CONVERGENCE OF COSTS IN AN OPTIMAL STOPPING PROBLEM FOR A PARTIALLY OBSERVABLE MODEL

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Abstract

The problem of optimal stopping of a stochastic process with incomplete data is reduced to the problem of optimal stopping of a stochastic process with complete data and the convergence of payoffs is proved when $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$, where ε_1 and ε_2 are small perturbation parameters of the observable process. Kalman–Bucys continuous and discrete models of partially observable random processes is considered.

Key words and phrases: Partially observable stochastic process, gain function, payoff, optimal stopping, stopping time.

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1 Introduction

Let us consider a partially observable stochastic process (θ_t, ξ_t), $0 \le t \le T < \infty$, of Kalman–Bucys model

$$d\theta_t = [a_0(t) + a_1(t)\theta_t + a_2(t)\xi_t]\,dt + b_1(t)dW_1(t) + b_2(t)dW_2(t), \quad (1.1)$$
$$d\xi_t = [A_0(t) + A_1(t)\theta_t + A_2(t)\xi_t]\,dt + \varepsilon_1 dW_1(t) + \varepsilon_2 dW_2(t), \quad (1.2)$$

where $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are constants, the coefficients $a_i(t)$, $A_i(t)$, $i = 0,1,2$, $b_k(t)$, $k = 1,2$, are nonrandom measurable functions, $W_1(t)$ and $W_2(t)$ are independent Wiener processes. It is assumed that in model (1.1), (1.2), θ_t is the nonobservable process and ξ_t is the observable process, where $A_1(t) \neq 0$ [1].

We consider a linear gain function

$$g(t,x) = f_0(t) + f_1(t)x, \quad x \in R, \quad (1.3)$$

and define payoffs by the equalities

$$s^\circ = \sup_{\tau \in \mathfrak{M}^\theta} Eg(\tau, \theta_\tau), \quad s = \sup_{\tau \in \mathfrak{M}^\xi} Eg(\tau, \theta_\tau), \quad (1.4)$$

where $f_0(t)$, $f_1(t)$ are nonrandom measurable functions; $\mathfrak{M}^\theta$, $\mathfrak{M}^\xi$ are the classes of stopping times with respect to the families of σ-algebras $\mathcal{F}_t^\theta = \sigma\{\theta_s : 0 \le s \le t\}$ and $\mathcal{F}_t^\xi = \sigma\{\xi_s : 0 \le s \le t\}$.

The following notation is introduced:

$$m_t = E(\theta_t |\ \mathcal{F}^\xi), \quad \gamma_t = E(\theta_t - m_t)^2.$$

2. *Continuous model*

For explain the quite part of this work we should use some notations and restrictions. We assume, that for every t, $0 \le t \le T$, the following conditions are satisfied

$$\begin{aligned} &1)\ 0 \le f(t) \le F < \infty; \\ &2)\ b_1^2(t) + b_2^2(t) \le B^2 < \infty; \\ &3)\ 0 < \underline{A} \le A_1(t) \le \overline{A} < \infty; \\ &4)\ 0 < \underline{a} \le a_2(t) \le \overline{a} < \infty; \\ &5)\ \Phi_{i1}(t) = \exp\left[\int_0^t a_1(s)\,ds\right] < c_1 < \infty, \ \ i = 1,2; \\ &6)\ \Phi_{j2}(t) = \exp\left[\int_0^t A_2(s)\,ds\right] < c_2 < \infty, \ \ j = 1,2. \end{aligned} \tag{2.1}$$

Theorem 2.1 *Let partially aftervable random process* (θ_t, ξ_t)*,* $0 \le t \le T$*, be given by* (1.1), (1.2) *expressions and payoffs are explained. If conditions* (2.1) *are satisfied, then the following estimate*

$$s^0 - s^{\varepsilon_1,\varepsilon_2} \le F[2B\pi^{-1}(\underline{A}^{-1} + \underline{a}^{-1})]^{-1/2}(c_1 + c_2)(\varepsilon_1 + \varepsilon_2)^{1/2} \tag{2.2}$$

is valid.

Poof. Let's use following notations

$$m_t = E(\theta_t |\ \mathcal{F}_t^\xi), \quad \gamma_t = E(\theta_t - m_t)^2, \tag{2.3}$$

$$\widetilde{m}_t = E(\xi_t |\ \mathcal{F}_t^\theta), \quad \widetilde{\gamma}_t = E(\xi_t - \widetilde{m}_t)^2. \tag{2.4}$$

According to Theorem 12.1, [1], there exist $W = (W(t), \mathcal{F}_t^\xi)$ Wiener's process, witle which process θ_t and $\widetilde{m}_t$ can be considered by following stochastic differentiable equalities

$$d\theta_t \ = [a_0(t) + a_1(t)\theta_t + a_2(t)\widetilde{m}_t]\,dt + \tfrac{a_2(t)\widetilde{\gamma}_t}{\sqrt{b_1^2(t)+b_2^2(t)}}\,dW(t), \tag{2.5}$$

$$d\widetilde{m}_t \ = [A_0(t) + A_1(t)\theta_t + A_2(t)\widetilde{m}_t]\,dt + \tfrac{a_2(t)\widetilde{\gamma}_t}{\sqrt{b_1^2(t)+b_2^2(t)}}\,dW(t) \tag{2.6}$$

and according to Theorem 10.3 of the same reference

$$\begin{aligned}\frac{d\gamma_t}{dt} &= 2a_1(t)\gamma_t + b_1^2(t) + b_2^2(t) - \tfrac{A_1^2(t)\gamma_t^2}{\varepsilon_1^2+\varepsilon_2^2}\,, &(2.7)\\ \frac{d\widetilde{\gamma}_t}{dt} &= 2A_1(t)\widetilde{\gamma}_t + b_1^2(t) + b_2^2(t) - \tfrac{a_1^2(t)\widetilde{\gamma}_t^2}{\varepsilon_1^2+\varepsilon_2^2}\,. &(2.8)\end{aligned}$$

Now let us define random process $\widetilde{\theta}_t$ and n_t by following equalities:

$$\begin{aligned}d\widetilde{\theta}_t &= [a_0(t) + a_1(t)\widetilde{\theta}_t + a_2(t)n_t]\,dt + \tfrac{A_2(t)\gamma(t)}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\,dW(t), &(2.9)\\ dn_t &= [A_0(t) + A_1(t)\widetilde{\theta}_t + A_2(t)n_t]\,dt + \sqrt{\varepsilon_1^2+\varepsilon_2^2}\,dW(t). &(2.10)\end{aligned}$$

As [4], we can show, that

$$s^{\varepsilon_1,\varepsilon_2} = \sup_{\tau\in\mathfrak{M}^\xi} Eg(\tau, m_\tau) = \sup_{\tau\in\mathfrak{M}^\theta} Eg(\tau, \widetilde{\theta}_\tau), \tag{2.11}$$

where $\widetilde{\theta}_t$ process is defined with (2.9). Then with notice 1) of (2.1) we have

$$s^0 = s_1^{\varepsilon_1,\varepsilon_2} \le FE[\sup_{t\le T}(\theta_t - \widetilde{\theta}_t)]. \tag{2.12}$$

Now, to proof of theorem we need to estimate supremum of mathematical expectation of the deferens $\theta_t - \widetilde{\theta}_t$ in the interval $[0,T]$. For this reason we use Theorem 4.10 [1] and then consider random process θ_t and $\widetilde{\theta}_t$ as the decision of according to system (2.5), (2.6). Then we can write

$$\begin{aligned}\theta_t - \widetilde{\theta}_t &= \Phi_{11}(t)\Bigg[\int_0^t \Phi_{11}^{-1}(s)\Big(\sqrt{b_1^2(s)+b_2^2(s)} - \tfrac{A(s)\gamma_s}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\Big)\,dW(s) + \\ &\quad + \int_0^t \Phi_{12}(t)\Big(\tfrac{a_2(s)\widetilde{\gamma}_s}{\sqrt{b_1^2(s)+b_2^2(s)}} - \sqrt{\varepsilon_1^2+\varepsilon_2^2}\Big)\,dW(s)\Bigg] + \\ &\quad + \Phi_{12}(t)\Bigg[\int_0^t \Phi_{21}^{-1}(s)\Big(\sqrt{b_1^2(s)+b_2^2(s)} - \tfrac{A_1(s)\gamma_s}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\Big)\,dW(s) + \\ &\quad + \int_0^t \Phi_{22}^{-1}(s)\Big(\tfrac{a_2(s)\widetilde{\gamma}_s}{\sqrt{b_1^2(s)+b_2^2(s)}} - \sqrt{\varepsilon_1^2+\varepsilon_2^2}\Big)\,dW(s)\Bigg]. &(2.13)\end{aligned}$$

Now, for $i = 1, 2$, take in following notations

$$\begin{aligned}Q_i(t) &= \int_0^t \Phi_{i1}^{-1}\Big(\sqrt{b_1^2(s)+b_2^2(s)} - \tfrac{A_1(s)\gamma_s}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}}\Big)\,dW(s), &(2.14)\\ \widetilde{Q}_i(t) &= \int_0^t \Phi_{i2}^{-1}\Big(\tfrac{a_2(s)\widetilde{\gamma}_s}{\sqrt{b_1^2(s)+b_2^2(s)}} - \sqrt{\varepsilon_1^2+\varepsilon_2^2}\Big)\,dW(s). &(2.15)\end{aligned}$$

Besides, we can notice, that from the differential equalities (2.7) and (2.8) we get such estimation for values γ_t and $\widetilde{\gamma}_t$ as

$$\gamma_t = A^{-1}B(\varepsilon_1 + \varepsilon_2), \quad \widetilde{\gamma}_t \le \underline{a}^{-1}B(\varepsilon_1 + \varepsilon_2). \tag{2.16}$$

It is easy to show with these estimation, that

$$E\sup_{t\le T}\Phi_{1i}(t)Q_i(t) \quad \le c_1\sqrt{2B\underline{A}^{-1}\pi^{-1}(\varepsilon_1+\varepsilon_2)}\,, \tag{2.17}$$

$$E\sup_{t\le T}\Phi_{1i}(t)\widetilde{Q}_i(t) \quad \le c_2\sqrt{2B\underline{a}^{-1}\pi^{-1}(\varepsilon_1+\varepsilon_2)}\,. \tag{2.18}$$

To sum (2.18), (2.18), estimations $\Phi_{11}(t) = \Phi_{21}(t)$, $\Phi_{12}(t) = \Phi_{22}(t)$ and dependents (2.12), (2.13) we get the proof of theorem. ■

3. *Discrete model*

Let us now consider Kalman-Busy's discrete model for the sequence (θ_n, ξ_n), $n = 0, 1, 2, \ldots, N < \infty$, where

$$\theta_{n+1} \quad = a_0(n) + a_1(n)\theta_n + b_1(n)\eta_1(n+1) + b_2(n)\eta_2(n+1), \tag{3.1}$$

$$\xi_{n+1} \quad = A_0(n) + A_1(n)\theta_n + \varepsilon_1\eta_1(n) + \varepsilon_2\eta_2(n), \tag{3.2}$$

where $a_i(n)$, $A_i(n)$, $i = 0, 1$, $b_k(n)$, $k = 1, 2$, are nonrandom functions, $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ are constants, $\eta_n(n)$ and $\eta_2(n)$ are independent standard normal random values.

We consider a linear gain function

$$g(n, x) = f_0(n) + f_1(n)x, \quad x \in R \tag{3.3}$$

and define payoffs of by the equalities

$$\begin{aligned} s^0 \quad &= \sup\nolimits_{\tau\in\mathfrak{M}^\theta} Eg(\tau, m_\tau), \\ s \quad &= \sup\nolimits_{\tau\in\mathfrak{M}^\xi} Eg(\tau, \theta_\tau), \end{aligned} \tag{3.4}$$

where $f_0(n)$, $f_1(n)$ are nonrandom functions, $\mathfrak{M}^\theta$, $\mathfrak{M}^\xi$ are the classes of stopping times with values in $\{0, 1, 2, \ldots, N\}$ with respect to the families of σ-algebras $\mathcal{F}_n^\theta = \sigma\{\theta_i,\ o \le n\}$ and $\mathcal{F}_n^\xi = \sigma\{\xi_i,\ i \le n\}$.

The following notation is introduced:

$$m_n = E(\theta_n|\ \mathcal{F}_n^\xi), \quad \gamma_n = E[(\theta_n - m_n)|\ \mathcal{F}_n^\xi]. \tag{3.5}$$

Let us assume, that the following conditions are satisfied

$$\begin{gathered} 1)\ \ a_2^2(n) < \infty,\ \ b_1^2(n) < \infty,\ \ i = 0, 1,\ \ A_0^2(n) < \infty, \\ 2)\ \ a_1^2(n) \le q,\ \ 0 < q < 1, \\ 3)\ \ 0 < \underline{A}_1 \le A_1^2(n) \le \overline{A}_1 < \infty, \\ 4)\ \ 0 \le f_1(n) \le H < \infty. \end{gathered} \tag{3.6}$$

The following theorem is analogously in result of Fermann's theorem.

Theorem 3.1 *Let, partially observable random sequence* (θ_n, ξ_n) *is given by system* (3.1), (3.2) *and conditions* (3.5) *are satisfied. Then the following estimate*

$$0 < s^0 - s \le 2H\sqrt{B(1+C)} \cdot (\varepsilon_1 + \varepsilon_2) \tag{3.7}$$

is valid, where B *and* C *constants are defined by equalities*

$$B = \underline{A}_1^{-1} \max\Big\{1; 2\max\big[\max_{0\le n\le N} a_1^2(n); q\big]\Big\}, \tag{3.8}$$

$$C = \max\Big\{ \max_{1\le n\le N}\Big[\sum_{k=1}^{n}\Big(\prod_{j=k}^{n} a_1^2(j)\Big)\Big]; \tfrac{q}{1-q}\Big\}. \tag{3.9}$$

Poof. At first notice that following expression of payoff is

$$s = \sup_{\tau\in\mathfrak{M}^{\xi}} Eg(\tau, m_\tau) = \sup_{\tau\in\mathfrak{M}^{\tilde\theta}} Eg(\tau, \widetilde{\theta}_\tau) \tag{3.10}$$

is valid, where $\widetilde{\theta}_n$ sequence is explained by means of following recurrence equalities (see [1])

$$\widetilde{\theta}_{n+1} = a_0(n) + a_1(n)\widetilde{\theta}_n + \beta_n \eta_{n+1}, \tag{3.11}$$

$$\beta_n = P(n)Q^{-1/2}(n), \tag{3.12}$$

$$P(n) = (b_1^2(n) + b_2^2(n))A_1(n+1) + a_1^2(n)A_1(n+1)\cdot\gamma_n, \tag{3.13}$$

$$Q(n) = (b_1^2(n) + b_2^2(n))A_1^2(n+1) + a_1^2(n)A_1^2(n+1)\cdot\gamma_n + \varepsilon_1^2 + \varepsilon_2^2 \tag{3.14}$$

After, we notice, that value γ_n satisfies the recurrence equality (see [1])

$$\gamma_{n+1} = a_1^2(n)\gamma_n + b_1^2(n) + b_2^2(n) - P^2(n)\cdot Q^{-1}(n), \tag{3.15}$$

from which we easily get the estimation of γ_n

$$\gamma_n \le \underline{A}_1^{-1}(\varepsilon_1^2 + \varepsilon_2^2). \tag{3.16}$$

Now, estimate the deference $s^0 - s$. We have

$$s^0 - s \le HE\Big\{\sup_{n\le N}(\theta_n - \widetilde{\theta}_n)\Big\}, \tag{3.17}$$

where random sequence $\widetilde{\theta}_n$ is defined by (3.11)–(3.14) and θ_n sequence can be considered as follows (see [1])

$$\theta_n = a_0(n) + a_1(n)\theta_n + \sqrt{b_1^2(n) + b_2^2(n)}\cdot\eta_{n+1}, \tag{3.18}$$

where as in (3.11) η_n is the standard normal random values.

Using the equalities (3.11) and (3.18), we get that

$$\theta_{n+1}-\widetilde{\theta}_{n+1}=a_1(n)(\theta_n-\widetilde{\theta}_n)+(\sqrt{b_1^2(n)+b_2^2(n)}-\beta_n)\eta_{n+1},$$

then with notice that $\theta_0=\widetilde{\theta}_0=0$ we can write

$$\theta_n-\widetilde{\theta}_n=\sum_{i=1}^{n}\delta_i^{(n-1)}(\sqrt{b_1^2(i-1)+b_2^2(i-1)}-\beta_{i-1})\eta_i, \tag{3.19}$$

where

$$\delta_i^{(n)}=\prod_{j=1}^{n}a_1(j).$$

Besides, we can see that

$$Q(n) \geq \underline{A}_1^2(b_1^2(n)+b_2^2(n)), \tag{3.20}$$

$$Q(n) \geq A_1^2(n+1)a_1^2(n)\gamma_n. \tag{3.21}$$

According to condition 2) of (3.6) we get

$$a_1^2(n)\leq \max\left\{q;\max_{0\leq k\leq N}a_2^2(k)\right\},$$

and according to (3.12), (3.19), (3.21) we can write

$$(\sqrt{b_1^2(n)+b_2^2(n)}-\beta_n)^2\leq B(\varepsilon_1^2+\varepsilon_2^2). \tag{3.22}$$

Using the description of constant c in the last estimate, with simple transformation we get estimate (3.7). ■

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