_1^{(3)}(R_2^2-R_1^2)$$
$$-[(\alpha_3-\beta_3)R_3^2+2\beta_3R_4^2]A_3^{(3)}=(\nu_2-\nu_3)R_3^2,$$
$$-A_3^{(3)}=\frac{1}{(R_4^4-R_3^4)[A_1^{(3)}(R_2^2-R_1^2)+A_2^{(3)}(R_3^2-R_2^2)]\nu}$$

From here we get

$$\{(R_4^2-R_3^2)[(\alpha_2-\beta_2)R_3^2+2\beta_2R_2^2]+(R_3^2-R_2^2)[(\alpha_3+\beta_3)R_3^2+2\beta_3(R_4^2-R_3^2)]\}A_2^{(3)}+$$
$$\{(R_2^2-R_1^2)[(\alpha_3+\beta_3)R_3^2+2\beta_3(R_4^2-R_3^2)$$
$$-2\beta_2(R_2^2-R_1^2)(R_4^2-R_3^2)\}A_1^{(3)}=R_3^2(R_4^2-R_3^2)(\nu_2-\nu_3),$$
$$\alpha[(\alpha_1+\beta_1)R_2^2+2(R_2^2-R_1^2)(\beta_2-\beta_1)]A_2^{(3)}-$$
$$R_2^2(\alpha_2+\beta_2)A_1^{(3)}=R_2^2(\nu_1-\nu_2).$$

It is easy to show that for the determinant ∇ of this system we have

$$\nabla=(R_2^2-R_1^2)[(\alpha_3+\beta_3)R_3^2+2\beta_3(R_4^2-R_3^2)+2\beta_3(R_4^2-R_3^2)-$$
$$2\beta_2(R_2^2+R_1^2)(R_4^2-R_3^2)][(\alpha_1+\beta_1)R_2^2+2(R_2^2-R_1^2)(\beta_2-\beta_1)]+(\alpha_2+\beta_2)R_2^2$$
$$\times\{(R_4^2-R_3^2)[(\alpha_2-\beta_2)R_3^2+2\beta_2R_2^2]+(R_2^2-R_1^2)[(\alpha_3+\beta_3)R_3^2+2\beta_3(R_4^2-R_3^2)]\}>0.$$

The other coefficients will be determined directly. Therefore, a third auxiliary problem is solved completely.

3 Extension by longitudinal force and bending due to couples of forces

a) *The problem of tension.*

Let the external forces applied to the "upper" base z=l of the body is statically equivalent to the force f_3 parallel to the axis Ox_3 and to the bending forces acting on the planes Oyz and Oxz respectively with the moments m_1 and m_2.This force we apply to the point $(0,0,l)$. The stress-strain-state of composed concentric circular body we seek in the following form

$$u_j=\sum_{i=1}^{3}a_i(g_j^{(i)}-u_j^{(i)})-2^{-1}a_jx_3^2,\ (j=1,2),$$

$$u_3=x_3\sum_{i=1}^{3}a_ix^{(i)},$$

$$\tau_{jj} = \sum_{i=1}^{3} [\tau_{jj}^{(i)} + 2^{-1}(j-1)(j-2)x^{(i)}], \quad (j = 1, 2, 3),$$

$$\tau_{12} = \sum_{i=1}^{3} a_i \tau_{12}^{(i)}, \; \tau_{13} = \tau_{23} = 0, \tag{3.1}$$

where constants a_j will be determined , $u_j^{(i)}$ and $\tau_{jk}^{(i)}$ are solutions of three auxiliary problems.

Previously was calculated some expressions,given by equalities (1.4)-(1.5)

$$J_{11} = J_{22} = \pi(4)^{-1}[E_1(R_2^4 - R_1^4) + E_2(R_3^4 - R_2^4) + E_3(R_4^4 - R_3^4)],$$

$$J_{33} = \pi[E_1(R_2^2 - R_1^2) + E_2(R_3^2 - R_2^2) + E_3(R_4^2 - R_3^2)],$$

$$K_{11} = 2\pi[A(1)_1\nu_1(R_2^4 - R_1^4) + A(1)_2\nu_2(R_3^4 - R_2^4) + A(1)_3\nu_3(R_4^4 - R_3^4)],$$

$$K_{22} = 2\pi[A(2)_1\nu_1(R_2^4 - R_1^4) + A(2)_2\nu_2(R_3^4 - R_2^4) + A(2)_3\nu_3(R_4^4 - R_3^4)],$$

$$K_{33} = 2\pi[A(3)_1\nu_1(R_2^2 - R_1^2) + A(3)_2\nu_2(R^2)_3 - R_2^2) + A(3)_3\nu_3(R_4^2 - R_3^2)],$$

$$J_{12} = J_{13} = J_{23} = K_{12} = K_{13} = K_{23} = 0. \tag{3.2}$$

Substituting this expressions into (1.7) we see that it is satisfied identically. Putting the components τ_{jk} into Saint-Venant's conditions (1.8) we get

$$\sum_{k=1}^{3} a_k B_{kj} = m_{j^0}^0, \; (j = 1, 2, 3), \tag{3.3}$$

where

$$m_1^0 = -m_2, \quad m_2^0 = m_1, \quad m_3^0 = p_3. \tag{3.4}$$

Taking into the account sixth equalities from (3.2) we get

$$a_1 = -m_2 B_{11}^{-1}, \;\; a_2 = m_2 B_{22}^{-1} \;\; a_3 = p_3 B_{33}^{-1}, \tag{3.5}$$

Therefore the problem of extension by longitudinal force and bending by couple of forces three-layered circular tube is solved completely.

b) *The problem of torsion.*

It is well known that the torsion function F_0 of three-layered circular tube $\omega = \omega_1 + \omega_2 + \omega_3$ is harmonic in each of domains ω_j and satisfies the following boundary-contact conditions

$$[D_n F_0]_1 \equiv [n_1 D_1 F_0 + n_2 D_2 F_0]_1 = (x_2 n_1 - x_1 n 2)_1$$

on Γ_1,

$$[D_n F_0]_3 = (x_2 n_1 - x_1 n_2)_4$$

on Γ_4,

$$[\mu_j D_n F_0]_j = [\mu_{j+1} F_0]_{j+1}$$

on Γ_{j+1},

$$[F_0]_j = [F_0]_{j+1}. \tag{3.6}$$

As on each circle $r = R_j$ we have

$$(x_2 n_1 - x_1 n_2)_j = R_j(\sin\vartheta\cos\vartheta - \cos\theta\sin\vartheta) = 0,$$

then for the torsion function F_0 we have $F_0 = C^* = constant$ and the torsion function has no effect on the stress condition.

C)*Bending by transverse forces*

It is assumed that external forces applied to the upper end $x_3 = l$ of three-layered concentric tube is statically equivalent to two bending forces P_1 and P_2 parallel to the axis Ox_1 and Ox_2 respectively and are applied at the point $(0, 0, l)$. Therefore, in the conditions (1.8) we take

$$P_1 \neq 0, \; P_2 \neq 0, \; P_3 = m_1 = m_2 = m_3 = 0. \tag{3.7}$$

As in above, we propose that the origin of the system of coordinates $Ox_1x_2x_3$ coincide with the generalized center and the generalized principal axis of the inertia of the domain $\omega = \omega_1 + \omega_2 + \omega_3$. On the "under" base $x_3 = 0$ the following equalities are true

$$B_{jk} \equiv J_{jk} + K_{jk} = 0, \; j \neq k,$$

where expressions J_{jk} and K_{jk} are given by (3.2).

As the torsion function F_0 is a constant the components of displacement and stress in each domain ω_j we seek in the following form

$$u_j = -x_3 \sum_{k=1}^{3} a_k (g_j^{(k)} - u_j^{(k)}) - 6^{-1} a_j x_3^3, \quad (j = 1, 2),$$

$$u_3 = (1/2)x_3^2 \sum_{k=1}^{3} a_k x^{(k)} + a_1 F_1 + a_2 F_2 - (1/3)(a_1 x_1^3 + a_2 x_2^3) - (1/2) a_3 (x_1^2 + x_2^2),$$

$$\tau_{jj} = x_3 \sum_{k=1}^{3} a_k [\tau_{jj}^{(k)} + (1/2)E(k-1)(k-2)x^{(k)}], \; (j = 1, 2, 3),$$

$$\tau_{12} = x_3 \sum_{k=1}^{3} a_k \tau_{12}^{(k)},$$

$$\tau_{j3} = \mu[a_1 D_j F_1 + a_2 D_j F_2 - a_j x_j^2 - a_3 x_j - \sum_{k=1}^{3} a_k (g_j^{(k)} - u_j^{(k)})], \ (j = 1, 2), \tag{3.8}$$

where the functions $u_j^{(k)}$ are the solutions of auxiliary problems given above, coefficients a_k and functions F_j will be determined.

At first we will write out the components of displacements and stresses corresponding to three auxiliary problems given in sections a), b) ,c) of the paragraph 2

$$(u_1^{(k)})_j = \frac{1}{2\mu_j}\{A_j^{(k)}\kappa_j(x_1^2 - x_2^2) - A_j^{(k)}X^2 - A_{-j}^{(k)}\frac{(x_1^2 - x_2^2)}{X^4}\},$$

$$(u_2^{(k)})_j = \frac{1}{\mu_j}[A_j^{(k)}\kappa_j x_1 x_2 - A_{-j}^{(k)}\frac{x_1 x_2}{X^2}],$$

$$(\tau_{11}^{(k)})_j = 2A_j^{(k)} x_1 - A_{-j}^{(k)}\frac{x_1^3 - 3x_1 x_2^2}{X^3},$$

$$(\tau_{22}^{(k)})_j = 6A_j^{(k)} x_2 + A_{-J}^{(k)}\frac{x_1^3 - 3x_1 x_2^2}{X^3},$$

$$(\tau_{12}^{(k)})_j = 2A_j^{(k)} x_2 - A_{-j}^{(k)}\frac{x_2^3 - 3x_1^2 x_2}{X^3},$$

$$2\mu(u_1^{(3)})_j = A_j^{(3)}\kappa_j - A_j^{(3)} x_1 - A_{-j}^{(3)}\frac{x_1}{X^2},$$

$$u_2^{(3)} = A_j^{(3)} x_2 \frac{2\mu}{\lambda_j + \mu_j} - A{-j}^{(3)}\frac{x_2}{X^2},$$

$$(\tau_{11}^{(3)})_j = 2A_j^{(3)} + 2A_{-j}^{(3)}\frac{x_1^2 - x_2^2}{X^2},$$

$$(\tau_{22}^{(3)})_j = 2A_j^{(3)} - 2A_{-j}^{(3)}\frac{x_1^2 - x_2^2}{X^2},$$

$$(\tau_{12}^{(3)})_j = 2A_{-j}^{(3)}\frac{x_1 x_2}{X^2}, \tag{3.9}$$

where $X^2 = x_1^2 + x_2^2$, the constants $A_j^{(k)}$ and $A_{-j}^{(k)}$ are given in sections a),b) and c) of paragraph 2.

It is easy to obtain that the components u_j and τ_{jm}, $(j, m = 1, 2)$, satisfy the following boundary-contact conditions

$$\tau_{nj} \equiv \tau_{1j} n_1 + \tau_{2j} n_2 = 0,$$

on γ_1 and γ_4,

$$\tau_{nj_m} = \tau_{nj_{m+1}}, \;\; u_{j_m} = u_{j_{m+1}},$$

on γ_{m+1}.

Putting u_3 and τ_{j3} from (3.8) into (3.5) for the functions F_1 and F_2 in each of domains ω_j, $j = 1, 2, 3$ we get the following boundary value problem

$$\Delta F_k^{(j)} = \xi_k^{(j)}(x_1, x_2), \;\; [D_n F_k^{(j)} + \eta_k^{(j)}]_j = 0; \;\; k = 1, 2; j = 1, 3; \tag{3.10}$$

on γ_1 and γ_4,

$$[D_n F_k^{(j)} + \eta_k^{(j)}]_j = [ident_k^{(j)}]_{j+1}, \;\; [u_3^{(3)}]_j = [u_3^{(3)}]_{j+1}, \tag{3.11}$$

on γ_{j+1}, where

$$D_n F = n_1 D_1 F + n_2 D_2 F, \;\; \mu\xi = (\lambda + \mu)\Theta,$$

$$\eta_k = [x_k^2 + (-1)^k \nu (x_2^2 - x_1^2)] n_k + \nu x_1 x_2 n_{3-k} - u_k^{(k)} n_k, \;\; \Delta = D_1^2 + D_2^2. \tag{3.12}$$

It is easy to see, that in this case

$$\xi_k^{(j)} = \frac{\lambda_j + \mu_j}{\mu_j} \Theta_k^{(j)} \equiv A_j^{(k)} x_k$$

N.I.Muskhelishvili [1] has proved that for the existence of the solution of boundary value problem (3.10)–(3.11) is necessary and sufficient

$$\int\int_\omega \mu\xi d\omega = \oint_{\gamma_1} \mu\eta_1 d\gamma + \oint_{\gamma_4} \mu\eta_4 d\gamma + \sum_m = 2^3 \oint \{[\mu\eta]_{m-1} - [\mu\eta]_m\} d\gamma. \tag{3.13}$$

In our case the condition (3.13) is fulfilled. The solution of the equation (3.10) in each of domains ω_j we seek in the form

$$F_k^{(j)} = (-2/3)\mu_j A_j^{(k)} x_k^3 + f_k^{(j)}, \tag{3.14}$$

where the function $f_k^{(j)}(x_1, x_2)$ is harmonic in each of domains ω_j.

We note that the functions $\eta_k^{(j)}$ are continuous over the interfaces γ_2 and γ_3, that is $[\eta_k^{(j)}]_{m-1} = [\eta_k^{(j)}]_m$, $(k = 1, 2; m = 2, 3)$.

Taking into the account this remark and putting the expression (3.14) into equations (3.10)–(3.12) for the functions $f_k^{(j)}$ we get the following boundary value problem:

$$\Delta f_k^{(j)} = 0$$

in each of domains ω_j,

$$D_n f_k^{(j)} = [2\mu A_j^{(k)} x_k^2 + x_k^2 + (-1)^k 2\nu(x_2^2 - x_1^2) - u_1^{(k)}]n_k + (\nu x_1 x_2 - u_2^{(k)})n_{3-k}$$

on γ_1 and γ_4,

$$[D_n f_k^{(m-1)}]_{m-1} - [D_n f_k^{(m)}]_m = 2((1/\mu_{m-1})A_{m-1}^{(k)} - (1/\mu_m)A_m^{(k)})x_k^2 n_k,$$

$$[f_k^{(m-1)}]_{m-1} - [f_k^{(m)}]_m = \frac{2}{3}(\frac{1}{\mu_{m-1}}A_{m-1}^{(k)} - \frac{1}{\mu_m}A_m^{(k)})x_k^3, \qquad (3.15)$$

$$(m = 2, 3; k = 1, 2.j = 1, 2, 3),$$

on γ_2.

Taking into the account

$$n_1 = \cos\vartheta, n_2 = \sin\vartheta,$$

$$\cos^3\vartheta = \frac{1}{3}(3\cos\vartheta + \cos 3\vartheta),$$

$$\sin^3\vartheta = \frac{1}{4}(3\sin\vartheta - \sin 3\vartheta),$$

the equalities (3.15) take the form

$$4r^{-2} D_n f_1^{(j)} = \zeta_{11}^{(j)} \cos\vartheta + \zeta_{31}^{(1)} \cos 3\vartheta,$$

$$4r^{-2} D_n f_2^{(j)} = \zeta_{12}^{(j)} \sin\vartheta + \zeta_{32} + 2\nu_j) \sin 3\vartheta,$$

$$4\{[D_n f_k^{m-1}]_{m-1} - [D_n f_k^{(m)}]_m\} = 2r^2\{[\phi_k^{m-1}]_{m-1} - [\phi_k^{(m)}]_m\}$$
$$\times[(2-k)(3\cos\vartheta + \cos 3\vartheta) + (k-1)(3\sin\vartheta - \sin 3\vartheta)],$$

$$[f_k^{(m-1)}]_{m-1} - [f_k^{(m)}]_m = (2/3)r^3\{[\phi_k^{(m-1)}]_{m-1} - [\phi_k^{(m)}]_m\}$$
$$\times\{(2-k)(3\cos\theta + \cos 3\vartheta) + (k-1)(3\sin\vartheta + \sin 3\theta)\},$$

where

$$\zeta_{1k}^{(j)} = \sum_{m=1}^{5} \chi_{mk}^{(j)}, \quad \zeta_{3k}^{j)} = (-1)^k \chi_{1k}^{(j)} + \chi_k 2^{(j)} + \chi_{3k}^{(j)} + (-1)^k \chi_{4k}^{(j)} + \chi_{5k}^{(j)};$$

$$\chi_{1k}^{(j)} = \frac{2}{\mu_j} + 1 + \frac{1}{2}\nu_j, \ \ \chi_{2k}^{(j)} = \frac{3}{2\mu_j}[(\kappa_j - 1)A_j^{(k)} - \frac{1}{r^4}A_{-j}^{(k)}],$$

$$\chi_{3k}^{(j)} = (\kappa_j + 1)A_j^{(k)} - \frac{1}{r^4}A_{-j}^{(k)},$$

$$\chi_{4k}^{(j)} = \frac{1}{\mu_j}(A_j^{(k)}\kappa_j - \frac{1}{r^4}A_{-j}^{(k)}), \ \chi_{5k}^{(j)} = -\frac{3}{2}\nu_j, \ j = 1, 3;$$

on γ_1 and γ_4.

The harmonic function $f_k^{(j)}$ we seek in the following form

$$f_1^{(j)} = b_1^{(j)}x_1 + b_3^{(j)}x_1^3 - 3b_3^{(j)}x_1x_2^2 + b_{-1}^{(J)}x_1x_1^2 + x_2^2 + b_{-3}^{(j)}\frac{(x_1^3 - 3x_1x_2^2)}{(x_1^2 + x_2^2)^3},$$

$$f_2^{(j)} = H_1^{(j)}x_2 + H_3^{(j)}x_2^3 - 3H_3^{(j)}x_1^2x^2 + \frac{H_{-1}^{(j)}x_2}{x_1^2 + x_2^2} + H_{-3}^{(j)}\frac{(x_2^3 - 3x_1^2x_2)}{(x_1^2 + x_2^2)^3},$$

where constants b_m and H_m will be determined from boundary conditions.

Putting the expressions (3.8) into the conditions (3.12) for the definition of the constants a_j we get the following system of algebraic equations

$$B_{11}a_1 + B_{21}a_2 + B_{31}a_3 = P_1,$$

$$B_{12}a_1 + B_{22}a_2 + B_{32}a_3 = P_2,$$

$$B_{13}a_1 + B_{23}a_2 + B_{33}a_3 = 0,$$

where $B_{jk} = J_{jk} + K_{jk}$ is given by equalities (3.8).It is easy to calculate that

$$B_{11} = J_{11} + K_{11} = \pi 4^{-1}\Sigma_{j=1}^3[(E_j + 2\nu_j A_j^{(1)})(R_{j+1}^4 - R_j^4)]$$

$$B_{22} = J_{22} + K_{22} = \pi 4^{-1}\Sigma_{j=1}^3[E_j + 2\nu_j A_2][(R_{j+1}^4 - R_j^4))],$$

$$B_{33} = J_{33} + K_{33} = \pi\Sigma_{j=1}^3[E_j(R_{j+1}^2 - R_2)].$$

Hence, all constants are defined and the problem is solved completely.

References

1. N.I.Muskhelishvili. Some basic problems mathematical theory of elasticity. Nauka, Moskow, 1966, pp. 707.

2. G.M.Khatiashvili. Homogenous and composed elastic cylinders. Mecniereba, Tbilisi, 1991, pp. 180.