THE FOURTH ORDER ACCURACY DECOMPOSITION SCHEME FOR A MULTI-DIMENSIONAL EVOLUTION PROBLEM

J. Rogava, M. Tsiklauri

I. Vekua Institute of Applied Mathematics of
Iv. Javakhishvili Tbilisi State University
0143 University Street 2, Tbilisi, Georgia

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Abstract

In the present work, the sequential-parallel type decomposition scheme with fourth order accuracy for a multidimensional evolution problem is constructed. The fourth order accuracy is attained by introducing a complex parameter. For the constructed scheme, there is obtained the explicit a priori estimate for the error of solution to the approximated problem. The relation between two and multi-dimensional decomposition formulas is established.

Key words and phrases: Decomposition method,Semigroup, Operator split method, Trotter formula, Cauchy abstract problem.

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1 Introduction

One of the most effective methods to solve multi-dimensional evolution problems is a decomposition method. Decomposition schemes with first and second order accuracy were constructed in the sixties of the XX century (see [5], [7] and references therein). Q. Sheng has proved that in the real number field there do not exist automatically stable decomposition schemes with an accuracy order higher than two (see [8]). Decomposition schemes are called automatically stable if a sum of the absolute values of its split coefficients (coefficients of exponentials' products) equals to one, and the real parts of exponential powers are positive. In the work [1] there is constructed decomposition schemes with the higher order accuracy, but their corresponding decomposition formulas are not automatically stable. In the works [2]-[4], introducing the complex parameter, we have constructed automatically stable decomposition schemes with third order accuracy for two- and multi-dimensional evolution problems and with fourth order accuracy for two-dimensional evolution problem (evolution problem with the operator A is called m-dimensional, if it can be represented as a sum of

m summands $A = A_1 + ... + A_m$). The new idea is an introduction of a complex parameter, which allows us to break the order 2 barrier. Let us remark that in the work of Schatzman there are constructed decomposition formulas for two-dimensional evolution problem.

Decomposition formulas constructed in the works [2]-[4] represent formulas of exponential splitting. Exponential splitting is called a splitting which approximates a semigroup by a combination of semigroups generated by the summands of the operator generating the given semigroup.

In the present work there is constructed an automatically stable decomposition scheme with the fourth order precision for a multidimensional evolution problem. For the solution error there is obtained an explicit *a priori* estimate. This work naturally proceeds from the articles [2]-[4] and it summarizes these articles from some point of view. In our opinion, the proposition given in the fourth chapter is quite interesting and, taking it as a base, from a two-dimensional decomposition formula, one can obtain a multidimensional decomposition formula with the same order. However for the moment we cannot prove this proposition.

2 Statement of the Problem

Let us consider the Cauchy problem for an evolution equation in the Banach space X:

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = \varphi, \tag{2.1}$$

where A is a linear closed operator with a definition domain $D(A)$, which is everywhere dense in X, φ is a given element from $D(A)$.

Suppose that the operator $(-A)$ generates a strongly continuous semigroup $\{exp(-tA)\}_{t\geq 0}$. Then the solution of problem (2.1) is given by the following formula (see for example [6]):

$$u(t) = U(t, A)\varphi, \tag{2.2}$$

with $U(t, A) = exp(-tA)$.

Let $A = A_1 + A_2 + ... + A_m$, $m \geq 2$, where A_i $(i = 1, ..., m)$ are closed operators, densely defined in X.

Let us introduce a grid set:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Together with problem (2.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$
\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\alpha}{2}A_i v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dv_k^{(m)}(t)}{dt} + \frac{1}{2}A_m v_k^{(m)}(t) &= 0, \quad v_k^{(m)}(t_{k-1}) = v_k^{(m-1)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_{2m-i} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= m+1, ..., 2m-2, \\
\frac{dv_k^{(2m-1)}(t)}{dt} + \overline{\alpha}A_1 v_k^{(2m-1)}(t) &= 0, \quad v_k^{(2m-1)}(t_{k-1}) = v_k^{(2m-2)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_{i-2m+2} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 2m, ..., 3m-3, \\
\frac{dv_k^{(3m-2)}(t)}{dt} + \frac{1}{2}A_m v_k^{(3m-2)}(t) &= 0, \quad v_k^{(3m-2)}(t_{k-1}) = v_k^{(3m-3)}(t_k), \\
\frac{dv_k^{(i)}(t)}{dt} + \frac{\alpha}{2}A_{4m-i-2} v_k^{(i)}(t) &= 0, \quad v_k^{(i)}(t_{k-1}) = v_k^{(i-1)}(t_k), \\
i &= 3m-1, ..., 4m-4, \\
\frac{dv_k^{(4m-4)}(t)}{dt} + \frac{\alpha}{2}A_1 v_k^{(4m-4)}(t) &= 0, \quad v_k^{(4m-4)}(t_{k-1}) = v_k^{(4m-3)}(t_k),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dw_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_m w_k^{(1)}(t) &= 0, \quad w_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\alpha}{2}A_{m-i+1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 2, ..., m-1, \\
\frac{dw_k^{(m)}(t)}{dt} + \frac{1}{2}A_1 w_k^{(m)}(t) &= 0, \quad w_k^{(m)}(t_{k-1}) = w_k^{(m-1)}(t_k), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_{i-m+1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= m+1, ..., 2m-2, \\
\frac{dw_k^{(2m-1)}(t)}{dt} + \overline{\alpha}A_m w_k^{(2m-1)}(t) &= 0, \quad w_k^{(2m-1)}(t_{k-1}) = w_k^{(2m-2)}(t_k),
\end{aligned}
$$

$$
\begin{aligned}
\frac{dw_k^{(i)}(t)}{dt} + \frac{\overline{\alpha}}{2} A_{3m-i-1} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 2m, ..., 3m-3, \\
\frac{dw_k^{(3m-2)}(t)}{dt} + \frac{1}{2} A_1 w_k^{(3m-2)}(t) &= 0, \quad w_k^{(3m-2)}(t_{k-1}) = w_k^{(3m-3)}(t_k), \\
\frac{dw_k^{(i)}(t)}{dt} + \frac{\alpha}{2} A_{i-3m+3} w_k^{(i)}(t) &= 0, \quad w_k^{(i)}(t_{k-1}) = w_k^{(i-1)}(t_k), \\
i &= 3m-1, ..., 4m-4, \\
\frac{dw_k^{(4m-4)}(t)}{dt} + \frac{\alpha}{2} A_m w_k^{(4m-4)}(t) &= 0, \quad w_k^{(4m-4)}(t_{k-1}) = w_k^{(4m-3)}(t_k),
\end{aligned}
$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Let the operators $(-A_j), (-\alpha A_j), (-\overline{\alpha} A_j)$, $j = 1, ..., m$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, ..,$ is defined on each interval $[t_{k-1}, t_k]$, as follows:

$$
u_k(t) = \frac{1}{2}\left[v_k^{(4m-4)}(t) + w_k^{(4m-4)}(t)\right]. \tag{2.3}
$$

We declare function $u_k(t)$ as an approximated solution of problem (2.1) on each interval $[t_{k-1}, t_k]$.

3 Estimate of Error of the Approximated Solution.

It is obvious that the definition domain $D(A^s)$ of the operator A^s represents an intersection of the definition domains of its addends.

Let us introduce the following notations:

$$
\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\| + ... + \|A_m\varphi\|, \quad \varphi \in D(A), \\
\|\varphi\|_{A^2} &= \sum_{i,j=1}^{m} \|A_i A_j \varphi\|, \quad \varphi \in D\left(A^2\right),
\end{aligned}
$$

where $\|\cdot\|$ is a norm in X; Analogously is defined $\|\varphi\|_{A^s}$, $(s = 3, 4, 5)$.

Theorem 3.1 *Let the following conditions be fulfilled:*

(a) $\alpha = \frac{1}{2} \pm i\frac{1}{2\sqrt{3}}$ $\left(i = \sqrt{-1}\right)$;

(b) Let the operators $(-\gamma A_j)$, $\gamma = 1, \alpha, \overline{\alpha}$ $(j = 1, ..., m, \quad m \geq 2)$ *and* $(-A)$ *generate strongly continuous semigroups, for which the following estimates are true:*

$$
\begin{aligned}
\|U(t, \gamma A_j)\| &\leq e^{\omega t}, \\
\|U(t, A)\| &\leq M e^{\omega t}, \quad M, \omega = const > 0;
\end{aligned}
$$

(c) $U(s,A)\varphi \in D\left(A^5\right)$ *for each fixed* $s \geq 0$ *.*
Then the following estimate holds:

$$\|u(t_k) - u_k(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^5},$$

where c *and* ω_0 *are positive constants.*

Proof. According to the following formula (see [6], p 603),

$$A\int_r^t U(s,A)\,ds = U(r,A) - U(t,A), \quad 0 \leq r \leq t,$$

we can obtain the following expansion:

$$U(t,A) = \sum_{i=0}^{k-1}(-1)^i \frac{t^i}{i!}A^i + R_k(t,A), \tag{3.1}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} U(s,A)ds ds_{k-1}\ldots ds_1. \tag{3.2}$$

From formula (2.3) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{3.3}$$

where

$$V(\tau) = \frac{1}{2}\left[V_1(\tau) + V_2(\tau)\right], \tag{3.4}$$

and

$$\begin{aligned}
V_1(\tau) \;=\;& U\left(\tau, \frac{\alpha}{2}A_1\right)\ldots U\left(\tau, \frac{\alpha}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\
&\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)\ldots U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \overline{\alpha}A_1\right) \\
&\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\ldots U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\
&\times U\left(\tau, \frac{\alpha}{2}A_{m-1}\right)\ldots U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right),
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
V_2(\tau) \;=\;& U\left(\tau, \frac{\alpha}{2}A_m\right)\ldots U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) \\
&\times U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\ldots U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right) U\left(\tau, \overline{\alpha}A_m\right) \\
&\times U\left(\tau, \frac{\overline{\alpha}}{2}A_{m-1}\right)\ldots U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) \\
&\times U\left(\tau, \frac{\alpha}{2}A_2\right)\ldots U\left(\tau, \frac{\alpha}{2}A_{m-1}\right) U\left(\tau, \frac{\alpha}{2}A_m\right).
\end{aligned} \tag{3.6}$$

Remark 3.1 *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem 3.1. In this case, for the solving operator, the following estimate holds:*

$$\left\| V^k(\tau) \right\| \le e^{\omega_1 t_k}, \tag{3.7}$$

where ω_1 is positive constant.

Let us introduce the following notations for combinations (sum, product) of semigroups: Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1, ..., m)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (3.1), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $\left(\tau^2 A_i A_j\right)$, $\left(-\tau^3 A_i A_j A_k\right)$ and $\left(\tau^4 A_i A_j A_k A_l\right)$ $(i, j, k, l = 1, ..., m)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (3.1) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{m} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{m} [V(\tau)]_{i,j} A_i A_j \\ &\quad -\tau^3 \sum_{i,j,k=1}^{m} [V(\tau)]_{i,j,k} A_i A_j A_k \\ &\quad +\tau^4 \sum_{i,j,k,l=1}^{m} [V(\tau)]_{i,j,k,l} A_i A_j A_k A_l + R_5(\tau). \end{aligned} \tag{3.8}$$

According to the first inequality of the condition *(b)* of the Theorem 3.1, for $R_5(\tau)$, the following estimate holds:

$$\| R_5(\tau) \varphi \| \le c e^{\omega_0 \tau} \tau^5 \| \varphi \|_{A^5}, \quad \varphi \in D\left(A^5\right), \tag{3.9}$$

where c and ω_0 are positive constants. It is obvious that, for the coefficients in formula (3.8), according to formula (3.4), we have:

$$\begin{aligned} [V(\tau)]_i &= \frac{1}{2}\left([V_1(\tau)]_i + [V_2(\tau)]_i\right), & (3.10) \\ i &= 1, ..., m, \\ [V(\tau)]_{i,j} &= \frac{1}{2}\left([V_1(\tau)]_{i,j} + [V_2(\tau)]_{i,j}\right), & (3.11) \\ i, j &= 1, ..., m, \end{aligned}$$

$$\begin{aligned} [V(\tau)]_{i,j,k} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k} + [V_2(\tau)]_{i,j,k}\right), \qquad (3.12)\\ i,j,k &= 1,...,m,\\ [V(\tau)]_{i,j,k,l} &= \frac{1}{2}\left([V_1(\tau)]_{i,j,k,l} + [V_2(\tau)]_{i,j,k,l}\right), \qquad (3.13)\\ i,j,k,l &= 1,...,m. \end{aligned}$$

Let us state the auxiliary lemma, which will be basis of the proof of the Theorem 3.1.

From the theorem proven in the work [6] it follows that if conditions (a) and (b) of the Theorem 3.1 are fulfilled and $m = 2$, then the following expansion is true:

$$V(\tau) = I - \tau A + \frac{\tau^2}{2}A^2 - \frac{\tau^3}{6}A^3 + \frac{\tau^4}{24}A^4 + R_5(\tau), \qquad (3.14)$$

where for the remainder term $R_5(\tau)$, the following estimate takes place:

$$\|R_5(\tau)\varphi\| \leq ce^{\omega_0\tau}\tau^5 \sup_{s\in[0,\tau]} \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right).$$

Let us make a remark which will simplify a calculation of coefficients in decomposition (3.8).

Remark 3.2 *The operators $V_1(\tau)$ and $V_2(\tau)$ are symmetric in the sense that in their expressions the factors equally remote from the ends coincide with each other. Therefore we have:*

$$\begin{aligned} [V(\tau)]_{i,j} &= [V(\tau)]_{j,i}, \quad i,j = 1,...,m;\\ [V(\tau)]_{i,j,k} &= [V(\tau)]_{k,j,i}, \quad i,j,k = 1,...,m;\\ [V(\tau)]_{i,j,k,l} &= [V(\tau)]_{l,k,j,i}, \quad i,j,k,l = 1,...,m. \end{aligned}$$

Let us calculate the coefficients $[V(\tau)]_i$ $(i = 1,...,m)$ corresponding to the first order members in formula (3.8). It is obvious that the members, corresponding to these coefficients, can be obtained from the decomposition of only those factors (semigroups) of the operators $V_1(\tau)$ and $V_2(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identity operators) will participate.

According to formulas (3.5) and (3.6), for any i have:

$$[V_1(\tau)]_i = [U(\tau, A_i)]_i = 1, \quad [V_2(\tau)]_i = [U(\tau, A_i)]_i = 1.$$

From here, according to formula (3.10), we obtain:

$$[V(\tau)]_i = 1. \qquad (3.15)$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i,j=1,...,m)$ corresponding to the second order members in formula (3.8). It is obvious that the members, corresponding to these coefficients, can be obtained from the decomposition of only those factors (semigroups) of the operators $V_1(\tau)$ and $V_2(\tau)$, which are generated by the operators $(-\gamma A_i)$ and $(-\gamma A_j)$, and from the decomposition of other semigroups only first addends (the members with identity operators) will participate. Let $i_1=\min(i,j)$ and $i_2=\max(i,j)$, then from formula (3.11), with account of (3.5) and (3.6), we obtain:

$$\begin{aligned} [V(\tau)]_{i,j} &= \frac{1}{2}\left(\left[U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\overline{\alpha}A_{i_1}\right)\right.\right. \\ &\quad \left.\times U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)\right]_{i,j} \\ &\quad + U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\overline{\alpha}A_{i_2}\right)\Big] \\ &\quad \left.\times U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)\Big]_{i,j}\right). \end{aligned}$$

From here, according to (3.14), we obtain:

$$[V(\tau)]_{i,j}=\frac{1}{2}. \tag{3.16}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k=1,...,m)$ corresponding to the third order members in formula (3.8). For $i=j=k$, according to formulas (3.5) and (3.6), we have:

$$\begin{aligned} [V_1(\tau)]_{i,i,i} &= [U(\tau,A_i)]_{i,i,i}=\frac{1}{6}, \\ [V_2(\tau)]_{i,i,i} &= [U(\tau,A_i)]_{i,i,i}=\frac{1}{6}. \end{aligned}$$

From here, according to formula (3.12), we obtain:

$$[V(\tau)]_{i,i,i}=\frac{1}{6}. \tag{3.17}$$

Let us consider the case when only two of the indices i,j and k differ from each other. Let $i_1=\min(i,j,k)$ and $i_2=\max(i,j,k)$, then from formula (3.12), with account of (3.5) and (3.6), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{2}\left(U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\overline{\alpha}A_{i_1}\right)\right.$$

$$U\left(\tau,\frac{1}{2}A_{i_2}\right)U\left(\tau,\frac{\alpha}{2}A_{i_1}\right)\Big]_{i,j,k}$$
$$+\Big[U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\overline{\alpha}A_{i_2}\right)$$
$$+U\left(\tau,\frac{1}{2}A_{i_1}\right)U\left(\tau,\frac{\alpha}{2}A_{i_2}\right)\Big]_{i,j,k}\Bigg).$$

From here, according to (3.14), we obtain:

$$[V(\tau)]_{i,j,k}=\frac{1}{6}, \tag{3.18}$$

for any indices i,j and k, where only two of them differ from each other.

Let us consider the case when the indices i,j and k differ from each other. If $i<j<k$ then, according to formula (3.5), the representation is valid:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k} &= \Big[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{1}{2}A_k\right)\\
&\times U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)U\left(\tau,\overline{\alpha}A_i\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)\\
&\times U\left(\tau,\frac{1}{2}A_k\right)U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{\alpha}{2}A_i\right)\Big]_{i,j,k}\\
&= \left[U\left(\tau,\frac{\alpha}{2}A_i\right)\right]_i\left[U\left(\tau,\frac{\alpha}{2}A_j\right)\right]_j\left[U\left(\tau,\frac{1}{2}A_k\right)\right]_k\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)\right]_i\left[U\left(\tau,\frac{\alpha}{2}A_j\right)\right]_j\left[U\left(\tau,\frac{1}{2}A_k\right)\right]_k\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)\right]_i\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)\right]_j\left[U\left(\tau,\frac{1}{2}A_k\right)\right]_k\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_i\right)\right]_i\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)\right]_j\left[U\left(\tau,\frac{1}{2}A_k\right)\right]_k\\
&+\left[U\left(\tau,\overline{\alpha}A_i\right)\right]_i\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)\right]_j\left[U\left(\tau,\frac{1}{2}A_k\right)\right]_k\\
&= \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}+\overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2}\\
&= \frac{\alpha^2+\alpha\overline{\alpha}+\overline{\alpha}^2}{4}=\frac{1}{6}.
\end{aligned} \tag{3.19}$$

Here we used the identities: $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$, $\alpha\overline{\alpha}=\frac{1}{3}$. Analogously from (3.21) we obtain:

$$[V_2(\tau)]_{i,j,k}=\frac{1}{6},\quad i<j<k. \tag{3.20}$$

From formula (3.12), with account of formulas (3.19) and (3.20), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i < j < k. \tag{3.21}$$

From here, due to **Remark 3.2**, we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad k < j < i. \tag{3.22}$$

Now consider the case $j < i < k$. Due to formula (3.5), the representation is valid:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k} &= \Big[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{1}{2}A_k\right) \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) U\left(\tau, \overline{\alpha}A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right) \\
&\quad \times U\left(\tau, \frac{1}{2}A_k\right) U\left(\tau, \frac{\alpha}{2}A_i\right) U\left(\tau, \frac{\alpha}{2}A_j\right)\Big]_{i,j,k} \\
&= \left[U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&\quad + \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha}A_j\right)\right]_j \left[U\left(\tau, \frac{1}{2}A_k\right)\right]_k \\
&= \frac{\alpha}{2}\overline{\alpha}\frac{1}{2} + \frac{\overline{\alpha}}{2}\overline{\alpha}\frac{1}{2} = \frac{\alpha\overline{\alpha} + \overline{\alpha}^2}{4} = \frac{\overline{\alpha}}{4}.
\end{aligned} \tag{3.23}$$

Analogously, from (3.21) we obtain:

$$[V_2(\tau)]_{i,j,k} = \frac{\alpha\overline{\alpha} + \alpha}{4}, \quad j < i < k. \tag{3.24}$$

From formula (3.12), with account of formulas (3.23) and (3.24), we obtain:

$$[V(\tau)]_{i,j,k} = \frac{\overline{\alpha} + \alpha\overline{\alpha} + \alpha}{8} = \frac{1}{6}, \quad j < i < k. \tag{3.25}$$

From here, due to **Remark 3.2**, we obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad k < i < j. \tag{3.26}$$

Now consider the case $j < k < i$. According to formula (3.5), the representation is valid:

$$[V_1(\tau)]_{i,j,k} = \Big[U\left(\tau, \frac{\alpha}{2}A_j\right) U\left(\tau, \frac{\alpha}{2}A_k\right) U\left(\tau, \frac{1}{2}A_i\right)$$

$$
\begin{aligned}
& \times U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) U\left(\tau, \overline{\alpha} A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) \\
& \left.\times U\left(\tau, \frac{1}{2} A_i\right) U\left(\tau, \frac{\alpha}{2} A_k\right) U\left(\tau, \frac{\alpha}{2} A_j\right)\right]_{i,j,k} \\
= & \left[U\left(\tau, \frac{1}{2} A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha} A_j\right)\right]_j \left[U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right)\right]_k \\
& + \left[U\left(\tau, \frac{1}{2} A_i\right)\right]_i \left[U\left(\tau, \overline{\alpha} A_j\right)\right]_j \left[U\left(\tau, \frac{\alpha}{2} A_k\right)\right]_k \\
= & \frac{1}{2}\overline{\alpha}\frac{\overline{\alpha}}{2} + \frac{1}{2}\overline{\alpha}\frac{\alpha}{2} = \frac{\overline{\alpha}^2 + \alpha\overline{\alpha}}{4} = \frac{\overline{\alpha}}{4}. \qquad (3.27)
\end{aligned}
$$

Analogously, from (3.21) we obtain:

$$
[V_2(\tau)]_{i,j,k} = \frac{\alpha\overline{\alpha} + \alpha}{4}, \quad j < k < i. \qquad (3.28)
$$

From formula (3.12), with account of formulas (3.27) and (3.28), we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{\overline{\alpha} + \alpha\overline{\alpha} + \alpha}{8} = \frac{1}{6}, \quad j < k < i. \qquad (3.29)
$$

From here, due to **Remark 3.2**, we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i < k < j. \qquad (3.30)
$$

Uniting formulas (3.17),(3.18),(3.21),(3.22),(3.25),(3.26),(3.29) and (3.30), we obtain:

$$
[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i,j,k = 1,...,m. \qquad (3.31)
$$

Let us calculate the coefficients $[V(\tau)]_{i,j,k,l}$ $(i,j,k,l = 1,...,m)$ corresponding to the fourth order members in formula (3.8). In the case when $i = j = k = l$, due to formulas (3.5) and (3.6), we obtain:

$$
\begin{aligned}
[V_1(\tau)]_{i,i,i,i} &= [U(\tau, A_i)]_{i,i,i,i} = \frac{1}{24}, \\
[V_2(\tau)]_{i,i,i,i} &= [U(\tau, A_i)]_{i,i,i,i} = \frac{1}{24}.
\end{aligned}
$$

From here, according to formula (3.13), we obtain:

$$
[V(\tau)]_{i,i,i,i} = \frac{1}{24}. \qquad (3.32)
$$

Let us consider the case when only two of the indices i, j, k and l differ from each other. Let $i_1 = \min(i,j,k,l)$ and $i_2 = \max(i,j,k,l)$, then from formula (3.13), with account of (3.5) and (3.6), the representation is valid:

$$
\begin{aligned}
[V(\tau)]_{i,j,k,l} \quad = \quad & \frac{1}{2}\left(\left[U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)U\left(\tau, \frac{1}{2}A_{i_2}\right)U\left(\tau, \overline{\alpha}A_{i_1}\right)\right.\right. \\
& \left. U\left(\tau, \frac{1}{2}A_{i_2}\right)U\left(\tau, \frac{\alpha}{2}A_{i_1}\right)\right]_{i,j,k,l} \\
& + \left[U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \overline{\alpha}A_{i_2}\right)\right. \\
& \left.\left. + \, U\left(\tau, \frac{1}{2}A_{i_1}\right)U\left(\tau, \frac{\alpha}{2}A_{i_2}\right)\right]_{i,j,k,l}\right).
\end{aligned}
$$

From here, due to (3.14), we obtain:

$$
[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \tag{3.33}
$$

for any indices i, j, k and l, where only two of them differ from each other.

Let us consider the case when only two of the indices i, j, k and l coincide with each other. On the whole, we have six cases, namely:

Case 1. $(i, j, k, l) = (i, j, k, i)$,
Case 2. $(i, j, k, l) = (i, j, i, k)$,
Case 3. $(i, j, k, l) = (i, i, j, k)$,
Case 4. $(i, j, k, l) = (i, j, k, j)$,
Case 5. $(i, j, k, l) = (i, j, j, k)$,
Case 6. $(i, j, k, l) = (i, j, k, k)$.

Comparing i, j and k indices we get six different subcases for each case. Let us consider **Case 1** and calculate its corresponding coefficients. The coefficients, corresponding to five other cases, can be calculated analogously.

Let us consider the subcases of **Case 1**:

Subcase 1.1. $i < j < k$. Due to formula (3.5) we have:

$$
\begin{aligned}
[V_1(\tau)]_{i,j,k,i} \quad = \quad & \left[U\left(\tau, \frac{\alpha}{2}A_i\right)U\left(\tau, \frac{\alpha}{2}A_j\right)U\left(\tau, \frac{1}{2}A_k\right)\right. \\
& \times U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right)U\left(\tau, \overline{\alpha}A_i\right)U\left(\tau, \frac{\overline{\alpha}}{2}A_j\right) \\
& \left.\times U\left(\tau, \frac{1}{2}A_k\right)U\left(\tau, \frac{\alpha}{2}A_j\right)U\left(\tau, \frac{\alpha}{2}A_i\right)\right]_{i,j,k,i} \\
= \quad & \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\overline{\alpha} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}\frac{\alpha}{2} \\
& + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} + \frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} + \overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2}\frac{\alpha}{2} \\
= \quad & \frac{\alpha^3 + 2\alpha^2\overline{\alpha} + \overline{\alpha}^2\alpha}{8} = \frac{\alpha^2\left(\alpha + \overline{\alpha}\right) + \overline{\alpha}\alpha\left(\alpha + \overline{\alpha}\right)}{8}
\end{aligned}
$$

$$= \frac{\alpha^2 + \overline{\alpha}\alpha}{8}. \tag{3.34}$$

Analogously we obtain

$$[V_2(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}^2}{8}. \tag{3.35}$$

From formula (3.13), with account of (3.34) and (3.35), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{\alpha^2 + \overline{\alpha}\alpha + \overline{\alpha}^2}{16} = \frac{1}{24}, \quad i < j < k. \tag{3.36}$$

Subcase 1.2. $k < j < i$. From formula (3.36), due to **Remark 3.2**, we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad k < j < i. \tag{3.37}$$

Subcase 1.3. $j < k < i$. According to formula (3.5), we have:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k,i} &= \Big[U\left(\tau, \frac{\alpha}{2} A_j\right) U\left(\tau, \frac{\alpha}{2} A_k\right) U\left(\tau, \frac{1}{2} A_i\right) \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) U\left(\tau, \overline{\alpha} A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_k\right) \\
&\quad \times U\left(\tau, \frac{1}{2} A_i\right) U\left(\tau, \frac{\alpha}{2} A_k\right) U\left(\tau, \frac{\alpha}{2} A_j\right) \Big]_{i,j,k,i} \\
&= \frac{1}{2} \overline{\alpha} \frac{\overline{\alpha}}{2} \frac{1}{2} = \frac{\overline{\alpha}^2}{8}.
\end{aligned} \tag{3.38}$$

Analogously we obtain:

$$[V_2(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}\alpha + \alpha^2}{8}. \tag{3.39}$$

From formula (3.13), with account of (3.38) and (3.39), we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{\overline{\alpha}^2 + \overline{\alpha}\alpha + \alpha^2}{16} = \frac{1}{24}, \quad j < k < i. \tag{3.40}$$

Subcase 1.4. $i < k < j$. From formula (3.40), due to **Remark 3.2**, we obtain:

$$[V(\tau)]_{i,j,k,i} = \frac{1}{24}, \quad i < k < j. \tag{3.41}$$

Subcase 1.5. $j < i < k$. According to formula (3.5), we have:

$$\begin{aligned}
[V_1(\tau)]_{i,j,k,i} &= \Big[U\left(\tau, \frac{\alpha}{2} A_j\right) U\left(\tau, \frac{\alpha}{2} A_i\right) U\left(\tau, \frac{1}{2} A_k\right) \\
&\quad \times U\left(\tau, \frac{\overline{\alpha}}{2} A_i\right) U\left(\tau, \overline{\alpha} A_j\right) U\left(\tau, \frac{\overline{\alpha}}{2} A_i\right)
\end{aligned}$$

$$\begin{aligned}
&\times U\left(\tau,\frac{1}{2}A_k\right)U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\frac{\alpha}{2}A_j\right)\Big]_{i,j,k,i}\\
=&\ \frac{\alpha}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2}+\frac{\overline{\alpha}}{2}\overline{\alpha}\frac{1}{2}\frac{\alpha}{2}=\frac{\alpha^2\overline{\alpha}+\overline{\alpha}^2\alpha}{8}\\
=&\ \frac{\alpha\overline{\alpha}\left(\alpha+\overline{\alpha}\right)}{8}=\frac{1}{24}. \qquad (3.42)
\end{aligned}$$

Analogously, from (3.21), we obtain:

$$[V_2(\tau)]_{i,j,k,i}=\frac{1}{24}. \qquad (3.43)$$

From formula (3.13), with account of (3.42) and (3.43), we obtain:

$$[V(\tau)]_{i,j,k,i}=\frac{1}{24},\quad j<i<k. \qquad (3.44)$$

Subcase 1.6. $k<i<j$. From formula (3.44), due to **Remark 3.2**, we obtain:

$$[V(\tau)]_{i,j,k,i}=\frac{1}{24},\quad k<i<j. \qquad (3.45)$$

Uniting formulas (3.36),(3.37),(3.40),(3.41),(3.44) and (3.45), we obtain:

$$[V(\tau)]_{i,j,k,i}=\frac{1}{24}, \qquad (3.46)$$

for any indices i,j and k different from each other. Analogously, for other five cases, we obtain:

$$\begin{aligned}
[V(\tau)]_{i,j,i,k} &= [V(\tau)]_{i,i,j,k}=[V(\tau)]_{i,j,k,j}\\
&= [V(\tau)]_{i,j,j,k}=[V(\tau)]_{i,j,k,k}=\frac{1}{24}, \qquad (3.47)
\end{aligned}$$

for any indices i,j and k different from each other.

Uniting formulas (3.46) and (3.47), we obtain:

$$[V(\tau)]_{i,j,k,l}=\frac{1}{24}, \qquad (3.48)$$

for any indices i,j,k and l, where only two of them coincide with each other.

Now let us consider the case when the indices i,j,k and l are different. It is obvious that comparing i,j,k and l indices we get twenty four different cases. Let us consider one of them and calculate its corresponding coefficients (the coefficients corresponding to other cases can be calculated analogously).

Let $i<j<k<l$, then according to formula (3.5), we obtain:

$$
\begin{aligned}
[V_1(\tau)]_{i,j,k,l} &= \Big[U\left(\tau,\frac{\alpha}{2}A_i\right)U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{\alpha}{2}A_k\right)\\
&\times U\left(\tau,\frac{1}{2}A_l\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)\\
&\times U\left(\tau,\overline{\alpha}A_i\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_j\right)U\left(\tau,\frac{\overline{\alpha}}{2}A_k\right)\\
&\times U\left(\tau,\frac{1}{2}A_l\right)U\left(\tau,\frac{\alpha}{2}A_k\right)\\
&\times U\left(\tau,\frac{\alpha}{2}A_j\right)U\left(\tau,\frac{\alpha}{2}A_i\right)\Big]_{i,j,k,l}\\
&= \frac{\alpha}{2}\frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\alpha}{2}\frac{\alpha}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}\\
&+\frac{\alpha}{2}\frac{\alpha}{2}\frac{\overline{\alpha}}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{\overline{\alpha}}{2}\frac{1}{2}+\frac{\alpha}{2}\frac{\overline{\alpha}}{2}\frac{\overline{\alpha}}{2}+\overline{\alpha}\frac{\overline{\alpha}}{2}\frac{1}{2}\\
&= \frac{\overline{\alpha}^2\alpha+\overline{\alpha}\alpha^2+\alpha^3+\overline{\alpha}^3}{8}\frac{\overline{\alpha}}{2}\\
&= \frac{\overline{\alpha}^2+\alpha^2}{8}=\frac{1}{24}. \qquad (3.49)
\end{aligned}
$$

Analogously, from (3.21), we obtain:

$$[V_2(\tau)]_{i,j,k,l}=\frac{1}{24}, \quad i<j<k<l. \qquad (3.50)$$

From formula (3.13), with account of formulas (3.49) and (3.50), we obtain:

$$[V(\tau)]_{i,j,k,l}=\frac{1}{24}, \quad i<j<k<l.$$

Analogously we can show that this equality is valid for other twenty three cases. Therefore we have:

$$[V(\tau)]_{i,j,k,l}=\frac{1}{24}, \qquad (3.51)$$

for any indices i,j,k and l, which differ from each other.

Uniting formulas (3.32), (3.33), (3.48) and (3.51), we obtain:

$$[V(\tau)]_{i,j,k,l}=\frac{1}{24}, \quad i,j,k,l=1,...,m. \qquad (3.52)$$

From equality (3.8), with account of formulas (3.15), (3.16), (3.31) and (3.52), we obtain:

$$V(\tau) = I-\tau\sum_{i=1}^{m}A_i+\frac{1}{2}\tau^2\sum_{i,j=1}^{m}A_iA_j-\frac{1}{6}\tau^3\sum_{i,j,k=1}^{m}A_iA_jA_k$$

$$
\begin{aligned}
& +\frac{1}{24}\tau^4 \sum_{i,j,k,l=1}^{m} A_i A_j A_k A_l + R_5(\tau) \\
= \; & I - \tau \sum_{i=1}^{m} A_i + \frac{1}{2}\tau^2 \left(\sum_{i=1}^{m} A_i \right)^2 \\
& -\frac{1}{6}\tau^3 \left(\sum_{i=1}^{m} A_i \right)^3 + \frac{1}{24}\tau^4 \left(\sum_{i=1}^{m} A_i \right)^4 + R_5(\tau) \\
= \; & I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_5(\tau). \qquad (3.53)
\end{aligned}
$$

According to formula (3.1), we have:

$$
U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_5(\tau, A). \qquad (3.54)
$$

According to the second inequality of condition *(b)* of the Theorem 3.1 for $R_5(\tau, A)$, the following estimate is valid:

$$
\begin{aligned}
\|R_5(\tau, A)\varphi\| & \leq ce^{\omega\tau}\tau^5 \left\|A^5\varphi\right\| \\
& \leq ce^{\omega\tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \qquad (3.55)
\end{aligned}
$$

According to equalities (3.53) and (3.54), we have:

$$
U(\tau, A) - V(\tau) = R_5(\tau, A) - R_5(\tau).
$$

From here, with account of inequalities (3.9) and (3.55), the following estimation can be obtained:

$$
\|[U(\tau, A) - V(\tau)]\varphi\| \leq ce^{\omega_2\tau}\tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D\left(A^5\right). \qquad (3.56)
$$

From equalities (2.2) and (3.3), with account of inequalities (3.7) and (3.56), we obtain:

$$
\begin{aligned}
\|u(t_k) - u_k(t_k)\| & = \left\| \left[U(t_k, A) - V^k(\tau) \right] \varphi \right\| \\
& = \left\| \left[U^k(\tau, A) - V^k(\tau) \right] \varphi \right\| \\
& = \left\| \sum_{i=1}^{k} V^{k-i}(\tau) \left[U(\tau, A) - V(\tau) \right] U((i-1)\tau, A)\varphi \right\| \\
& \leq \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \\
& \quad \times \|[U(\tau, A) - V(\tau)] U((i-1)\tau, A)\varphi\|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{i=1}^{k} e^{\omega_1 (k-i)\tau} c e^{\omega_2 \tau} \tau^5 \left\| U\left((i-1)\tau, A\right)\varphi \right\|_{A^5} \\
&\leq c e^{\omega_0 t_k} \tau^5 \sum_{i=1}^{k} \left\| U\left((i-1)\tau, A\right)\varphi \right\|_{A^5} \\
&\leq k c e^{\omega_0 t_k} \tau^5 \sup_{s\in[o,t_k]} \left\| U\left(s, A\right)\varphi \right\|_{A^5} \\
&\leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[o,t_k]} \left\| U\left(s, A\right)\varphi \right\|_{A^5} \qquad \blacksquare
\end{aligned}
$$

4 Relation between two-dimensional and multi-dimensional decomposition formulas

In this section we propose a method by means of which in our opinion it is available on the basis of two-dimensional decomposition formula to construct a multi-dimensional decomposition formula with the same precision order. Let the two-dimensional decomposition formula has the following form:

$$
V^{(2)}\left(\tau; A_1, A_2\right) = \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} U\left(\tau, \alpha_j^{(i)} A_1\right) U\left(\tau, \beta_j^{(i)} A_2\right), \tag{4.1}
$$

where parameters σ_i, $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ satisfy the following conditions (weights σ_i are real numbers, and $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ are generally complex numbers):

$$
\sum_{i=1}^{q} \sigma_i = 1, \tag{4.2}
$$

$$
\sum_{i=1}^{q} \sigma_i \sum_{j=1}^{m_i} \alpha_j^{(i)} = \sum_{i=1}^{q} \sigma_i \sum_{j=1}^{m_i} \beta_j^{(i)} = 1. \tag{4.3}
$$

In the formula (4.1) we mean, that $U\left(\tau, \gamma A_l\right) = I$ $(l = 1, 2)$, when $\gamma = 0$.

For the given method it is necessary that the parameters $\alpha_j^{(i)}$ and $\beta_j^{(i)}$ additionally satisfy the following conditions:

$$
\sum_{j=1}^{m_i} \alpha_j^{(i)} = \sum_{j=1}^{m_i} \beta_j^{(i)}, \quad i = 1, ..., q. \tag{4.4}
$$

At the first step of the method the formula (4.1) is written in such a form that one can clearly see its generalization for the multi-dimensional case. For this reason the formula (4.1) is written in the following form:

$$\begin{aligned} V^{(2)}\left(\tau; A_1, A_2\right) &= \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} U\left(\tau, \mu_{1,j}^{(i)} A_1\right) U\left(\tau, \mu_{1,j}^{(i)} A_2\right) \\ &\quad \times U\left(\tau, \mu_{2,j}^{(i)} A_2\right) U\left(\tau, \mu_{2,j}^{(i)} A_1\right) \\ &= \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} \left(\prod_{l=1}^{2} U\left(\tau, \mu_{1,j}^{(i)} A_l\right) \right) \\ &\quad \times \left(\prod_{l=1}^{2} U\left(\tau, \mu_{2,j}^{(i)} A_{3-l}\right) \right). \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} \mu_{1,j}^{(i)} &= \beta_j^{(i)} + \sum_{k=1}^{j} \left(\alpha_k^{(i)} - \beta_k^{(i)} \right), \\ \mu_{2,j}^{(i)} &= \sum_{k=1}^{j} \left(\beta_k^{(i)} - \alpha_k^{(i)} \right). \end{aligned}$$

For the formula (4.5) to be the equivalent to the formula (4.1), it is necessary to fulfill the following equalities:

$$\begin{aligned} \mu_{1,j}^{(i)} + \mu_{2,j}^{(i)} &= \beta_j^{(i)}, \\ \mu_{2,j}^{(i)} + \mu_{1,j+1}^{(i)} &= \alpha_{j+1}^{(i)}, \\ \mu_{1,m_i}^{(i)} &= \beta_{m_i}^{(i)}, \\ \mu_{2,m_i}^{(i)} &= 0. \end{aligned}$$

It is easy to check that these equalities are fulfilled if the equalities (4.4) are fulfilled.

Let us construct the following decomposition formula on the basis of the formula (4.5):

$$\begin{aligned} V^{(m)}\left(\tau; A_1, ..., A_m\right) &= \sum_{i=1}^{q} \sigma_i \prod_{j=1}^{m_i} \left(\prod_{l=1}^{m} U\left(\tau, \mu_{1,j}^{(i)} A_l\right) \right) \\ &\quad \times \left(\prod_{l=1}^{m} U\left(\tau, \mu_{2,j}^{(i)} A_{m-l+1}\right) \right). \end{aligned} \tag{4.6}$$

Naturally the operators $A_3, ..., A_m$ $(m > 2)$ have to satisfy the same conditions as operators A_1 and A_2. In our opinion, the formula (4.6) constructed for m summands $(A = A_1 + A_2 + ... + A_m)$ will be of the same order as the decomposition formula (4.5) constructed for two summands ($A = A_1 + A_2$).

In the work [3] and present work, using this method there are constructed third and fourth order precision multi-dimensional decomposition formulas.

To illustrate the method, let us consider the following case of Streng formula in detail $\left(V\left(\tau; A_1, A_2\right) = U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, A_2\right) U\left(\tau, \frac{1}{2}A_1\right)\right)$. We write it in the form as (4.5):

$$V^{(2)}\left(\tau; A_1, A_2\right) = U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right).$$

Hence, for a multi-dimensional case we obtain the following formula:

$$\begin{aligned} V^{(m)}\left(\tau; A_1, ..., A_m\right) &= U\left(\tau, \frac{1}{2}A_1\right) ...U\left(\tau, \frac{1}{2}A_{m-1}\right) U\left(\tau, \frac{1}{2}A_m\right) \\ &\quad \times U\left(\tau, \frac{1}{2}A_m\right) U\left(\tau, \frac{1}{2}A_{m-1}\right) ...U\left(\tau, \frac{1}{2}A_1\right) \\ &= U\left(\tau, \frac{1}{2}A_1\right) ...U\left(\tau, \frac{1}{2}A_{m-1}\right) U\left(\tau, A_m\right) \\ &\quad \times U\left(\tau, \frac{1}{2}A_{m-1}\right) ...U\left(\tau, \frac{1}{2}A_1\right). \end{aligned}$$

The given method has not been proven yet, though below we prove the theorem which partially justifies this method.

Theorem 4.1 *Let the decomposition formula (4.6) has the precision order p (≥ 2) at $m = p$. Then the decomposition formula (4.6) will have the same precision order for any m (≥ 2).*

Proof. As following to the condition of the Theorem 4.1, the decomposition formula (4.6) has the precision order p at $p = m$, therefore the equalities are valid:

$$\left[V^{(p)}\left(\tau; A_1, ..., A_p\right)\right]_i = 1, \quad i = 1, ..., p, \tag{4.7}$$

$$\begin{aligned} \left[V^{(p)}\left(\tau, A_1, ..., A_p\right)\right]_{i_1,...,i_s} &= \frac{1}{s!}, \\ i_1, ..., i_s &= 1, ..., p, \quad s = 2, ..., p \end{aligned} \tag{4.8}$$

Therefore it follows that, for any $m \leq p$, the following equalities are valid:

$$\left[V^{(m)}\left(\tau; A_1, ..., A_m\right)\right]_i = 1, \quad i = 1, ..., m, \tag{4.9}$$

$$\begin{aligned} \left[V^{(m)}\left(\tau, A_1, ..., A_m\right)\right]_{i_1,...,i_s} &= \frac{1}{s!}, \\ i_1, ..., i_s &= 1, ..., m, \quad s = 2, ..., p. \end{aligned} \tag{4.10}$$

It means that the decomposition formula (4.6) has the order p for any $m \leq p$. Now let us show that equalities (4.9) and (4.10) are valid for any $m > p$. Validity of equalities (4.9) can be easily checked, as, according to formula (4.3), we have:

$$\begin{aligned}\left[V^{(m)}(\tau; A_1, ...A_m)\right]_i &= \sum_{i=1}^{q} \sigma_i \sum_{j=1}^{m_i} \left(\mu_{1,j}^{(i)} + \mu_{2,j}^{(i)}\right) \\ &= \sum_{i=1}^{q} \sigma_i \sum_{j=1}^{m_i} \beta_j^{(i)} = 1. \end{aligned} \tag{4.11}$$

Let us prove the validity of equalities (4.10) for any $m > p$. Coefficients $\left[V^{(m)}(\tau, A_1, ...A_m)\right]_{i_1,...,i_s}$ can be obtained from the decomposition of only those semigroups which are generated by the operators $(-A_{j_1}), ..., (-A_{j_r})$, where $(j_1, .., j_r)$ is a system of different indices from $(i_1, ..., i_s)$ sorted ascending (for example, if $s = 5$ and $(i_1, i_2, i_3, i_4, i_5) = (3, 3, 1, 2, 1)$, then $r = 3$ and $(j_1, j_2, j_3) = (1, 2, 3)$). From the decompositions of other semigroups, there will participate only first summands (terms with identity operators). Therefore we have:

$$\left[V^{(m)}(\tau, A_1, ..., A_m)\right]_{i_1,...,i_s} = \left[V^{(r)}(\tau, A_{j_1}, ..., A_{j_r})\right]_{i_1,...,i_s}. \tag{4.12}$$

As $r \leq s \leq p$ in the right-hand side of equality (4.12), therefore, according to (4.10) we have:

$$\left[V^{(r)}(\tau, A_{j_1}, ..., A_{j_r})\right]_{i_1,...,i_s} = \frac{1}{s!}, \quad s = 2, ..., p. \tag{4.13}$$

From (4.12) and (4.13) we obtain:

$$\begin{aligned}\left[V^{(m)}(\tau, A_1, ..., A_m)\right]_{i_1,...,i_s} &= \frac{1}{s!}, \\ i_1, ..., i_s &= 1, ..., m, \quad s = 2, ..., p, \quad m > p.\end{aligned} \tag{4.14}$$

From (4.10), (4.11) and (4.14) it follows that decomposition formula (4.6) has a precision order p for any $m \geq 2$. ■

From this theorem it follows that if formula (4.1) has second order precision, then decomposition formula (4.6) will automatically have second order precision (obviously, according to conditions (4.2) and (4.3), decomposition formula (4.6) will always have first order precision).

Below, on basis of the above-described method, we will construct a generalization of third and fourth order precision Schatzman decomposition formulas for any number m (≥ 2) of summands. In case of two summands, these formulas have the following form (see [1]):

$$\begin{aligned} V_1^{(2)}(\tau; A_1, A_2) &= \frac{2}{3}\left[U\left(\tau, \frac{1}{2}A_1\right)U(\tau, A_2)\,U\left(\tau, \frac{1}{2}A_1\right)\right. \\ &\quad \left. + U\left(\tau, \frac{1}{2}A_2\right)U(\tau, A_1)\,U\left(\tau, \frac{1}{2}A_2\right)\right] \\ &\quad -\frac{1}{6}\left(U(\tau, A_1)\,U(\tau, A_2)\right. \\ &\quad \left.\times U(\tau, A_2)\,U(\tau, A_1)\right). \end{aligned} \tag{4.15}$$

$$\begin{aligned} V_2^{(2)}(\tau; A_1, A_2) &= \frac{4}{3}U\left(\tau, \frac{1}{4}A_1\right)U\left(\tau, \frac{1}{2}A_2\right) \\ &\quad \times U\left(\tau, \frac{1}{2}A_1\right)U\left(\tau, \frac{1}{2}A_2\right)U\left(\tau, \frac{1}{4}A_1\right) \\ &\quad -\frac{1}{3}U\left(\tau, \frac{1}{2}A_1\right)U(\tau, A_2)\,U\left(\tau, \frac{1}{2}A_1\right). \end{aligned} \tag{4.16}$$

Decomposition formula (4.15) has third order precision, and decomposition formula (4.16) has fourth order precision. Generalization of these formulas for any number m (≥ 2) of summands will be written as follows:

$$\begin{aligned} V_1^{(m)}(\tau; A_1, ..., A_m) &= \frac{2}{3}\left[U\left(\tau, \frac{1}{2}A_1\right)...U\left(\tau, \frac{1}{2}A_{m-1}\right)U(\tau, A_m)\right. \\ &\quad \times U\left(\tau, \frac{1}{2}A_{m-1}\right)...U\left(\tau, \frac{1}{2}A_1\right) \\ &\quad + U\left(\tau, \frac{1}{2}A_{m-1}\right)...U\left(\tau, \frac{1}{2}A_2\right)U(\tau, A_1) \\ &\quad \left.\times U\left(\tau, \frac{1}{2}A_2\right)...U\left(\tau, \frac{1}{2}A_m\right)\right] \\ &\quad -\frac{1}{6}\left(U(\tau, A_1)...U(\tau, A_m)\right. \\ &\quad \left.\times U(\tau, A_m)...U(\tau, A_1)\right). \end{aligned} \tag{4.17}$$

$$\begin{aligned} V_2^{(m)}(\tau; A_1, ..., A_m) &= \frac{4}{3}U\left(\tau, \frac{1}{4}A_1\right)...U\left(\tau, \frac{1}{4}A_{m-1}\right) \\ &\quad \times U\left(\tau, \frac{1}{2}A_m\right)U\left(\tau, \frac{1}{4}A_{m-1}\right)...U\left(\tau, \frac{1}{4}A_2\right) \\ &\quad U\left(\tau, \frac{1}{2}A_1\right)U\left(\tau, \frac{1}{4}A_2\right)....U\left(\tau, \frac{1}{4}A_{m-1}\right) \end{aligned}$$

$$\times U\left(\tau, \frac{1}{2}A_m\right) U\left(\tau, \frac{1}{4}A_{m-1}\right) ...U\left(\tau, \frac{1}{4}A_1\right)$$
$$-\frac{1}{3}U\left(\tau, \frac{1}{2}A_1\right) ...U\left(\tau, \frac{1}{2}A_{m-1}\right) U\left(\tau, A_m\right)$$
$$\times U\left(\tau, \frac{1}{2}A_{m-1}\right) ...U\left(\tau, \frac{1}{2}A_1\right) \quad (4.18)$$

As a result of some calculations, we have obtained that decomposition formula (4.17) has third order precision for $m = 3$ summands, and decomposition formula (4.18) has fourth order precision for $m = 4$ summands. From here, due to Theorem 4.1 it follows that decomposition formulas (4.17) and (4.18) have respectively third and fourth order precision for any number m (≥ 2) of summands.

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