BASIC PROPERTIES OF THE FUNDAMENTAL SOLUTION IN THE THEORY OF MICROPOLAR THERMOELASTICITY WITHOUT ENERGY DISSIPATION

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Abstract

In this article the linear theory of micropolar thermoelasticity without energy dissipation is considered. Some basic properties of the fundamental solution of the system of differential equations in the case of steady oscillations are estabilished.

Key words and phrases: Micropolar thermoelasticity without energy dissipation; Steady oscillations; Fundamental solution.

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1 Introduction

The theory of micropolar thermoelasticity based on the Fourier law for the heat conduction (parabolic-type heat equation) was presented by Nowacki [1] and Eringen[2]. This theory permits the transmission of heat as thermal waves at infinite speed. In recent years there has been very much written on the subject of this theory. The basic results, historical information and an extensive review on the theory of micropolar thermoelasticity can be found in the books of Burchuladze and Gegelia [3], Eringen [4], Nowacki [5] and Dyszlewicz [6].

In [7], Boschi and Iesan extended a generalized theory of micropolar thermoelasticity that permits the transmission of heat as thermal waves at finite speed (hyperbolic-type heat equation). Recently, Green and Naghdi [8] introduced a theory of thermoelasticity without energy dissipation. This theory permits the transmission of heat as thermal waves at finite speed, and the heat flow does not involve energy dissipation. In [9], Ciarletta presented a linear theory of micropolar thermoelasticity without energy dissipation.

For investigating boundary value problems of the theory of micropolar thermoelasticity without energy dissipation it is necessary to estabilish the basic properties of the fundamental solution of the system of differential equations in the case of steady oscillations (steady vibrations).

The fundamental solutions in the theories of thermoelasticity without energy dissipation and micropolar thermoelasticity are constructed by Iesan [10] and Nowacki [11], respectively. The fundamental solution in the theory of micropolar thermoelasticity without energy dissipation is presented in [12]. The fundamental solutions in the microcontinuum fields theories are constructed by de Boer and Svanadze [13], Svanadze [14–17], and Svanadze and De Cicco [18].

This paper concerns with the linear theory of micropolar thermoelasticity without energy dissipation [9]. Some basic properties of the fundamental solution of the system of differential equations in the case of steady oscillations are estabilished.

2 Basic Equations

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the point of the Euclidean three-dimensional space E^3, $|\mathbf{x}| = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$, $\mathbf{D_x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$, and let t denote the time variable.

The system of linearized equations of motion in the theory of micropolar thermoelasticity without energy dissipation can be written as [9]

$$
\begin{aligned}
&(\mu+\kappa)\,\Delta\widetilde{\mathbf{u}} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\widetilde{\mathbf{u}} + \kappa\,\mathrm{curl}\,\widetilde{\boldsymbol{\varphi}} - m\,\mathrm{grad}\,\widetilde{\theta} = \rho\left(\ddot{\widetilde{\mathbf{u}}} - \mathbf{G}'\right),\\
&\gamma\Delta\widetilde{\boldsymbol{\varphi}} + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\widetilde{\boldsymbol{\varphi}} + \kappa\,\mathrm{curl}\,\widetilde{\mathbf{u}} - 2\kappa\widetilde{\boldsymbol{\varphi}} = \rho_1\ddot{\widetilde{\boldsymbol{\varphi}}} - \rho\mathbf{G}'',\\
&k_0\Delta\widetilde{\theta} - aT_0\ddot{\widetilde{\theta}} - mT_0\,\mathrm{div}\,\ddot{\widetilde{\mathbf{u}}} = -\rho\dot{S},
\end{aligned}
\tag{2.1}
$$

where $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \widetilde{u}_2, \widetilde{u}_3)$ is the displacement vector, $\widetilde{\boldsymbol{\varphi}} = (\widetilde{\varphi}_1, \widetilde{\varphi}_2, \widetilde{\varphi}_3)$ is the microrotation vector, $\widetilde{\theta}$ is the temperature measured from the constant absolute temperature T_0 ($T_0 \neq 0$); λ, μ, κ, m, α, β, γ, k_0, a are constitutive coefficients, ρ is the reference mass density ($\rho > 0$), ρ_1 is a coefficient of inertia; $\mathbf{G}'$ is the body force density, $\mathbf{G}''$ is the body couple density, and S is the heat source density [9]; Δ is the Laplacian, and dot denotes differentiation with respect to t: $\dot{\widetilde{\mathbf{u}}} = \frac{\partial\widetilde{\mathbf{u}}}{\partial t}$, $\ddot{\widetilde{\mathbf{u}}} = \frac{\partial^2\widetilde{\mathbf{u}}}{\partial t^2}$.

If the body forces $\mathbf{G}'$, $\mathbf{G}''$ and the heat source density S are assumed to be absent, and the displacement vector $\widetilde{\mathbf{u}}$, the microrotation vector $\widetilde{\boldsymbol{\varphi}}$ and the temperature $\widetilde{\theta}$ are postulate to have a harmonic time variation, that is

$$\widetilde{\mathbf{u}}(\mathbf{x},t) = \mathrm{Re}\left[\mathbf{u}(\mathbf{x})\, e^{-i\omega t}\right],$$

$$\widetilde{\boldsymbol{\varphi}}(\mathbf{x},t) = \mathrm{Re}\left[\boldsymbol{\varphi}(\mathbf{x})\, e^{-i\omega t}\right],$$

$$\widetilde{\theta}(\mathbf{x},t) = \mathrm{Re}\left[\theta(\mathbf{x})\, e^{-i\omega t}\right],$$

then from system (2.1) we obtain the following system of equations of steady oscillations

$$\begin{aligned}
&(\mu+\kappa)\,\Delta\mathbf{u} + (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} + \kappa\,\mathrm{curl}\,\boldsymbol{\varphi} - m\,\mathrm{grad}\,\theta + \rho\omega^2\mathbf{u} = \mathbf{0},\\
&\gamma\Delta\boldsymbol{\varphi} + (\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi} + \kappa\,\mathrm{curl}\,\mathbf{u} + \mu_1\boldsymbol{\varphi} = \mathbf{0},\\
&k_0\Delta\theta + a_0\theta + m_0\,\mathrm{div}\,\mathbf{u} = 0,
\end{aligned} \tag{2.2}$$

where ω is the oscillation frequency ($\omega > 0$), and

$$\mu_1 = \rho_1\omega^2 - 2\kappa, \qquad a_0 = aT_0\omega^2, \qquad m_0 = mT_0\omega^2.$$

3 Fundamental Solution

We introduce the matrix differential operator

$$\mathbf{A}(\mathbf{D_x}) = \|A_{pq}(\mathbf{D_x})\|_{7\times 7},$$

where

$$A_{lj}(\mathbf{D_x}) = \left[(\mu+\kappa)\,\Delta + \rho\omega^2\right]\delta_{lj} + (\lambda+\mu)\,\frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{l,j+3}(\mathbf{D_x}) = A_{l+3,j}(\mathbf{D_x}) = \kappa \sum_{r=1}^{3} \varepsilon_{lrj}\,\frac{\partial}{\partial x_r},$$

$$A_{l7}(\mathbf{D_x}) = -m\frac{\partial}{\partial x_l}, \qquad A_{l+3,7}(\mathbf{D_x}) = A_{7,l+3}(\mathbf{D_x}) = 0,$$

$$A_{l+3,j+3}(\mathbf{D_x}) = (\gamma\Delta + \mu_1)\,\delta_{lj} + (\alpha+\beta)\,\frac{\partial^2}{\partial x_l \partial x_j},$$

$$A_{7l}(\mathbf{D_x}) = m_0\frac{\partial}{\partial x_l},$$

$$A_{77}(\mathbf{D_x}) = k_0\Delta + a_0, \qquad l,j = 1,2,3,$$

δ_{lj} is the Kronecker delta, and ε_{lrj} is the alternating symbol.

The system (2.2) can be written as

$$\mathbf{A}\left(\mathbf{D_x}\right)\mathbf{U}\left(\mathbf{x}\right)=\mathbf{0},$$

where $\mathbf{U}=(\mathbf{u},\boldsymbol{\varphi},\theta)$ is seven-component vector function on E^3.

We assume that the constitutive coefficients satisfy the condition

$$\gamma k_0\left(\mu+\kappa\right)\left(\lambda+2\mu+\kappa\right)\left(\alpha+\beta+\gamma\right)\neq 0. \tag{3.3}$$

Obviously, if condition (3.3) is satisfied, then $\mathbf{A}$ is the elliptic differential operator [19].

Definition. The fundamental solution of the system (2.2) (the fundamental matrix operator $\mathbf{A}$) is the matrix $\mathbf{\Gamma}\left(\mathbf{x}\right)=\left\|\Gamma_{lj}\left(\mathbf{x}\right)\right\|_{7\times7}$ satisfying condition [19]

$$\mathbf{A}\left(\mathbf{D_x}\right)\mathbf{\Gamma}\left(\mathbf{x}\right)=\delta\left(\mathbf{x}\right)\mathbf{J}, \tag{3.4}$$

where δ is the Dirac delta, $\mathbf{J}=\left\|\delta_{lj}\right\|_{7\times7}$ is the unit matrix, and $\mathbf{x}\in E^3$.

In what follows we use the notation

$$\begin{aligned}
&\Lambda_1(\Delta)=b_1\left[\left(\mu_0\Delta+\rho\omega^2\right)\left(k_0\Delta+a_0\right)+mm_0\Delta\right],\\
&\Lambda_2(\Delta)=b_2\left\{\left[\left(\mu+\kappa\right)\Delta+\rho\omega^2\right]\left(\gamma\Delta+\mu_1\right)+\kappa^2\Delta\right\},\\
&\mu_0=\lambda+2\mu+\kappa,\qquad b_1=\frac{1}{\mu_0k_0},\qquad b_2=\frac{1}{\gamma\left(\mu+\kappa\right)}.
\end{aligned} \tag{3.5}$$

It is easily seen that

$$\Lambda_1(\Delta)=\left(\Delta+k_1^2\right)\left(\Delta+k_2^2\right),$$

$$\Lambda_2(\Delta)=\left(\Delta+k_3^2\right)\left(\Delta+k_4^2\right),$$

where k_1^2,k_2^2 and k_3^2,k_4^2 are the roots of the equations $\Lambda_1(-\chi)=0$ and $\Lambda_2(-\chi)=0$ (with respect to χ), respectively. We assume that $k_l\neq0,\ k_l^2\neq k_j^2,\ l,j=1,2,\cdots,5,\ l\neq j.$

Let

$$\begin{aligned}
&\mathbf{Y}(\mathbf{x})=\left\|Y_{pq}(\mathbf{x})\right\|_{7\times7}, && Y_{ll}(\mathbf{x})=\sum_{j=1}^{4}p_{1j}\gamma_j(\mathbf{x}),\\
&Y_{l+3,l+3}(\mathbf{x})=\sum_{j=3}^{5}p_{2j}\gamma_j(\mathbf{x}), && Y_{77}(\mathbf{x})=\sum_{j=1}^{2}p_{3j}\gamma_j(\mathbf{x}),\\
&Y_{pq}(\mathbf{x})=0,\qquad l=1,2,3, && p,q=1,2,...,7,\qquad p\neq q,
\end{aligned} \tag{3.6}$$

where

$$
\begin{aligned}
&\gamma_p(x) = -\frac{1}{4\pi\,|\mathbf{x}|}e^{ik_p|\mathbf{x}|}, \qquad p = 1, 2, ..., 5,\\
&p_{1j} = \prod_{\substack{l=1\\ l\neq j}}^{4}(k_l^2 - k_j^2)^{-1}, \qquad j = 1, 2, 3, 4,\\
&p_{2q} = \prod_{\substack{l=3\\ l\neq q}}^{5}(k_l^2 - k_q^2)^{-1}, \qquad q = 3, 4, 5,\\
&p_{31} = -p_{32} = \frac{1}{k_2^2 - k_1^2}, \qquad k_5^2 = \frac{\mu_1}{\gamma_0}, \qquad \gamma_0 = \alpha + \beta + \gamma.
\end{aligned}
\tag{3.7}
$$

We introduce the matrix

$$
\mathbf{\Gamma}(\mathbf{x}) = \mathbf{L}(\mathbf{D_x})\mathbf{Y}(\mathbf{x}), \tag{3.8}
$$

where $\mathbf{L}(\mathbf{D_x})$ is the following matrix differential operator

$$
\mathbf{L}(\mathbf{D_x}) = \|L_{pq}(\mathbf{D_x})\|_{7\times 7} = \left\| \begin{matrix} \mathbf{L^{(1)}}(\mathbf{D_x}) & \mathbf{L^{(2)}}(\mathbf{D_x}) & \mathbf{L^{(5)}}(\mathbf{D_x}) \\ \mathbf{L^{(3)}}(\mathbf{D_x}) & \mathbf{L^{(4)}}(\mathbf{D_x}) & \mathbf{L^{(6)}}(\mathbf{D_x}) \\ \mathbf{L^{(7)}}(\mathbf{D_x}) & \mathbf{L^{(8)}}(\mathbf{D_x}) & L^{(9)}(\mathbf{D_x}) \end{matrix} \right\|_{7\times 7},
$$

$$\mathbf{L}^{(1)}(\mathbf{D_x}) = \|L_{lj}(\mathbf{D_x})\|_{3\times 3} = b_2(\gamma\Delta + \mu_1)\Lambda_1(\Delta)\mathbf{I} + n'(\Delta)\,\mathrm{grad}\,\mathrm{div},$$

$$\mathbf{L}^{(2)}(\mathbf{D_x}) = \left\| L^{(2)}_{lj}(\mathbf{D_x})\right\|_{3\times 3} = -b_2\kappa(\Delta + k_5^2)\,\mathrm{curl},$$

$$\mathbf{L}^{(3)}(\mathbf{D_x}) = \left\| L^{(3)}_{lj}(\mathbf{D_x})\right\|_{3\times 3} = -b_2\kappa\Lambda_1(\Delta)\,\mathrm{curl},$$

$$\mathbf{L}^{(4)}(\mathbf{D_x}) = \left\| L^{(4)}_{lj}(\mathbf{D_x})\right\|_{3\times 3} = b_2(\Delta + k_5^2)\left[(\mu+\kappa)\,\Delta + \rho\omega^2\right]\mathbf{I} + n''(\Delta)\,\mathrm{grad}\,\mathrm{div},$$

$$\mathbf{L}^{(5)}(\mathbf{D_x}) = \left\| L^{(5)}_{lj}(\mathbf{D_x})\right\|_{3\times 1} = b_1 m\,\mathrm{grad},$$

$$\mathbf{L}^{(6)}(\mathbf{D_x}) = \left\| L^{(6)}_{lj}(\mathbf{D_x})\right\|_{3\times 1} = \mathbf{0},$$

$$\mathbf{L}^{(7)}(\mathbf{D_x}) = \left\| L^{(7)}_{lj}(\mathbf{D_x})\right\|_{1\times 3} = -b_1 m_0\Lambda_2(\Delta)\,\mathrm{grad},$$

$$\mathbf{L}^{(8)}(\mathbf{D_x}) = \left\| L^{(8)}_{lj}(\mathbf{D_x})\right\|_{1\times 3} = \mathbf{0},$$

$$L^{(9)}(\mathbf{D_x}) = L_{77}(\mathbf{D_x}) = b_1(\mu_0\Delta + \rho\omega^2), \qquad \mathbf{I} = \|\delta_{lj}\|_{3\times 3}, \tag{3.9}$$

and

$$n'(\Delta) = -b_1 b_2\{(k_0\Delta + a_0)[(\lambda+\mu)(\gamma\Delta+\mu_1) - \kappa^2] + m m_0(\gamma\Delta + \mu_1)\},$$

$$n''(\Delta) = -\frac{b_2}{\gamma}\{(\alpha+\beta)[(\mu+\kappa)\Delta + \rho\omega^2] - \kappa^2\}. \tag{3.10}$$

In [12], the matrix $\mathbf{\Gamma}(\mathbf{x})$ is constructed and the following theorem is proved.

Theorem 1. *The matrix $\mathbf{\Gamma}(\mathbf{x})$ defined by (3.8) is the fundamental solution of system (2.2), that is, $\mathbf{\Gamma}(\mathbf{x})$ is a solution to Eq. (3.4).*

4 Basic properties of the matrix $\mathbf{\Gamma}(\mathbf{x})$

Obviously, if condition (3.3) is satisfied, then the fundamental solution of the system

$$(\mu+\kappa)\,\Delta\mathbf{u}+(\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u}=\mathbf{0},$$

$$\gamma\Delta\boldsymbol{\varphi}+(\alpha+\beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\varphi}=\mathbf{0},$$

$$k_0\Delta\theta=0$$

is the matrix [20, 21]

$$\mathbf{\Psi}\,(\mathbf{x})=\|\Psi_{pq}\,(\mathbf{x})\|_{7\times 7}=\begin{Vmatrix}\mathbf{\Psi}^{(1)} & \mathbf{\Psi}^{(2)} & \mathbf{\Psi}^{(5)}\\ \mathbf{\Psi}^{(3)} & \mathbf{\Psi}^{(4)} & \mathbf{\Psi}^{(6)}\\ \mathbf{\Psi}^{(7)} & \mathbf{\Psi}^{(8)} & \Psi^{(9)}\end{Vmatrix}_{7\times 7},$$

where

$$\mathbf{\Psi}^{(1)}(\mathbf{x})=\left\|\Psi^{(1)}_{lj}\,(\mathbf{x})\right\|_{3\times 3}=\left(\frac{1}{\mu_0}\,\mathrm{grad}\,\mathrm{div}-\frac{1}{\mu+\kappa}\,\mathrm{curl}\,\mathrm{curl}\right)\eta_1(\mathbf{x}),$$

$$\mathbf{\Psi}^{(2)}(\mathbf{x})=\mathbf{\Psi}^{(3)}(\mathbf{x})=\left\|\Psi^{(3)}_{lj}\,(\mathbf{x})\right\|_{3\times 3}=\mathbf{0},$$

$$\mathbf{\Psi}^{(4)}(\mathbf{x})=\left\|\Psi^{(4)}_{lj}\,(\mathbf{x})\right\|_{3\times 3}=\left(\frac{1}{\gamma_0}\,\mathrm{grad}\,\mathrm{div}-\frac{1}{\gamma}\,\mathrm{curl}\,\mathrm{curl}\right)\eta_1(\mathbf{x}),$$

$$\mathbf{\Psi}^{(5)}(\mathbf{x})=\mathbf{\Psi}^{(6)}(\mathbf{x})=\left\|\Psi^{(6)}_{lj}\,(\mathbf{x})\right\|_{3\times 1}=\mathbf{0},$$

$$\mathbf{\Psi}^{(7)}(\mathbf{x})=\mathbf{\Psi}^{(8)}(\mathbf{x})=\left\|\Psi^{(8)}_{lj}\,(\mathbf{x})\right\|_{1\times 3}=\mathbf{0},$$

$$\Psi^{(9)}(\mathbf{x})=\frac{1}{k_0}\eta_2(\mathbf{x}),\qquad \eta_1(\mathbf{x})=-\frac{|\mathbf{x}|}{8\pi},\qquad \eta_2(\mathbf{x})=\Delta\eta_1(\mathbf{x})=-\frac{1}{4\pi\,|\mathbf{x}|}. \tag{4.11}$$

As one may be easily verify, on the basis of Eqs. (3.6), (3.8) and (3.9) the matrix $\mathbf{\Gamma}(\mathbf{x})$ can be written in the from

$$\mathbf{\Gamma}\,(\mathbf{x})=\|\Gamma_{pq}\|_{7\times 7}=\begin{Vmatrix}\mathbf{\Gamma}^{(1)} & \mathbf{\Gamma}^{(2)} & \mathbf{\Gamma}^{(5)}\\ \mathbf{\Gamma}^{(3)} & \mathbf{\Gamma}^{(4)} & \mathbf{\Gamma}^{(6)}\\ \mathbf{\Gamma}^{(7)} & \mathbf{\Gamma}^{(8)} & \Gamma^{(9)}\end{Vmatrix}_{7\times 7},$$

where

$$
\begin{aligned}
&\boldsymbol{\Gamma}^{(p)}(\mathbf{x}) = \mathbf{L}^{(p)}(\mathbf{D}_{\mathbf{x}})Y_{11}(\mathbf{x}), \qquad && p = 1,3,7,\\
&\boldsymbol{\Gamma}^{(q)}(\mathbf{x}) = \mathbf{L}^{(q)}(\mathbf{D}_{\mathbf{x}})Y_{44}(\mathbf{x}), \qquad && q = 2,4,\\
&\boldsymbol{\Gamma}^{(r)}(\mathbf{x}) = \mathbf{L}^{(r)}(\mathbf{D}_{\mathbf{x}})Y_{77}(\mathbf{x}), \qquad && r = 5,9,\\
&\boldsymbol{\Gamma}^{(6)}(\mathbf{x}) = \left\|\Gamma^{(6)}_{lj}(\mathbf{x})\right\|_{3\times 1} = \mathbf{0}, \qquad && \boldsymbol{\Gamma}^{(8)}(\mathbf{x}) = \left\|\Gamma^{(8)}_{lj}(\mathbf{x})\right\|_{1\times 3} = \mathbf{0}.
\end{aligned}
\tag{4.12}
$$

In what follows we shall use the following lemma.

Lemma. If condition (3.3) is satisfied, then

$$
\begin{aligned}
&\Delta n'(\Delta) = b_1\left(k_0\Delta + a_0\right)\Lambda_2(\Delta) - b_2\left(\gamma\Delta + \mu_1\right)\Lambda_1(\Delta),\\
&\Delta n''(\Delta) = \frac{1}{\gamma_0}\Lambda_2(\Delta) - b_2\left(\Delta + k_5^2\right)[(\mu+\kappa)\Delta + \rho\omega^2].
\end{aligned}
\tag{4.13}
$$

Proof. Taking into account the equality $(3.10)_1$ we have

$$
\begin{aligned}
\Delta n'(\Delta) &= -b_1 b_2\left\{(k_0\Delta + a_0)\left[(\lambda+\mu)\Delta(\gamma\Delta+\mu_1) - \kappa^2\Delta\right] + m m_0\Delta(\gamma\Delta+\mu_1)\right\}\\
&= -b_1 b_2(\gamma\Delta+\mu_1)\left(k_0\Delta+a_0\right)\left\{(\mu_0\Delta+\rho\omega^2) - \left[(\mu+\kappa)\Delta+\rho\omega^2\right]\right\}\\
&- m m_0 b_1 b_2(\gamma\Delta+\mu_1)\Delta + b_1 b_2\kappa^2\Delta(k_0\Delta+a_0)\\
&= -b_1 b_2(\gamma\Delta+\mu_1)\left\{\frac{1}{b_1}\Lambda_1(\Delta) - (k_0\Delta+a_0)\left[(\mu+\kappa)\Delta+\rho\omega^2\right]\right\}\\
&+ b_1 b_2\kappa^2\Delta(k_0\Delta+a_0)\\
&= b_1 b_2\left\{(\gamma\Delta+\mu_1)\left[((\mu+\kappa)\Delta+\rho\omega^2\right] + \kappa^2\Delta\right\}(k_0\Delta+a_0)\\
&- b_2\left(\gamma\Delta+\mu_1\right)\Lambda_1(\Delta)\\
&= b_1\left(k_0\Delta+a_0\right)\Lambda_2(\Delta) - b_2\left(\gamma\Delta+\mu_1\right)\Lambda_1(\Delta).
\end{aligned}
$$

Equation $(4.13)_2$ is proven in a quite similar manner. □

Using the equality

$$
(\Delta + k_j^2)\gamma_j(\mathbf{x}) = \delta(\mathbf{x}), \qquad j = 1,2,...,5, \tag{4.14}
$$

from (4.13) we obtain the following result.

Corollary. If $\mathbf{x} \in E^3 \setminus \{0\}$, than

$$n'(-\kappa_j^2)\gamma_j(\mathbf{x}) = \frac{1}{k_j^2}\left[b_1(k_0 k_j^2 - a_0)\Lambda_2(-k_j^2) - b_2(\gamma k_j^2 - \mu_1)\Lambda_1(-k_j^2)\right]\gamma_j(\mathbf{x}),$$

$$n''(-k_j^2)\gamma_j(\mathbf{x}) = \left\{-\frac{1}{\gamma_0 k_j^2}\Lambda_2(-k_j^2) - \frac{b_2}{k_j^2}(k_5^2 - k_j^2)\left[(\mu+\kappa)k_j^2 - \rho\omega^2\right]\right\}\gamma_j(\mathbf{x}),$$

$j = 1, 2, ..., 5.$

On the basis of Eqs. (3.5) and (3.7) we have

$$\begin{gathered}
p_{1j}\Lambda_1(-k_j^2) = \begin{cases} 0, & \text{for} \quad j = 1,2, \\ (-1)^j (k_3^2 - k_4^2)^{-1}, & \text{for} \quad j = 3,4, \end{cases} \\
p_{1j}\Lambda_2(-k_j^2) = \begin{cases} (-1)^j (k_1^2 - k_2^2)^{-1}, & \text{for} \quad j = 1,2, \\ 0, & \text{for} \quad j = 3,4, \end{cases} \\
p_{2j}(k_5^2 - k_j^2) = \begin{cases} (-1)^j (k_3^2 - k_4^2)^{-1}, & \text{for} \quad j = 3,4, \\ 0, & \text{for} \quad j = 5, \end{cases} \\
p_{2j}\Lambda_2(-k_j^2) = \begin{cases} 0, & \text{for} \quad j = 3,4, \\ 1, & \text{for} \quad j = 5, \end{cases}
\end{gathered} \tag{4.15}$$

Let

$$\begin{gathered}
d_{1j} = \frac{(-1)^j\, b_1\left(k_0 k_j^2 - a_0\right)}{k_j^2(k_1^2 - k_2^2)}, \qquad d_{1l} = \frac{(-1)^l\, b_2(\gamma k_l^2 - \mu_1)}{k_l^2(k_3^2 - k_4^2)}, \\
d_{20} = \frac{\kappa b_2}{k_3^2 - k_4^2}, \qquad d_{4l} = \frac{(-1)^l\, b_2}{k_l^2(k_3^2 - k_4^2)}[(\mu+\kappa)k_l^2 - \rho\omega^2], \\
d_{50} = \frac{m b_1}{k_2^2 - k_1^2}, \qquad d_{70} = \frac{m_0 b_1}{k_1^2 - k_2^2}, \\
d_{9j} = \frac{(-1)^j\, b_1}{k_2^2 - k_1^2}\left(\mu_0 k_j^2 - \rho\omega^2\right), \qquad j = 1,2, \qquad l = 3,4.
\end{gathered} \tag{4.16}$$

Obviously, by Eqs. (4.16) we find that

$$\sum_{j=1}^{2} d_{1j} = \sum_{l=3}^{4} d_{1l} = -\frac{1}{\rho\omega^2}, \qquad \sum_{j=1}^{2} k_j^2 d_{1j} = -\frac{1}{\mu_0}, \qquad \sum_{l=3}^{4} k_l^2 d_{1l} = -\frac{1}{\mu+\kappa}, \tag{4.17}$$

and

$$\sum_{l=3}^{4} d_{4l} = -\frac{1}{\mu_1}, \qquad \sum_{l=3}^{4} k_l^2 d_{4l} = -\frac{1}{\gamma}. \tag{4.18}$$

Theorem 2. If $\mathbf{x} \in E^3 \setminus \{0\}$ then

$$\begin{aligned}
&\mathbf{\Gamma}^{(1)}(\mathbf{x}) = \operatorname{grad}\operatorname{div}\sum_{j=1}^{2} d_{1j}\gamma_j(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\sum_{l=3}^{4} d_{1l}\gamma_l(\mathbf{x}),\\
&\mathbf{\Gamma}^{(2)}(\mathbf{x}) = \mathbf{\Gamma}^{(3)}(\mathbf{x}) = d_{20}\operatorname{curl}[\gamma_3(\mathbf{x}) - \gamma_4(\mathbf{x})],\\
&\mathbf{\Gamma}^{(4)}(\mathbf{x}) = -\frac{1}{\mu_1}\operatorname{grad}\operatorname{div}\gamma_5(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\sum_{l=3}^{4} d_{4l}\gamma_l(\mathbf{x}),\\
&\mathbf{\Gamma}^{(5)}(\mathbf{x}) = d_{50}\operatorname{grad}[\gamma_1(\mathbf{x}) - \gamma_2(\mathbf{x})], \qquad \mathbf{\Gamma}^{(6)}(\mathbf{x}) = \mathbf{0}, \qquad \mathbf{\Gamma}^{(8)}(\mathbf{x}) = \mathbf{0},\\
&\mathbf{\Gamma}^{(7)}(\mathbf{x}) = d_{70}\operatorname{grad}[\gamma_1(\mathbf{x}) - \gamma_2(\mathbf{x})], \qquad \Gamma^{(9)}(\mathbf{x}) = \sum_{j=1}^{2} d_{9j}\gamma_j(\mathbf{x}).
\end{aligned} \tag{4.19}$$

Proof. Taking into account the inequalities (4.14) and

$$\Delta\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u}$$

we have

$$\mathbf{I}\gamma_j(\mathbf{x}) = -\frac{1}{k_j^2}(\operatorname{grad}\operatorname{div} - \operatorname{curl}\operatorname{curl})\gamma_j(\mathbf{x}), \tag{4.20}$$

where $\mathbf{x} \in E^3 \setminus \{0\}$ and $j = 1, 2, ..., 5$. By virtue of Eqs. $(3.6)_2$, $(3.9)_2$, (4.20) and Corollary from $(4.12)_1$ we obtain

$$\mathbf{\Gamma}^{(1)}(\mathbf{x}) = \sum_{j=1}^{4} p_{1j}[b_2(\gamma\Delta + \mu_1)\Lambda_1(\Delta)\mathbf{I} + n'(\Delta)\,\mathrm{grad}\,\mathrm{div}]\gamma_j(\mathbf{x}) \qquad (4.21)$$

$$= \sum_{j=1}^{4} p_{1j}\left\{\left[\frac{b_2}{k_2^2}(\gamma k_j^2 - \mu_1)\Lambda_1(-k_j^2) + n'(-k_j^2)\right]\mathrm{grad}\,\mathrm{div}\right.$$

$$\left.-\frac{b_2}{k_j^2}(\gamma k_j^2 - \mu_1)\Lambda_1(-k_j^2)\,\mathrm{curl}\,\mathrm{curl}\right\}\gamma_j(\mathbf{x})$$

$$= \sum_{j=1}^{4} p_{1j}\left\{\frac{b_1}{k_j^2}(k_0 k_j^2 - a_0)\Lambda_2(-k_j^2)\,\mathrm{grad}\,\mathrm{div}\right.$$

$$\left.-\frac{b_2}{k_j^2}(\gamma k_j^2 - \mu_1)\Lambda_1(-k_j^2)\,\mathrm{curl}\,\mathrm{curl}\right\}\gamma_j(\mathbf{x})$$

Using identities (4.15) and (4.16) from (4.21) we have

$$\mathbf{\Gamma}^{(1)}(\mathbf{x}) = \sum_{j=1}^{2} \frac{(-1)^j\, b_1}{k_j^2(k_1^2 - k_2^2)}(k_0 k_j^2 - a_0)\,\mathrm{grad}\,\mathrm{div}\,\gamma_j(\mathbf{x})$$

$$-\sum_{l=3}^{4} \frac{(-1)^l\, b_2}{k_l^2(k_3^2 - k_4^2)}(\gamma k_l^2 - \mu_1)\,\mathrm{curl}\,\mathrm{curl}\,\gamma_l(\mathbf{x})$$

$$= \mathrm{grad}\,\mathrm{div}\sum_{j=1}^{2} d_{1j}\gamma_j(\mathbf{x}) - \mathrm{curl}\,\mathrm{curl}\sum_{l=3}^{4} d_{1l}\gamma_l(\mathbf{x}).$$

On the basis of $(3.6)_3$, (3.9) and (4.15) from $(4.12)_2$ we obtain Eq. $(4.19)_3$. The other formulae of (4.19) can be proven quite similarly. □

Theorem 3. The relations

$$\Gamma_{pq}(\mathbf{x}) - \Psi_{pq}(\mathbf{x}) = \mathrm{const} + O(|\mathbf{x}|), \qquad (4.22)$$

$$\frac{\partial^s}{\partial x_1^{s_1} \partial x_2^{s_2} \partial x_3^{s_3}} [\Gamma_{pq}(\mathbf{x}) - \Psi_{pq}(\mathbf{x})] = O(|\mathbf{x}|^{1-s}) \tag{4.23}$$

and

$$\begin{aligned}
&|\Gamma_{lj}(\mathbf{x})| < \text{const}\, |\mathbf{x}|^{-1}, \qquad |\Gamma_{l+3,j+3}(\mathbf{x})| < \text{const}\, |\mathbf{x}|^{-1},\\
&|\Gamma_{77}(\mathbf{x})| < \text{const}\, |\mathbf{x}|^{-1}, \qquad |\Gamma_{lr}(\mathbf{x})| < \text{const},\\
&|\Gamma_{rl}(\mathbf{x})| < \text{const},
\end{aligned} \tag{4.24}$$

hold in the neighborhood of the origin, where $s = s_1 + s_2 + s_3$, $s \geq 1$, $l, j = 1, 2, 3$, $r = 4, 5, 6, 7$, $p, q = 1, 2, ..., 7$.

Proof. In view of Eqs. $(4.11)_1$ and $(4.19)_1$ we obtain

$$\begin{aligned}
\mathbf{\Gamma}^{(1)}(\mathbf{x}) - \mathbf{\Psi}^{(1)}(\mathbf{x}) = \operatorname{grad}\operatorname{div}\left[\sum_{j=1}^{2} d_{1j}\gamma_j(\mathbf{x}) - \frac{1}{\mu_0}\eta_1(\mathbf{x})\right]\\
- \operatorname{curl}\operatorname{curl}\left[\sum_{l=3}^{4} d_{1l}\gamma_l(\mathbf{x}) - \frac{1}{\mu+\kappa}\eta_1(\mathbf{x})\right].
\end{aligned} \tag{4.25}$$

In the neighborhood of the origin from $(3.7)_1$ we have

$$\gamma_p(\mathbf{x}) = -\frac{1}{4\pi\,|\mathbf{x}|}\sum_{n=0}^{\infty}\frac{(ik_p\,|\mathbf{x}|)^n}{n!} = \eta_2(\mathbf{x}) - \frac{ik_p}{4\pi} - k_p^2\eta_1(\mathbf{x}) + \widetilde{\gamma}_p(\mathbf{x}), \tag{4.26}$$

where $\widetilde{\gamma}_p(\mathbf{x}) = -\frac{1}{4\pi\,|\mathbf{x}|}\sum_{n=3}^{\infty}\frac{(ik_p\,|\mathbf{x}|)^n}{n!}$, $p = 1, 2, 3, 4$. Obviously,

$$\begin{aligned}
&\widetilde{\gamma}_p(\mathbf{x}) = O(|\mathbf{x}|^2), \qquad \frac{\partial}{\partial x_l}\widetilde{\gamma}_p(\mathbf{x}) = O(|\mathbf{x}|),\\
&\frac{\partial^2}{\partial x_l \partial x_j}\widetilde{\gamma}_p(\mathbf{x}) = \text{const} + O(|\mathbf{x}|), \qquad l, j = 1, 2, 3, \qquad p = 1, 2, 3, 4.
\end{aligned} \tag{4.27}$$

On the basis of Eqs. (4.17) from (4.26) we get

$$\sum_{j=1}^{2} d_{1j}\gamma_j(\mathbf{x}) - \frac{1}{\mu_0}\eta_1(\mathbf{x}) = \sum_{j=1}^{2} d_{1j}\eta_2(\mathbf{x}) - \left[\sum_{j=1}^{2} k_j^2 d_{1j} + \frac{1}{\mu_0}\right]\eta_1(\mathbf{x})$$

$$+\sum_{j=1}^{2} d_{1j}\left[-\frac{ik_j}{4\pi} + \widetilde{\gamma}_j(\mathbf{x})\right] = -\frac{1}{\rho\omega^2}\eta_2(\mathbf{x}) + \sum_{j=1}^{2} d_{1j}\left[-\frac{ik_j}{4\pi} + \widetilde{\gamma}_j(\mathbf{x})\right],$$

$$\sum_{l=3}^{4} d_{1l}\gamma_l(\mathbf{x}) - \frac{1}{\mu+\kappa}\eta_1(\mathbf{x}) = \sum_{l=3}^{4} d_{1l}\eta_2(\mathbf{x}) - \left[\sum_{l=3}^{4} k_l^2 d_{1l} + \frac{1}{\mu+\kappa}\right]\eta_1(\mathbf{x})$$

$$+\sum_{l=3}^{2} d_{1l}\left[-\frac{ik_j}{4\pi} + \widetilde{\gamma}_l(\mathbf{x})\right] = -\frac{1}{\rho\omega^2}\eta_2(\mathbf{x}) + \sum_{l=3}^{2} d_{1l}\left[-\frac{ik_j}{4\pi} + \widetilde{\gamma}_l(\mathbf{x})\right]. \tag{4.28}$$

Taking into account the equalities (4.28) and $\Delta\eta_2(\mathbf{x}) = 0$ $(\mathbf{x} \neq \mathbf{0})$ from (4.25) we have

$$\begin{aligned}\mathbf{\Gamma}^{(1)}(\mathbf{x}) - \mathbf{\Psi}^{(1)}(\mathbf{x}) &= \operatorname{grad}\operatorname{div}\left[-\frac{1}{\rho\omega^2}\eta_2(\mathbf{x}) + \sum_{j=1}^{2} d_{1j}\widetilde{\gamma}_j(\mathbf{x})\right] \\ &- \operatorname{curl}\operatorname{curl}\left[-\frac{1}{\rho\omega^2}\eta_2(\mathbf{x}) + \sum_{l=3}^{4} d_{1l}\widetilde{\gamma}_l(\mathbf{x})\right] \\ &= \operatorname{grad}\operatorname{div}\sum_{j=1}^{2} d_{1j}\widetilde{\gamma}_j(\mathbf{x}) - \operatorname{curl}\operatorname{curl}\sum_{l=3}^{4} d_{1l}\widetilde{\gamma}(\mathbf{x}).\end{aligned} \tag{4.29}$$

Obviously, in view of (4.27) from (4.29) we obtain the relation $(4.22)_1$ for $p, q = 1, 2, 3$. On the basis of $(4.11)_3$, (4.18) and $(4.19)_3$ we get the relation $(4.22)_1$ for $p, q = 4, 5, 6$. The other formulae of Eqs. (4.22) and (4.23) can be proven quite similar manner.

Inequalities (4.24) can be obtained easily from (4.19) and estimates [20]

$$|\Psi_{lj}(\mathbf{x})| < \mathrm{const}\,|\mathbf{x}|^{-1}, \qquad |\Psi_{l+3,j+3}(\mathbf{x})| < \mathrm{const}\,|\mathbf{x}|^{-1},$$

$$|\Psi_{77}(\mathbf{x})| < \mathrm{const}\,|\mathbf{x}|^{-1}, \qquad l, j = 1, 2, 3.$$

Thus, the matrix $\mathbf{\Psi}(\mathbf{x})$ is the singular part of the fundamental matrix $\mathbf{\Gamma}(\mathbf{x})$ in the neighborhood of the origin. $\square$

5 Concluding remark

By the basic properties of the fundamental solution $\mathbf{\Gamma}(\mathbf{x})$ of the system (2.2) it is possible to investigate three-dimensional boundary value problems of the theory of micropolar thermoelasticity without energy dissipation with the potential method. The main results obtained in the classical theory of elasticity, thermoelasticity and micropolar theory of elasticity with the potential method are given in the book of Kupradze and al. [20].

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