ON THE SIMULATION OF THE AMERICAN OPTION PRICING PROCESS

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Abstract

Using the Black-Sholes model of a financial market, the problem of the American type option pricing theory is reduced to the optimal stopping problem for a diffusion process on a finite time interval $[0, T]$. A partial differential equation is derived for a price function. Analogous results are also obtained for the Cox-Ross-Rubinstein binomial model of a financial market. The obtained results make it possible to construct a complex of programs for numerical examples.

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1 Introduction

Let us consider the Black-Sholes model of a financial market

$$\begin{aligned} dS_t &= \mu S_t\, dt + \sigma S_t\, dW_t, \\ dB_t &= r B_t\, dt, \end{aligned} \tag{1.1}$$

where $S = (S_t)$ are stocks, $B = (B_t)$ are bonds, $0 \le t \le T$, $B_0 > 0$, $S_0 > 0$, $\sigma > 0$, $r > 0$, and the American option payoff function is given by the equality

$$f_t = \max_{u \le t} S_u - S_t. \tag{1.2}$$

Lemma 1.1 *The function $u(y, t)$ of a rational price of the American option defined by equality* (1.2) *is a solution of the following optimal stopping problem*

$$u(y, t) = \sup_{0 \le r \le t} E_y(y_\tau - 1), \tag{1.3}$$

where the random process $y_t = \max_{u \le t} S_u / S_t$.

Proof. An American type put option with the payoff function (1.2) is of the so-called "Russian type" put option. Since its fulfillment may occur at any (random) moment on the finite time interval $[0, T]$, an option pricing problem is actually an optimal stopping problem on a finite time interval. If we introduce the random process $y_t = \max_{u \le t} S_u / S_t$, then we can easily write the rational price function of an American put option in form (1.3) or as the price of the process y_t in an optimal stopping problem ■

Lemma 1.2 *In problem* (1.3) *the optimal stopping time can be written in the form*

$$\tau_t = \inf \{s, \ 0 \le s \le t : \ \ y_s \ge b(t-s)\}, \tag{1.4}$$

where the function $b(t)$ *is an optimal stopping boundary.*

Proof. The form of an optimal stopping time moment is well known from the results of the optimal stopping theory for a Markov random process on a finite time interval (see [2], [3]). The optimal stopping time moment defined by equality (1.4) is a particular case of the results obtained in these papers, where the optimal stopping problem for the process y_t is considered ■

2. *Basic results*

Theorem 2.1 *In the financial market model* (1.1) *we consider the American option defined by equality* (1.3)*. Then*

1) *In the domain* $\{(y,t) : \ 1 \le y \le b(t)\}$ *the function* $u(y,t)$ *satisfies the equation*

$$Lu(y,t) = 0, \quad u(y,0) = y - 1, \tag{2.1}$$

with the boundary conditions

$$u(b(t), t) = b(t) - 1, \quad \frac{\partial u}{\partial y}(1+, t) = 0, \quad \frac{\partial u}{\partial y}(b(t)-, t) = 1,$$

where L *is the following parabolic differential operator*

$$L = \frac{\sigma^2}{2}\, y^2 \,\frac{\partial^2 u}{\partial y^2} - ry\,\frac{\partial u}{\partial y} - \frac{\partial u}{\partial t}\,, \tag{2.2}$$

2) *The optimal stopping boundary* $b(t)$ *is a solution of the integral equation*

$$y = E_y y_t - r \int_0^t E_y(y_s I_{(y_s \ge b(t-s))})\, ds. \tag{2.3}$$

Proof. It is the well known fact that in the optimal stopping problem of a Markov process the price is the so-called solution of Stefan's problem with a free boundary. For this it is necessary that the price be twice continuously differentiable. In that case, using Lemmas 1.1 and 1.2 it is easy to show that the option price is given by the solution of equation (2.1), where the optimal stopping boundary $b(t)$ satisfies equation (2.3) ■

3. *Discrete model*

Consider the Cox–Ross–Rubinstein binomial model of a financial market

$$B_n=(1+r)B_{n-1}, \quad S_n=(1+\rho_n)S_{n-1}, \quad B_0>0, \quad S_0>0, \tag{3.1}$$

where the interest rate $r > 0$ and $\rho = (\rho_n)$ is a sequence of independent, identically distributed random variables taking only two values a and b, $-1 < a < r < b$, $n = 0, 1, \ldots, N$.

Let $f = f_n(x)$, $n = 0, 1, \ldots, N$, be some payoff function of the American option. We introduce the operator

$$Tf(x) = (1+r)^{-1}[pf((1+b)x) + (1-p)f((1+a)x)], \quad p = \frac{r-a}{b-a}. \tag{3.2}$$

Lemma 3.1 *An investors capital process is defined by the equality*

$$X_n = Tf_{n+1}(S_n).$$

Proof. Assume that at a moment of time n the investor's strategy is $\pi_n = (\beta_n, \gamma_n)$, the corresponding capital of which is

$$X_n^\pi = \beta_n B_n + \gamma_n S_n.$$

At a moment of time n we need to construct a strategy $\pi_{n+1} = (\beta_{n+1}, \gamma_{n+1})$. such that the equality

$$X_{n+1}^\pi = \beta_{n+1}B_{n+1} + \gamma_{n+1}S_{n+1} = f(S_{n+1})$$

be fulfilled at a moment of time $n + 1$.

Using the financial (B, S)-market model it is easy to show that the capital process can be written in the form

$$X_n^\pi = \frac{1}{1+r}\left[\frac{r-a}{b-a}\, f((1+b)S_n) + \frac{b-r}{b-a}\, f((1+a)S_n)\right]$$

which immediately implies equality (10) ■

Lemma 3.2 *A valid (rational) price $P_n(x)$ of the American option satisfies the recurrent equation*

$$P_n(x) = \max\left(f_n(x), TP_{n+1}(x)\right), \quad n = 0, 1, \dots, N-1, \quad P_N(x) = f_N(x).$$

Proof. The proof of Lemma 3.2 immediately follows from Lemma 3.1 if we use the recurrent equation from [1] for a price in the optimal stopping problem of a Markov random sequence ■

If we consider nonselffinancing strategies with some consumption, then the capital process is divided into two parts and the consumption process is defined by the equality

$$c_{n+1} = P_n(S_n) - TP_{n+1}(S_n),$$

while the sum $TP_{n+1}(S_n)$ is used to construct a portfolio $\pi^*_{n+1}=(\beta^*_{n+1}, \gamma^*_{n+1})$ whose capital $X^{\pi^*}_{n+1}$ will be equal to $P_{n+1}(S_{n+1})$. As to a rational time moment for the American option fulfillment, we can take, for instance, the time moment

$$\tau = \inf\{n : \ f_n(S_n) \geq TP_{n+1}(S_n)\}.$$

Moreover, the operator T can be used to establish a relation between the prices of standard European and American options and also, correspondingly to an optimal stopping problem, to define the "stopping", "uncertainty" and "continuation" domains.

Theorem 3.1 *Let the payoff functions of the sellers and buyers standard American option be respectively*

$$f(x) = \max(x - K, 0), \quad f(x) = \max(K - x, 0),$$

where $K > 0$ is an agreed price. Then

1) $f(S_n) > TP_{n+1}(S_n)$ *is the "stopping" domain;*

2) $f(S_n) < TP_{n+1}(S_n)$ *is the "continuation" domain;*

3) $f(S_n) = TP_{n+1}(S_n)$ *is the "uncertainty" domain.*

Proof. The proof of the theorem immediately follows from Lemmas 3.1 and 3.2 if we use representation (3.2) of the operator T ■

Example. Assume that the consumption is given by the equality

$$c_n = c_1\beta_n B_{n-1} + c_2\gamma_n S_{n-1}, \quad 0 < c_1 < 1, \quad 0 < c_2 < 1.$$

Then the operator T has the form

$$Tf(x) = \frac{1 + c_1}{1 + r}\left[p^* f((1 + b)x) + (1 - p^*) f((1 + a)x)\right], \tag{3.3}$$

where

$$p^* = \frac{r - c_1(1+a) + c_2(1+r) - a}{(b-a)(1+c_1)}\,. \tag{3.4}$$

As to a rational price, it can be calculated by the recurrent equality

$$P_{N-k,j} = \max\left\{ f_{N-k,j},\ \frac{1+c_1}{1+r}\left[p^* P_{N-k+1,j+1} + (1-p^*)P_{N-k+1,j}\right]\right\}, \tag{3.5}$$

where $k = 1, \dots, N$, $j = 0, 1, \dots, N-k$, p^* is defined by equality (3.4).

Remark. The facts given in Subsection 3 and in the Example can be easily proved also in the case of the sellers American option with the payoff function $f_n(x) = \beta^n \cdot \max(x-K, 0)$, $0 < \beta < 1$.

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