EXISTENCE OF PERIODS OF OCCUPATION AND FREEDOM IN STOCHASTIC NETWORKS

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Abstract

The work completed in this article is devoted to study the stability of open stochastic networks, and to seek the moments when the network becomes empty or occupied. One will limit oneself to the simple case of open networks i.e. a system of queueing made up of a finished number of stations taken into parallels and each station is equipped by several servers which serve the customers according to discipline FIFO (first int first out). Our goal is to give a detailed proof concerning the existence of occupation and freedom periods in this type of networks by using specific processes theory.

Key words and phrases: renewal processes, specific process, queueing networks.

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1 Introduction

The recent developments of the telecommunications technics make mathematical modelling by queueing networks from now on essential since the first study probabilities carried out into 1917 by Erlang on the modelling of a telephone systems.

Many problems were tackled ; let us quote some work: Moscox Songhurst D.J. (1884) "Subscriber repeat attempts, congestion and the quality of service" Brit-telecom.Technol. *J*.2 - regenerative J.W. Cohen "on regenerative processes in queueing theory" Springer- Verlag.

In the general case, there are two types of queueing networks, open and closed.

The study of queueing networks knew an enormous renewed interest since 1970, it is more and more indispensable to study the great systems starting

from the mathematical models, the queueing networks often are used to solve certain telephone or data-processing problems.
To return the study of this models's type easy, one calls upon the notions of a specific processes and Palm theory, while basing oneself on these two concepts, one can study the stationarity of the networks where the laws of entry and service are general. One needs two very useful properties, one is a property of the recurrence, it is satisfied by a good number of queueing systems, it acts of lapse memory property, the second is the coupling property. By connecting the two properties, one will speak about the **SWAP** i.e stable well autocoupled processes.

The work completed in this article is devoted to study the stability of open stochastic networks; to give the sufficient conditions so that the system is stable, and to seek the moments when the network becomes empty or occupied by considering a queueing system made up of a finished number of stations taken in parallels and each station is equipped by several servers which serve the customers according to discipline FIFO (first in first out).

In its article of 1981, Numelin [9] studied the recurrence in $GI/G/1$ cascade queueing networks, Sigman [11] generalized the results of Numelin and showed that the systems are regenerating by using the discipline RA (Random assignment) which consists in allotting to each customer arriving in the system a free server with an independent uniform probability.
We study a generalization of these results for the queueing systems quoted with the top.

One gives the stability conditions for the network made up of N stations taken in parallels, then a detailed proof of the theorem ensuring the existence of the freedom and occupation moments by making the study station by station.

2 Open queueing Networks with several identical servers

21. Stability study of the network

We consider an open queueing networks with m stations i.e a queueing system in cascade. Station l has q_l servers, $q_l \in \mathbb{N}^*$.

The entry-service process is a marked specific process of renewal with marks in $\mathbb{R}_+^m$, and is defined by $N = \sum_{n\in\mathbb{Z}} \delta_{T_n} \otimes \delta_{B_n}$, on a real flood

$(\Omega, \mathcal{A}, \mathbb{P}, \Theta)$.

$B_n = (B_n^1, \cdots, B_n^m), n \in \mathbb{Z}$ represent the services claimed successively at the m stations by the customer arriving at the moment T_n in the system, This customer is called the n-th customer, he passes successively in each m services at moments $T_n, T_n^1, \cdots, T_n^{m-1}$ where $T_n^l, l \in 1, \cdots, m-1$ indicates the exit moment of the n-th customer of the l-th station or the moment of arrival of this customer to the $(l+1)$-th station.

One notes

$$\mathcal{S}_l = \{s_l = (s_l^1, \cdots, s_l^{q_1}) \in \mathbb{R}^{q_l}/0 \leq s_l^1 \leq \cdots \leq s_l^{q_1}$$

$$\mathcal{S} = \Pi_{l=1}^m \mathcal{S}_l.$$

Let $s = (s_1, \cdots, s_m) \in \mathcal{S}$, for $l \in \{1, \cdots, m\}, \mathcal{W}^l(s,t)$ represents the load of the l-th station at moment $t-0$, with $\mathcal{W}^l(s,0) = s_l$ and we note

$$\mathcal{W}(s,t) = (\mathcal{W}^1(s,t), ..., \mathcal{W}^m(s,t)).$$

One poses

$$\mathcal{W} = (\mathcal{W}(s,t), t \in \mathbb{R}_+, s \in \mathcal{S}).$$

The moments of exit in transient state of the stations are defined by

$$\forall s \in \mathcal{S}, \forall n \in \mathbb{N}, \quad T_n^0(s) = T_n.$$

$$\forall s \in \mathcal{S}, \forall n \in \mathbb{N}, \forall l \in \{1, \cdots, m-1\},$$

$$T_n^l(s) = T_n^{l-1}(s) + W^{l,1}(s, T_n^{l-1}(s)) + B_n^l.$$

In the same way, if there is a stationary regime, one notes $\widehat{W}^l(t)$ the load in stationary regime of the l-th station at moment $t-0$, and we note

$$\widehat{W}(t) = (\widehat{W}^1(t), ..., \widehat{W}^m(t)).$$

The exit moments of l-th station in stationary regime are given by

$$\forall n \in \mathbb{Z}, \quad T_n^0 = T_n.$$

$$\forall n \in \mathbb{Z}, \forall l \in (i, ..., m-1), \quad T_n^l = T_n^{l-1} + \widehat{W}^{l,1}(T_n^{l-1}) + B_n^l.$$

$\sum_{n=0}^{+\infty} \delta_{T^l_n(s)} \otimes \delta_{B^l_n}$ is nonstationary specific process, and

$$\lim_{n \to +\infty} T^l_n(s) = +\infty$$

it makes it possible to define the process $(\mathcal{W}^l(s,t), t \geq 0)$.

$T^l_n(s)$ and T^l_n, with $n \in \mathbb{Z}$ are not inevitably ordered by order ascending of the indices and we don't have inevitably $T^l_0 \leq 0$.

The following theorem gives us a sufficient condition of stability.

Theorem 21..1 (6) *If for all C of the σ-algebra $\sigma\{A_i, i \in \mathbb{Z}\}$, for all $l \in \{1, \cdots, m\}$, for all D_l of the σ-algebra $\sigma\{B^l_i, i \leq 0\}$ and for all E_l of the σ-algebra $\sigma\{B^l_i, i \geq 1\}$:*

$$\widehat{\mathbb{P}}(C) > 0, \widehat{\mathbb{P}}(D_l) > 0 \quad et \quad \widehat{\mathbb{P}}(E_l) > 0 \Longrightarrow \widehat{\mathbb{P}}(C \cap \bigcap_{l=l}^{m} D_l \cap E_l) > 0$$

and if for all l, $\quad l \in \{1, \cdots, m\}$

$$\widehat{\mathbb{E}}(B^l_0) < q_l \widehat{\mathbb{E}}(A),$$

*then the process (W, Q) is a stable well autocoupled process **SWAP**.*

22. Occupation and freedom moments in open queueing networks with several identical servers

Freedom moments

In its article of 1981, Numelin studied the recurrence of $GI/G/1$ in cascade and showed that under the assumption $\sum_{l=1}^{m} b^l < a$, the system becomes empty an infinity of time. Us, we study a generalization of this result for open networks with several identical servers. One supposes here that $N = \sum_{n\mathbb{Z}} \varepsilon_{T_n} \otimes \varepsilon_{B_n}$ is a marked specific process of renewal and that the sequences of random variables $(A_n, n \in \mathbb{Z}), (B^1_n, n \in \mathbb{Z}), \ldots, (B^m_n, n \in \mathbb{Z})$ are independent and for all $l \in \{1, \ldots, m\}$, one defines

$$\begin{aligned}
&b^l = essinf\ B^l, \quad b = (b^1, \ldots, b^m) \\
&a = esssup\ A \\
&j_l = \max\{k \in \mathbb{N} / ka \leq b^l\}, \quad j = \textstyle\sum_{l=1}^{m} j_l \\
&k_l = \max\{k' \in \mathbb{N} / k'a \leq \textstyle\sum_{i=1}^{l} b^i\} \\
&h_l = j_l + k_{l-1} + 1.
\end{aligned}$$

We have

$$j_1 + \ldots + j_l \leq k_l \leq j_1 + \ldots + j_l + l - 1$$

While posing $h = h_m$, one will have (according to the assumption of stability)

$$\forall l \quad , 1 \leq l \leq m : \quad 0 \leq j_l < q_l.$$

Proposition 22..1 *Under the assumption of stability*

$$\widehat{\mathbb{E}}(B_0^l) < q_l \widehat{\mathbb{E}}(A), \quad \forall l, \quad 1 \leq l \leq m$$

and if $\sum_{l=1}^{m} b^l < a$ *, then for all initial state* x *in* $\mathcal{S}$*, with probability equalizes to* 1*, the system empties an infinity of time.*

Proof

So that the demonstration is clear, one considers initially the case where the sequences of random variables $(A_n, n \in \mathbb{Z})$ and $(B_n, n \in \mathbb{Z})$ are constant, then we look at the general case.

First stage : The variables are constant
In this stage, one poses for all $n \in \mathbb{Z}$, $A_n = a$ and $B_n = b$.
Initially, it is supposed that the loads of all the servers at the moment T_0 are equal to 0 (in other words, one brings back the beginning of the phenomenon to the moment T_0), i.e

$$\forall l = 1, \ldots, m; \quad W^l(T_0) = 0_{q_l}.$$

Let $W(T_0) = 0_{q_l} \times \ldots \times 0_{q_m}$ be the initial state of the process

$$W = (W^l, 1 \leq l \leq m).$$

The first customer who arrives just after the moment T_0 i.e. at $T_0 + 0$ to the system, he asks for a service b^1. Consequently, the load vector of the servers becomes :

$$W(T_0 + 0) = ((0_{q_1 - 1}, b^1); 0_{q_2}; \ldots, 0_{q_m}),$$

this because when the customer arrives, he chooses one among the q_1 servers, and when he is been useful, he leaves the first station at the moment $T_0 + b^1$, then he returns in the second station and request another service b^2. However, no arrival is recorded until the customer is served in all the stations , owing to the fact that $\displaystyle\sum_{l=1}^{m} b^l < a$. From where

$$W(T_0 + b^l) = W(T_0),$$

then at the last station, the load vector of the servers is given by

$$W(\sum_{l=1}^{m} b^l + 0) = (0_{q_1}, \dots, 0_{q_m - 1}, (0_{q_m - 1}, b^m)).$$

Consequently, at moment $\sum_{l=1}^{m} b^l$, one has

$$W(\sum_{l=1}^{m} b^l) = (0_{q_1}, \dots, 0_{q_m - 1}, (0_{q_m})),$$

and that wants to say that

$$W(\sum_{l=1}^{m} b^l) = W(T_0).$$

The customer leaves the system at the moment $(T_0 + \sum_{l=1}^{m} b^l)$, and as from this moment, a second customer can arrive at the moment T_2 , he does the same work as that of the first customer. Thus, by using the same stages one obtains $W(T_1) = W(T_0)$, and so on.
For all $n \in \mathbb{Z}$, the system remains empty at T_n, and consequently, the moments of freedom are the moments T_n of customers's arrivals.

It is supposed now that the initial loads of the services are not null, i.e

$$W_0^x = W(x, T_0) = x = (x_1, \dots, x_m) \in \mathcal{S}_1 \times \dots \times \mathcal{S}_m \quad , \quad x \neq (0_{\mathbb{R}^{q_1}}, \dots, 0_{\mathbb{R}^{q_m}}).$$

The arrival of a first customer at the first station at the moment $T_0 + 0$ results in the request for a service b^1 and the load vector of the servers checks

$$W(x, T_0 + 0) \leq ((x_1^{q_1} e_{q_1 - 1}, x_1^{q_1} + b^1), x_2^{q_2} e_{q_2}, \dots, x_m^{q_m} e_{q_m}),$$

then, at moment when this customer finishes his first service, the load vector of servers checks

$$W(x, b^1) \leq (((x_1^{q_1} - b^1)_+ e_{q_1} - 1, x_1^{q_1}), (x_{q_2}^{q_2} - b^1)_+ e_{q_2}, \dots, (x_m^{q_m} - b^1)_+ e_{q_m}).$$

One continues in the same way for the following stations, and when the customer completes his service, he leaves the network. From this moment, another arrival is awaited, saying at the moment $T_1 + 0$. But just before this arrival, i.e. at the moment T_1 the load vector of the servers checks

$$W(x, T_1) \leq (((x_1^{q_1} - a)_+ e_{q_1} - 1, (x_1^{q_1} + b^1 - a)_+), \dots,$$

$$((x_m^{q_m} - a)_+ e_{q_m} - 1, (x_m^{q_m} + b^m - a)_+)).$$

Thus, one finds oneself in the situation of the first case, i.e. $W(x, T_1) = 0_q$, under the condition

$$x_l^{q_l} + b^l - a \leq 0, \quad \forall l \in \{1, \dots, m\}.$$

If not, one continues to calculate the load vector of the servers until there in n such as

$$\exists l \in \{l, \dots, m\}; \quad x_l^{q_l} + n(b^l - q_l a) > 0,$$

and

$$\forall l \in \{1, \dots, m\}, \quad x_l^{q_l} + (n+1)(b^l - q_l a) \leq 0.$$

Thus, we have

$$W(x, T_{nq}) \leq (x_1^{q_1} + n(b^1 - q_1 a))e_{q_1}, \dots, (x_m^{q_m} + n(b^m - q_m a))e_{q_m}).$$

Of another share, we have

$$x_l^{q_l} + (n+1)(b^l - q_l a) \leq 0, \quad \forall l \in \{1, \dots, m\},$$

from where

$$W(x, T_{(n+1)q_1}) = 0_{q_l}, \quad \forall l \in \{1, \dots, m\}.$$

Thus,

$$W(x, T_{(n+1)q}) = 0_q.$$

Consequently, one will be able to determine the moments of freedom in the network while choosing n such as

$$n = \sup\{[\frac{x_l^{q_l}}{q_l a - b^l}], \quad 1 \leq l \leq m\} + 1.$$

Second stage : $(A_n, n \in \mathbb{Z})$ and $(B_n, n \in \mathbb{Z})$ are unspecified.
In this case, one defines for all $n \in \mathbb{N}$ the following event:

$$H_n = \cup_{k=1}^{n} \{B_{k-1} \leq b + \varepsilon e_m, A_k > a - \varepsilon\},$$

where ε is a strictly positive real number suitably chosen such as for all $l \in \{1, \dots, m\}$, one has $b^l + 2\varepsilon < a$.
In addition, one can notice that

$$\forall n \in \mathbb{N}, \ \widehat{\mathbb{P}}(H_n) > 0.$$

Let us suppose moreover than the initial loads of the servers are not null, and that the event H_n is carried out. It in result that

$$W(x, T_0) = x \neq 0_q.$$

When the first customer arrives at the moment $T_0 + 0$ at the first station, the load vector of the servers checks the inequality :

$$W(x, T_0 + 0) \leq ((x_1^{q_1} e_{q_1} + b^1 + \varepsilon, x_1^{q_1} e_{q_1 - 1}), x_2^{q_2} e_{q_2}, \ldots, x_m^{q_m} e_{q_m}),$$

and the same, at the moment when the customer finishes its first service, the load vector of the servers will check

$$W(x, T_0 + B_0^1) \leq ((x_1^{q_1} + \varepsilon, (x_1^{q_1} - b^1) e_{q_1 - 1}), (x_2^{q_2} - b^1) e_{q_2}, \ldots, (x_m^{q_m} - b^1) e_{q_m}).$$

The assumption $\sum_{l=1}^{m} b^l < a$ affirms us that no arrival is recorded before the moment T_1 . Thus, the load vector of the servers at this moment check

$$W(x, T_1) \leq (((x_1^{q_1} - a + 2\epsilon)_+ e_{q_1} - 1, (x_1^{q_1} + b^1 - a + 2\epsilon)_+)$$

$$, ..., ((x_m^{q_m} - a + 2\varepsilon)_+ e_{q_m} - 1, (x_m^{q_m} + b^1 - a + 2\varepsilon)_+)).$$

Thus, T_1 will be a freedom moment if

$$\forall l \in \{1, \ldots, m\}, \quad x_l^{q_l} + b^l - a + 2\varepsilon \leq 0.$$

If not, there is n such as

$$\exists l \in \{1, \ldots, m\}; x_l^{q_l} + n(b^l - q_l a) + 2\varepsilon > 0,$$

and

$$\forall l \in \{1, \ldots, m\}, \quad x_l^{q_l} + (n+1)(b^l - q_l a) + 2\varepsilon \leq 0.$$

To determine the moments of freedom T_n on H_n, we will choose n such as

$$n = \sup\{[\frac{x_l^{q_l}}{q_l a - b^l - 2\varepsilon}], \quad 1 \leq l \leq m\} + 1.$$

Thus, on H_n, we have $W(x, T_n) = 0_q$. That completes the proof of the proposition and gives the moments of freedom in the network.

23. Occupation moments

One sought in the preceding paragraph the moments of freedom in the queueing system (open network with several identical servers), the following paragraph will be devoted to the research of the occupation moments. In other words, moments when the system is not empty.

One supposes that $\forall t \geq T_0$, there is at least an occupied server, i.e. a station in load. From where, the contrary condition at times of freedom, i.e we are in the situation $\sum_{l=1}^{m} b^l \geq a$.

We will determine the moments of occupation by making the study station by station, because it will pose the problem of customers reclassification after having finished the service in each station.

Initially , we consider that the random variables $(A_n, n \in \mathbb{Z})$ and $(B_n, n \in \mathbb{Z})$ are constant, then we will treat the case of them where the variables are unspecified.

First stage : The variables are constant, $A_n = a$ and $B_n = b$.

For all $l \in \{1, \ldots, m\}$, one notes

$$v_1 = (0, \ldots, 0, b^l - j_l a, \ldots, b^l - a)$$

$$\overline{v}_l = (0, \ldots, 0, (\sum_{k=1}^{l} b^k - h_l a)_+, \sum_{k=1}^{l} b^k - (h_l - 1)a, \ldots, \sum_{k=1}^{l} b^k (K_{l-1} + 1)a).$$

By keeping the same notations used in preceding paragraph, one defines also the vector

$$\overline{v} = (\overline{v}_1, \ldots, \overline{v}_m).$$

The following lemma shows that starting from a certain number N, every moment are moments of occupation, i.e. at least, there is an occupied server.

Lemma 23..1 *For all initial state $x \in \mathcal{S}$, there is N such as*

$$\forall n \geq N, \quad W(x, T_n) = \overline{v}.$$

Proof: During the demonstration of the lemma, one will determine the moments of occupation. Because of customers's reclassification, one will show this lemma station by station.

First station: The network contains m stations, which are equipped by q_l servers respectively with $l \in \{1, \ldots, m\}$. Let $x_1 \in \mathcal{S}_1$ be the initial state of the process W^1, i.e

$$W^1(x, T_0) = x_1 = (x_1^1, \ldots, x_1^{q_1}) \in \mathcal{S}_1.$$

one tries to find N such as

$$\forall n \geq N, \quad W^1(x_1, T_n) = v_1$$

i.e. that starting from a certain number of customers, the first station becomes occupied. For that, one will discuss according to the initial state of the process in the first station. Two possibilities are taken into account :

First case : $(x_1^{q_1} < q_1 a)$
One has according to the definitions of the integers k_j and a that
$k_1 < q_1$ and $k_1 a \leq x_1^1 < (k_1+1)a$.
Thus, the load vector at T_{k_1} checks

$$W^1(x_1, T_{k_1}) \geq (x_1^1 - k_1 a, \ldots, x_1^1 - k_1 a, x_1^1 + b^1 - k_1 a, \ldots, x_1^1 + b^1 - k_1 a).$$

Since $x_1^1 - (k_1+1) + a < 0$ and $x_1^1 - k_1 a \geq 0$. Then, at moment T_{k_1+1}, this vector checks

$$W^1(x_1^1, T_{k_1+1}) \geq (0, \ldots, 0, (x_1^1 + b^1 - (k_1+1)a)_+, \ldots, (x_1^1 + b^1 - (k_1+1)a)_+).$$

Moreover, one has $x_1^1 - k_1 a \geq 0$, and $b^1 - a \geq 0$, so

$$x_1^1 + b^1 - (k_1+1)a \geq 0,$$

this returns the load vector at the moment T_{k_1+1} in the form :

$$W^1(x_1^1, T_{k_1+1}) \geq (0, \ldots, 0, x_1^1 + b^1 - (k_1+1)a, \ldots, (x_1^1 + b^1 - (k_1+1)a).$$

Since j_l being largest integer checking $\quad j_l a \leq b^l, \quad$ then

$$b^1 - j_1 a \leq b^1 - k_1 a,$$

and we have

$$x_1^1 + b^1 - (k_1+1)a > (x_1^1 - k_1 a) + b^1 - j_1 a.$$

Moreover, one has according to the stability condition that

$$\forall l \in \{1, \ldots, m\}, \quad 0 \leq j_l \leq q_l.$$

Thus, at $T_{k_1+q_1}$, one has

$$W^1(x_1, T_{k_1+q_1}) \geq (0, \ldots, 0, b^1 - j_1 a, \ldots, b^1 - a).$$

It is known in addition that the arrival of k_1-th customer was characterized by the load vector which checks

$$W^1(x_1, T_{k_1}) \leq (x_1^{q_1} - k_1 a, \ldots, x_1^{q_1} - k_1 a, (x_1^{q_1} + b^1 - k_1 a), \ldots, x_1^{q_1} + b^1 - k_1 a)).$$

By using the same reasoning as previously, but instead of the first server, one speaks about q_1-th server. Thus, one will have

$$W^1(x_1^1, T_{k_1+1}) \leq (0, \ldots, 0, (x_1^{q_1} + b^1 - (k_1+1)a)_+, \ldots, (x_1^{q_1} + b^1 - (k_1+1)a)_+$$

i.e

$$W^1(x_1, T_{k_1+q_1}) \leq (0, \ldots, 0, b^1 - j_1 a, \ldots, b^1 - a),$$

and this is not other than v_1 or $\overline{v}_1$. Thus, the load vectors $W^1(x_1, T_{k_1+q_1})$ and $W^1(x_1, T_{k_1+q_1+1})$ are raised and undervalued by the same vector, from where

$$W^1(x_1, T_{k_1+q_1}) = W^1(x_1, T_{k_1+q_1+1}) = v_1 = \overline{v}_1.$$

Consequently, one has

$$\forall n \geq k_1 + q_1 : \quad W^1(x_1, T_n) = \overline{v}_1.$$

Since the sequence (T_n) is increasing and $\overline{v}_1$ is not null, then $T_n, n \geq k_1 + q_1$ are a moments of occupation in the system.

Second case: $(x_1^{q_1} \geq q_1 a - b^1)$
In this case, instead of k_1 one uses $k_1^{'}$ which is defined by

$$k_1^{'} = [\frac{x_1^{q_1}}{q_1 a - b^1}] + 1.$$

Thus, we have

$$k_1^{'}(q_1 a - b^1) \leq x_1^{q_1} \leq (k_1^{'} + 1)(q_1 a - b^1)$$

and

$$W^1(x_1, T_{k_1^{'}+q_1}) \leq (x_1^{q_1} + k_1^{'}(b^1 - q_1 a))e_{q_1}.$$

By using the condition $x_1^{q_1} \geq q_1 a - b^1$ and the definition of j_1, we will obtain

$$W^1(x_1, T_{(k_1^{'}+q_1)+j_1}) \leq (0, \ldots, 0, b^1 - j_1 a, \ldots, b^1 - a).$$

In addition, and by using the same stages that those of the first case, one obtains

$$W^1(x_1, T_{(k_1^{'}+1)q_1+j_1}) \geq (0, \ldots, 0, b^1 - j_1 a, \ldots, b^1 - a).$$

From where

$$W^1(x_1, T_{(k_1^{'}+1)q_1+j_1}) \geq \overline{v}_1.$$

Thus, we choose $N = \max(k_1 + q_1, (k_1^{'} + 1)q_1 + j_1)$ to have

$$\forall n \geq N, \quad W^1(x_1, T_n) = \overline{v}_1.$$

Then, the network is occupied in every moments T_n with $n \geq N$.

After having completed their services at the first station, the customers arrive at the second station at the moments

$$T_n^1(x_1) = na + b^1 + W^{1,1}(x_1, T_n).$$

But, the order of the customers will change. For that, one must order the customers after their exits of the first station, as follows:
one has according to previously

$$\forall n \geq N, W^1(x_1, T_n) = \overline{v}_1.$$

The first component of $\overline{v}_1$ is null, and the latter is nothing other than $W^{1,1}(x_1, T_n)$. Thus, one has

$$T_n^1(x_1) = na + b^1 + W^{1,1}(x_1, T_n) = na + b^1.$$

However, $T_n(x_1), n \geq N$ are ordered, since the sequence $(na + b^1)_{n \geq N}$ is increasing. One can thus find oneself in the case of the first station while posing

$$T_n^1(x_1) = T_n^1(0).$$

In other words, starting from a certain number N, the process forgets its initial state.

Second station: The second station contains q_2 servers, reclassification which was made before enables us to remake the same reasoning in the second station. Let $x_2 \in \mathcal{S}_2$ be the initial state of W^2 and $x_2^{'}$ the state of W^2 at moment T_N. We consider

$$k_2^{'} = [\frac{x_2^{q_2}}{q_2 a - b^2}] + 1.$$

For $N^{'} = \max(k_2 + q_2, (k_2^{'} + 1)q_2 + j_2 + N^{'})$, we have

$$\forall n \geq N^{'} : W^2(x_1, x_2, T_n 1(x_1)) = v_2.$$

Customers leave the second station at moments $T_n^2(x_1, x_2)$, such that

$$T_n^2(x_1, x_2) = na + b^1 + b^2 + W^{2,2}(x_1, x_2, T_{N^{'}}^1(x_1)).$$

However, for $n \geq N^{'}$, one has $W^{2,2}(x_1, x_2, T_{N^{'}}^1(x_1)) = 0$, this owing to the fact that it is the second component of v_2 which is worth

$$(0, \ldots, 0, b^2 - j_2 a, \ldots, b^2 - a).$$

From here, one has

$$T_n^2(x_1, x_2) = na + b^1 + b^2.$$

Since the sequence $(na + b^1 + b^2)_{n \geq N'}$ is increasing, then the moments $T_n^2(x_1, x_2)$ are ordered. An identical work is done until the m-th station.

Let us seek now, the moments of occupation overall for all the stations. When the customer number n enters in the system at the moment T_n, we suppose that the customer number N' returns in the second station before the $(r_1 + 1)$-th customer did not arrive in the system, where r_1 is a natural number higher than N' which corresponds to the number of the customer arriving at the system at T_{r_1}. Consequently,

$$T_{r_1} \leq T_{N'}^1(x_1) < T_{r_1+1}$$

and the time of service claimed at the first station checks

$$(r_1 - N')a \leq b^1 < (r_1 - N' + 1)a.$$

In addition, one has

$$j_1 = \max\{k \in \mathbb{N}, k \leq \frac{b^1}{a}\} = [\frac{b^1}{a}]$$

Then,

$$j_1 \leq \frac{b^1}{a} < j_1 + 1,$$

i.e

$$j_1 a \leq b^1 < (j_1 + 1)a.$$

Owing to the fact that $j_1 a \leq b^1 < (r_1 - N' + 1).a$, one has

$$j_1 < r_1 - N' + 1,$$

and

$$(j_1 + 1)a > b^1 \geq (r_1 - N')a,$$

i.e

$$j_1 > r_1 - N' - 1.$$

Of another share, and since these numbers are positive integers, therefore

$$r_1 = j_1 + N'.$$

Before the customer number $r_1 + 1$ returns in the system, an arrival of a customer at the moment $T_{N'}^1 + 0$ at the second station results in the load vector of the servers given by

$$W^2(x_1, x_2, T_{N'}^1 + 0) = (0, \ldots, 0, b^2 - j_2 a, \ldots, b^2 - a, b^2),$$

and when the r_1+1-th customer enters the first station, the load vector is written as

$$W^2(x_1,x_2,T_{r_1+1}) = (0,\dots,0, b^1+b^2-(j_1+j_2+1)a)_+, (b^1+b^2-(j_1+j_2a)_+,$$
$$\dots, b^1+b^2-(j_1+1)a).$$

Since $b^2 \geq j_2 a$, then one has

$$b^1+b^2 \geq (j_1+j_2)a.$$

That entrains

$$(b^1+b^2-(j_1+j_2)a)_+ = b^1+b^2-(j_1+j_2)a,$$

and we obtain

$$W^2(x_1,x_2,T_{r_1+1}) = (0,0,\dots,0, b^1+b^2-(j_1+j_2+1)a)_+, b^1+b^2-(j_1+j_2a),$$
$$\dots, b^1+b^2-(j_1+1)a).$$

We have

$$k_2 = \max\{k \in \mathbb{N}, k \leq \frac{b^1+b^2}{a}\} = [\frac{b^1+b^2}{a}],$$

thus,

$$k_2 a \leq b^1+b^2 < (k_2+1)a,$$

and as

$$(j_1+j_2)a \leq b^1+b^2 < (j_1+j_2+2)a,$$

then, for $h_2 = k_1+j_2+1 = j_1+j_2+1$ (since $j_1 = k_1$), one will have

$$W^2(x_1,x_2,T_{r_1+1}) = (0,0,\dots,0, b^1+b^2-h_2 a)_+, \dots, b^1+b^2-(k_1+1)a).$$

But this quantity is nothing other than $\overline{v}_2$. And in the same way, one continues for each station until the existence of a natural number $N^{(m)}$ such as

$$\forall n \geq N^{(m)} : W(x_1,x_m,T_n) = \overline{v},$$

i.e the moments $(T_n)_{n\geq N^{(m)}}$ are an occupation moments in the system under the condition $\sum_{l=1}^{m} b^l \geq a$.

Now, one looks at the case which the variables A_n and B_n, $n \in \mathbb{Z}$ are not constant and one seeks inter alia the moments of occupation.

Second stage: The variables are unspecified.
While looking at the event

$$H_n = \cap_{k=1}^{n}\{B_{k-1} \leq b + \varepsilon e_m, A_k > a - \varepsilon\}.$$

One has of course for all n , $\widehat{\mathbb{P}}(H_n) > 0$.
One defines the vector v_1^ε, and for all l, $(2 \leq l \leq m)$ the vectors v_l^ε by

$$v_1^\varepsilon = \{0, \ldots, 0, b^1 - j_1 a + (j_1+1)\varepsilon, \ldots, b^1 - a + 2\varepsilon)$$
$$v_l^\varepsilon = \{0, \ldots, 0, b^l - j_l a + (2j_l+1)\varepsilon, \ldots, b^l - a + 3\varepsilon.)$$

Then, one defines for all l, $1 \leq l \leq m$ the vectors $v_l^{\shortmid\varepsilon}$, $\overline{v}_l^\varepsilon$, $\underline{v}_l^\varepsilon$ by

$$v_l^{\shortmid\varepsilon} = (0, \ldots, 0, b^l - j_l(a+\varepsilon), \ldots, b^l - a - \varepsilon)$$
$$\overline{v}_l^\varepsilon = (0, \ldots, 0, (\textstyle\sum_{i=1}^{l} b^i - h_l(a-2\varepsilon))_+, \ldots, \sum_{i=1}^{l} b^i - (k_{l-1}+1)(a-2\varepsilon))$$
$$\underline{v}_l^\varepsilon(0, \ldots, 0, (\textstyle\sum_{i=1}^{l} b^i - h_l(a-\varepsilon))_+, \ldots, \sum_{i=1}^{l} b^i - (k_{l-1}+1)(a+\varepsilon)))$$

One poses for all l, $1 \leq l \leq m$

$$\overline{V}_l^\varepsilon = \{x_l \in \mathcal{S}_l / \underline{v}_l^\varepsilon \leq x_l \leq \overline{v}_l^\varepsilon\}$$

and $\overline{V}^\varepsilon = \overline{V}_1^\varepsilon \times \ldots \times \overline{V}_m^\varepsilon$.

Theorem 23..1 *For all $x \in \mathcal{S}, W(x, T_n, n \in \mathbb{N})$ reaches $\overline{V}^\varepsilon$ with $\widehat{\mathbb{P}}$-strictly positive probability.*

Proof: Let $x \in \mathcal{S}$, and $W^1(T_0) = (x_1^1, \ldots, x_1^{q_1}) \in \mathcal{S}_1$. One follows the same stages as those of the case where the variables were constant .

First station: for clarifying the demonstration well one discusses the two following cases :

First case: $(x_1^{q_1} < q_1(a-\varepsilon))$
One chooses $k_1 = [\frac{x_1^{q_1}}{q_1(a-\varepsilon)}] + 1$, one will have

$$W^1(x_1, T_{k_1+q_1}) \leq (0, \ldots, 0, b^1 - j_1(a-2\varepsilon), \ldots, b^1 - (a-2\varepsilon)),$$

then

$$W^1(x_1, T_{k_1+q_1}) \geq (0, ..., 0, b^1 - j_1(a+\varepsilon), \ldots, b^1 - (a+\varepsilon)).$$

The two inequalities are true only on H_n, and according to the definition of the two vectors $\overline{v}_1^\varepsilon$ et $\underline{v}_1^\varepsilon$, one has

$$\underline{v}_1^\varepsilon \leq W^1(x_1, T_{k_1+q_1}) \leq \overline{v}_1^\varepsilon$$

i.e

$$W^1(x_1, T_{k_1+q_1}) \in \overline{V}_1^{\varepsilon}.$$

In other words, at the moment $T_{k_1+q_1}$, the process enters as a minimum set $\overline{V}_1^{\varepsilon}$. Then, $T_{k_1+q_1}$ is an occupation moment.

Second case: $(x_1^{q_1} \geq q_1(a-\varepsilon))$

In this case, one is interested in

$$k_1^{'} = [\frac{x_1^{q_1}}{q_1(a-\varepsilon) - b^1}] + 1.$$

One will have in the same way on H_n that

$$\underline{v}_1^{\varepsilon} \leq W^1(x_1, T_{(k_1^{'})q_1+j_1}) \leq \overline{v}_1^{\varepsilon}.$$

Thus, there exists a natural number N, $N \geq \max((k_1^{'}+1)q_1 + j_1, (k_1 + j_1))$ such as

$$\forall k : \max((k_1^{'}+1)q_1 + j_1, (k_1 + j_1)) \leq k \leq n, \quad W^1(x_1, T_k) \in \overline{V}_1^{\varepsilon},$$

thus, the moments $T_k, \quad N \leq k \leq n$ are an occupation moments under the condition

$$x_1^{q_1} \geq q_1(a-\varepsilon).$$

While passing to the second station, customers leave the first station at moments $T_k^1(x_1)$ such that on H_n, one has

$$\forall k, \quad N \leq k \leq n; \quad T_k^1(x_1) = T_k^1(0).$$

Moments $T_k^1(0)$ are ordered, one will thus proceed of the same reasoning as that of the first station.

Second station: One poses $x_2^{'} = W^2(x_1, x_2, T_N(x_1))$. The two following cases are discussed :

First case: $(x_2^{'q_2} < q_2(a-2\varepsilon))$.

We choose $k_2^{'} = [\frac{x_2^{'q_2}}{q_2(a-\varepsilon)-b^2}] + 1$, thus one has for n enough large, and on H_n,

$$W^2(x_1, x_2, T^1_{N+k_2^{'}+q_2}(x_1)) \in \overline{V}_2^{\varepsilon}.$$

i.e the moment $T^1_{N+k_2^{'}+q_2}$ is an occupation moment.

Second case: $(x_2^{\prime q_2} > q_2(a-2\varepsilon) - b^2)$.

We have

$$k_2^{\prime}(a_2(a-2\varepsilon)b^2) \leq x_2^{\prime q_2} < (k_2^{\prime}+1)(q_2(a-2\varepsilon)-b^2),$$

and the same for

$$N^{\prime} = \max(k_2+q_2, (k_2^{\prime}+1)q_2+j_2)+N.$$

Moreover, one has for all k, $N^{\prime} \leq k \leq n$

$$W^2(x_1,x_2,T_k^1(0)) \in \overline{V}_2^{\varepsilon}.$$

i.e the moments $T_k^1(0)$, $N^{\prime} \leq k \leq n$ are an occupation moments in the system.

We study now, the load vector of servers at moments T_n. Let r_1 (such that $r_1 \geq N^{\prime}$) be a natural integer which corresponds to the customer's number arriving at the system at the moment T_{r_1}. One supposes that the $N^{\prime}$-th customer enters the second station before the (r_1+1)-th customer arrives at the network, it results in :

$$T_{r_1} \leq T_{N^{\prime}}^1(0) < T_{r_1+1}.$$

Thus,

$$(p_1-N_2)(a-\varepsilon) < (p_1-N_2)(a+\varepsilon) \leq b^1 < (p_1-N_2+1)a,$$

and for ε enough small, we will have

$$r_1 = j_1 + N^{\prime}.$$

When a customer arrives at moment $T_{N^{\prime}}^1 + 0$ at the second station, one has

$$W^2(x_1,x_2,T_{N^{\prime}}^1+0) = (0,\ldots,0,b^2-j_2a+(2j_2+1)\varepsilon,\ldots,b^2-(a-3\varepsilon),b^2+\varepsilon),$$

and when the (r_1+1)-th customer enters the system, the load vector checks

$$W^2(x_1,x_2,T_{r_1+1}) = (0,\ldots,0,(b^1+b^2-(j_1+j_2+1)a+(j_1+2j_2+3)\varepsilon)_+,$$

$$,\ldots,b^1+b^2-(j_1+1)a+(j_1+2)\varepsilon).$$

Since $h_2 = j_1+j_2+1$, then

$$W^2(x_1,x_2,T_{r_1+1}) = (0,\ldots,0,(b^1+b^2-h_2(a-2\varepsilon))_+,\ldots,b^1+b^2-(h_1+1)(a-2\varepsilon_1)).$$

In the same way,

$$W^2(x_1,x_2,T_{r_1+1}) \geq (0,\ldots,0,(b^1+b^2-(j_1+j_2+1)a-j_2\varepsilon)_+,$$

$$\ldots,b^1+b^2-(j_1+2)a-\varepsilon, b^1+b^2-(j_1+1)(a+\varepsilon))$$

i.e.

$$W^2(x_1,x_2,T_{r_1+1}) \geq (0,\ldots,0,(b^1+b^2-h_2(a+\varepsilon))_+,\ldots,b^1+b^2-$$

$$(k_1+2)(a+\varepsilon), b^1+b^2-(k_1+1)(a+\varepsilon)).$$

Thus, one has

$$W^2(x_1,x_2,T_{r_1+1}) \in \overline{V}^{\varepsilon}_2.$$

One continues stage by stage for each m station until there is an integer n such as

$$\widehat{\mathbb{P}}(W(x,T_n) \in \overline{V}^{\varepsilon}) \geq \widehat{\mathbb{P}}(H_n) > 0.$$

Thus, with a strictly positive probability and starting from a certain row, T_n are a moments of occupation in the system.

Conclusion

The phenomenon of waiting, in particular the queueing systems is a very wide field. It contains several branches, each one can be interpreted various ways. Our work was devoted to the recurrence study in open networks. Our goal was to seek the freedom moments in this type of networks, i.e. when all the servers are free. In this case, the customers any more will not wait, it is the hoped situation.

Reciprocally, and with a little difficulty of reordonnement of customers after the exit of each station, one established the moments when the system is occupied, i.e. the servers's load in one of the stations is different from zero (there is waiting).

Our work can intervene in fields such as the process of taking out of bond, the consecutive operations within the data-processing framework, the machines which break down in ateliers, the people who solicitant a communication by the intermediary of a telephone switchboard, etc $\cdots$.

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