METHODS OF INTERPOLATION OF THE DEFINITE CLASS OF NON-LINEAR FUNCTIONS WITH PRACTICAL EXAMPLES

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(Received: 12.08.05; accepted: 17.12.05)

Abstract

In the paper there are considered one-parameter families of functions from polynomials and a set of non-linear functions of real variables depended on many parameters. A general method of determination of unknown parameters values for both equidistant and non-equidistant values of argument is offered. The method allows to reduce the interpolation task to solving of system of non linear equations (consisted of one or two equations) and finding the initial approximations for roots of these equations, for which the monotonous convergence of the iteration sequence to the unknown root of the system is guaranteed.

Key words and phrases: non-linear function; interpolation; system of equations; initial approximation; iteration method.

Introduction

The problem of interpolating of non-linear functions arises at the solution of the many miscellaneous problems of science and practice. For example, at restoration of non-linear functional connections under the data which do not contain accidental errors, at solving of problems by methods of the spline-analysis, at definition of initial intervals of parameters in problems of identification of non-linear regressions [1, 2, 3], etc. Therefore elaboration of optimum algorithms of interpolation of non-linear functional connections and their research is an actual problem having a wide practical use.

Below are given algorithms of interpolation of one-parameter families of functions from polynomials and some non-linear functions, which are most frequently used at restoration of functional connections in many practical applications. These algorithms are used by the authors in designed by them universal application package of processing of the experimental data (SD-pro) for a PC-compatible personal computers in the section of identification of functional dependencies for finding optimum initial intervals for definition of unknown parameters of these relations [1, 2, 4] and at the solution of many practical problems connected with the identification of statistical models of propagation of pollutants in the rivers.

2. One-parameter families of functions from polynomials

Let $[z_0, z_1, ..., z_N]$ and $[w_0, w_1, ..., w_N]$ be sequences of complex numbers; $z_j \neq z_k$ at $j \neq k$, $j, k = 0, 1, ..., N$; $\psi(\gamma, z, w)$ is a complex function of three variables. It is required to determine a polynomial of degree $N-1$

$$\alpha(z) = \sum_{k=0}^{N-1} \alpha_k \, z^k$$

and numeric parameter γ, for which values of the function $\psi\big(\gamma, z, \alpha(z)\big)$ in given nodes z_0, z_1, ..., z_N coincide with numbers w_0, w_1, ..., w_N, accordingly.

Let us introduce auxiliary functions

$$\varrho^{(m)}(\xi_1, \xi_2, ..., \xi_m) \quad \text{and} \quad \lambda_{jk}^{(m)}(\xi_1, \xi_2, ..., \xi_m)$$

$(m = 1, 2, ...;\ j, k = 1, ..., m)$, defined for a sequence of m complex numbers ξ_1, ξ_2, ..., ξ_m as follows: the values $\lambda_{jk}^{(m)}(\xi_1, ..., \xi_m)$ are elements of the matrix, inverse of the square $m \times m$ matrix

$$\mu^{(m)} = \begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{m-1} \\ 1 & \xi_2 & \xi_2^2 & \cdots & \xi_2^{m-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \xi_m & \xi_m^2 & \cdots & \xi_m^{m-1} \end{bmatrix}$$

and

$$\varrho^{(m)}(\xi_1, \xi_2, ..., \xi_m) \equiv \det\big(\mu^{(m)}\big) = \prod_{k=2}^{m} \prod_{j=1}^{k-1} (\xi_k - \xi_j).$$

is a Vandermonde determinant for the numbers ξ_1, ξ_2, ..., ξ_m [5].

The theorem 1: If the function $\Psi(\xi) = \psi(\gamma, z, \xi)$ has inverse function $\Phi(\eta) = \varphi(\gamma, z, \eta)$ for any fixed values γ and z, then the parameters α_0, α_1, ..., α_{N-1} are expressed in terms of the parameter γ and initial data of the problem by relation

$$\alpha_{k-1} = \sum_{j=1}^{N} \lambda_{kj}^{(N)}(z_1, z_2, ..., z_N) \cdot \varphi(\gamma, z_j, w_j), \tag{2.1}$$

and parameter γ is a solution of the equation

$$\sum_{k=0}^{N} \beta_k \cdot \varphi(\gamma, z_k, w_k) = 0, \tag{2.2}$$

where

$$\beta_k = (-1)^k \cdot \varrho^{(N)}(z_0, ..., z_{k-1}, z_{k+1}, ..., z_N).$$

In particular, at $N = 2$

$$\beta_0 = z_2 - z_1; \quad \beta_1 = z_0 - z_2; \quad \beta_2 = z_1 - z_0;$$

at $N = 3$

$$\beta_0 = (z_3 - z_1)(z_3 - z_2)(z_2 - z_1); \quad \beta_1 = -(z_3 - z_2)(z_3 - z_0)(z_2 - z_0);$$
$$\beta_2 = (z_3 - z_1)(z_3 - z_0)(z_1 - z_0); \quad \beta_3 = -(z_2 - z_1)(z_2 - z_0)(z_1 - z_0).$$

The proof: It is obvious that the parameters γ, α_0, ..., α_{N-1} can be determined as a solution to the system of equations

$$\alpha(z_k) = \varphi(\gamma, z_k, w_k), \quad k = 0, 1, 2, ..., N. \tag{2.3}$$

To prove (2.1) it is sufficient to note, that the system of equations (2.3) at $k = 1, 2, ..., N$ is equivalent to the system of linear equations for coefficients of the polynomial $\alpha(z)$, whose determinant is equal to Vandermonde determinant for numbers z_1, z_2, ..., z_N. According to conditions of the problem, this determinant is equal to zero, whence follows uniqueness of the solution.

Substituting in equations (2.3) coefficients of the polynomial $\alpha(z)$, defined by the formula (2.1), we receive

$$\sum_{k=0}^{N} \beta_k \cdot \varphi(\gamma, z_k, w_k) = 0,$$

where β_0 is an arbitrary nonzero constant and

$$\beta_k = -\beta_0 \sum_{j=1}^{N} z_0^{j-1} \cdot \lambda_{jk}^{(N)}(z_1, z_2, ..., z_N), \quad k = 1, 2, ..., N.$$

Multiplying these equalities on z_k^{L-1} and summarizing on k in the interval from 1 up to N, we receive the following system of equations for coefficients β_k:

$$\sum_{k=0}^{N} \beta_k \, z_k^{L-1} = 0 \qquad (L = 1, ..., N).$$

We select the coefficient β_0 so that the following additional relation be obtained:

$$\sum_{k=0}^{N} \beta_k \, z_k^N = \varrho_0,$$

where $\varrho_0 = \varrho^{(N+1)}(z_0, ..., z_N)$. As a result the system of equations for required coefficients becomes

$$\sum_{k=0}^{N} \beta_k \, z_k^L = \varrho_0 \cdot \delta_{LN} \qquad (L = 0, ..., N).$$

where δ_{LN} is Kronecker symbol. This system is solvable uniquely

$$\beta_k = \varrho_0 \cdot \lambda^{(N+1)}_{(N+1),(k+1)}(z_0, z_1, ..., z_N), \qquad k = 0, ..., N.$$

Let us compare these parameters to the coefficients

$$\beta'_k \equiv (-1)^k \cdot \varrho^{(N)}(z_0, ..., z_{k-1},\ z_{k+1}, ..., z_N).$$

Allowing that β'_{k+1} represents a cofactor of element $\mu_{k,N+1}$ of the matrix $[\mu_{jk}]^{(N+1)\times(N+1)}$, where $\mu_{jk} = z_{j-1}^{k-1}$, we receive $\beta_k = \beta'_k$ $(k = 0, ..., N)$, whence follows (2.2).

Examples: 1. One-parameter functions from polynomials

a) At interpolation of logarithmic-polynomial relation

$$\psi\big(\gamma, z, \alpha(z)\big) = \gamma \cdot \ln\left(\sum_{k=0}^{N} \alpha_k \, z^k\right)$$

parameter γ can be determined as the solution of an equation

$$\sum_{k=0}^{N} \beta_k \cdot e^{w_k/\gamma} = 0,$$

and the coefficients of the polynomial are equal to

$$\alpha_{k-1} = \sum_{j=1}^{N} \lambda^{(N)}_{kj}(z_1, z_2, ..., z_N) \cdot e^{w_j/\gamma}, \qquad k = 1, ..., N.$$

b) At interpolation of geometric-polynomial relation

$$\psi\big(\gamma, z, \alpha(z)\big) = z^\gamma \cdot \sum_{k=0}^{N} \alpha_k \, z^k$$

and exponential-polynomial relations

$$\psi\big(\gamma, z, \alpha(z)\big) = e^{\gamma z} \cdot \sum_{k=0}^{N} \alpha_k \, z^k$$

, parameter γ and coefficients of the polynomial are determined by relation

$$\sum_{k=0}^{N} \beta_k \, w_k \cdot z_k^{-\gamma} = 0;$$

$$\alpha_{k-1} = \sum_{j=1}^{N} \lambda_{kj}^{(N)}(z_1, z_2, ..., z_N) \cdot w_j \cdot z_j^{-\gamma}, \qquad k = 1, ..., N,$$

or, accordingly,

$$\sum_{k=0}^{N} \beta_k \, w_k \cdot e^{-\gamma z_k} = 0;$$

$$\alpha_{k-1} = \sum_{j=1}^{N} \lambda_{kj}^{(N)}(z_1, z_2, ..., z_N) \cdot w_j \cdot e^{-\gamma z_j}, \quad k = 1, ..., N.$$

3. Function $a + b \cdot e^{cx}$

Let it be required to interpolate the function $a + b\, e^{cx}$ on three pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$; in other words, it is required to solve the system of equations

$$a + b \cdot e^{cx_1} = y_1; \quad a + b \cdot e^{cx_2} = y_2; \quad a + b \cdot e^{cx_3} = y_3$$

regarding the parameters a, b, c. The conditions $x_1 \neq x_2 \neq x_3$ and $y_1 \neq y_2 \neq y_3$ are supposed to be satisfied. Let us introduce an auxiliary quantity $s = \frac{y_3 - y_2}{y_2 - y_1}$; then $s + 1 = \frac{y_3 - y_1}{y_2 - y_1}$. Parameter c is a nonzero solution of the equation

$$(y_2 - y_3) \cdot e^{cx_1} + (y_3 - y_1) \cdot e^{cx_2} + (y_1 - y_2) \cdot e^{cx_3} = 0.$$

The parameters a and b are determined by the relation

$$a = \frac{y_2 \cdot e^{cx_1} - y_1 \cdot e^{cx_2}}{e^{cx_1} - e^{cx_2}} = \frac{y_3 \cdot e^{cx_2} - y_2 \cdot e^{cx_3}}{e^{cx_2} - e^{cx_3}};$$

$$b = \frac{y_1 - y_2}{e^{cx_1} - e^{cx_2}} = \frac{y_2 - y_3}{e^{cx_2} - e^{cx_3}}.$$

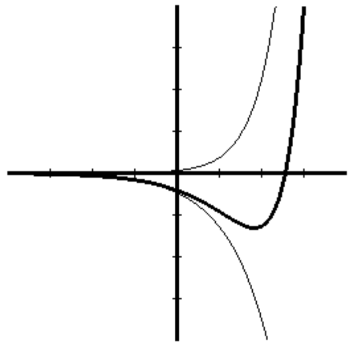

Figure 0.1: Plots of the functions $y = a \cdot e^{cx} + b \cdot e^{dx}$ (bold line), $y = a \cdot e^{cx}$, $y = b \cdot e^{dx}$ (thin lines) at $a < 0$, $b > 0$, $c > 0$, $d > 0$.

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 \equiv \Delta x$, then the initial system of equations has the solution in that and only in that case, if $s > 0$; thus $c = \ln s / \Delta x$.

Let $x_1 < x_2 < x_3$. The initial system of equations has a solution in that and only that case, if $s > 0$. The equation, which should be satisfied by the parameter c, conveniently can be represented in the form $f(c) = 0$, where

$$f(c) = s - (s+1) \cdot e^{(x_2 - x_1) \cdot c} + e^{(x_3 - x_1) \cdot c}.$$

The plot of the function $f(c)$ with an accuracy of parallel shift looks as introduced in the fig. 0.1. The number

$$c_0 = (x_3 - x_2)^{-1} \cdot \ln \left((s+1) \cdot \frac{x_2 - x_1}{x_3 - x_1} \right)$$

is a minimum point of the function $f(c)$; the number

$$c_1 = (x_3 - x_2)^{-1} \cdot \ln \left((s+1) \cdot \left(\frac{x_2 - x_1}{x_3 - x_1} \right)^2 \right)$$

is an abscissa of inflection point of the plot of the function $f(c)$; the number

$$c_2 = (x_3 - x_2)^{-1} \cdot \ln(s+1)$$

is an upper bound for c.

As an initial approximation for c at the solution of equation $f(c) = 0$ by iteration method of Newton can be taken the number c_1 (at $c_0 < 0$) or c_2 (at $c_0 > 0$).

Let us also note that $c/c_0 > 1$.

4. Function $(a + b\,x) \cdot e^{cx}$

Let it be required to interpolate the function $(a+b\,x)\cdot e^{cx}$ on three pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, i.e. to solve the system of equations

$$(a + b\,x_1) \cdot e^{cx_1} = y_1; \quad (a + b\,x_2) \cdot e^{cx_2} = y_2; \quad (a + b\,x_3) \cdot e^{cx_3} = y_3$$

regarding parameters a, b, c.

The parameter c is a solution to the equation

$$(x_2 - x_3) \cdot y_1 \cdot e^{-cx_1} + (x_3 - x_1) \cdot y_2 \cdot e^{-cx_2} + (x_1 - x_2) \cdot y_3 \cdot e^{-cx_3} = 0.$$

The parameters a and b are determined by the relation

$$a = \frac{1}{x_2 - x_1} \cdot \Big(x_2 \cdot y_1 \cdot e^{-cx_1} - x_1 \cdot y_2 \cdot e^{-cx_2}\Big);$$
$$b = \frac{1}{x_2 - x_1} \cdot \Big(-y_1 \cdot e^{-cx_1} + y_2 \cdot e^{-cx_2}\Big).$$

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 \equiv \Delta x$, then the value $u \equiv \exp(c \cdot \Delta x)$ satisfies to the quadratic equation

$$y_1 \cdot u^2 - 2y_2 \cdot u + y_3 = 0.$$

Let $x_1 < x_2 < x_3$. The equation, which should be satisfied by the parameter c, conveniently can be represented in the form $f(c) = 0$, where

$$f(c) = H_1 \cdot e^{(x_3 - x_1)c} + H_2 \cdot e^{(x_3 - x_2)c} + H_3;$$
$$H_1 = y_1 \cdot (x_2 - x_3); \quad H_2 = y_2 \cdot (x_3 - x_1); \quad H_3 = y_3 \cdot (x_1 - x_2).$$

Depending on signs of the numbers y_1, y_2 and y_3 some cases are possible:

a) $y_1 y_2 \le 0$ and $y_2 y_3 \le 0$. The function $f(c)$ has no zero points.

b) $y_1 y_2 < 0$ and $y_2 y_3 > 0$. The function $f(c)$ has unique zero point, as an initial approximation for which, at the solution of an equation $f(c) = 0$ by iteration method of Newton, the following number can be taken

$$\max\left\{\frac{1}{x_3 - x_2} \cdot \ln\Big(\frac{-H_3}{H_1 + H_2}\Big),\ \frac{1}{x_3 - x_1} \cdot \ln\Big(\frac{-H_3}{H_1 + H_2}\Big)\right\}$$

(being upper bound for c).

c) $y_1 y_2 > 0$ and $y_2 y_3 < 0$. The function $f(c)$ has unique zero point, as an initial approaching for which one at the solution of an equation $f(c) = 0$ by iteration method of Newton the following number can be taken

$$\max\left\{\frac{1}{x_2 - x_1} \cdot \ln\Big(\frac{H_2 + H_3}{-H_1}\Big),\ \frac{1}{x_3 - x_1} \cdot \ln\Big(\frac{H_2 + H_3}{-H_1}\Big)\right\}$$

(being upper bound for c).

d) $y_1 y_2 > 0$ and $y_2 y_3 > 0$. Let us introduce supplementary notations:

$$c_0 = (x_2 - x_1)^{-1} \cdot \ln(y_2/y_1); \quad \Delta c = (x_2 - x_1)^{-1} \cdot \ln\left(\frac{x_3 - x_2}{x_3 - x_1}\right);$$

$$v_0 = \left(\frac{-H_3}{f(c_0) - H_3}\right)^{x_2 - x_1} = y_1^{x_3 - x_2} \cdot y_2^{x_1 - x_3} \cdot y_3^{x_2 - x_1} = \\ = \big(y_3/y_2\big)^{x_2 - x_1} \cdot \big(y_1/y_2\big)^{x_3 - x_2}.$$

(c_0 is an extremum point of the function $f(c)$; $c_0 - \Delta c$ is an abscissa of inflection point of this function).

At $v_0 < 1$ the function $f(c)$ has two zero points c' and c''. Let us assume for determinacy that $c' < c''$; then $c' < c_0 < c'' < c_0 + \Delta c$. The numbers $c_0 - \Delta c$ and $c_0 + \Delta c$ can be utilized as initial approximations, accordingly, for c' and c'' at solution of the equation $f(c) = 0$ by an iteration method of Newton.

At $v_0 = 1$ the function $f(c)$ has unique zero point conterminous with c_0.

At $v_0 > 1$ the function $f(c)$ have no zero points.

5. Function $h + (a + bx) \cdot e^{cx}$

Let it be required to interpolate the function $h + (a + bx) \cdot e^{cx}$ on four pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_4, y_4\}$, i.e., to solve the system of equations

$$h + (a + bx_1) \cdot e^{cx_1} = y_1; \quad h + (a + bx_2) \cdot e^{cx_2} = y_2;$$
$$h + (a + bx_3) \cdot e^{cx_3} = y_3; \quad h + (a + bx_4) \cdot e^{cx_4} = y_4$$

regarding parameters h, a, b, c.

The parameter c is a nonzero solution of the equation $f(c) = 0$, where

$$f(c) = \\ (y_3 - y_4) \cdot (x_2 - x_1) \cdot e^{c \cdot (x_1 + x_2)} + (y_2 - y_4) \cdot (x_1 - x_3) \cdot e^{c \cdot (x_1 + x_3)} + \\ + (y_2 - y_3) \cdot (x_4 - x_1) \cdot e^{c \cdot (x_1 + x_4)} + (y_1 - y_4) \cdot (x_3 - x_2) \cdot e^{c \cdot (x_2 + x_3)} + \\ + (y_1 - y_3) \cdot (x_2 - x_4) \cdot e^{c \cdot (x_2 + x_4)} + (y_1 - y_2) \cdot (x_4 - x_3) \cdot e^{c \cdot (x_3 + x_4)}.$$

The number 0 always is zero point of functions $f(c)$ and $\dot{f}(c)$.

The parameters h, a and b are determined by the relation

$$a = \frac{1}{D}\Big(y_1 \cdot \big(x_2 \cdot e^{cx_2} - x_3 \cdot e^{cx_3}\big) + y_2 \cdot \big(x_3 \cdot e^{cx_3} - x_1 \cdot e^{cx_1}\big) + \\ + y_3 \cdot \big(x_1 \cdot e^{cx_1} - x_2 \cdot e^{cx_2}\big)\Big);$$

$$b = -\frac{1}{D}\Big(y_1 \cdot \big(e^{cx_2} - e^{cx_3}\big) + y_2 \cdot \big(e^{cx_3} - e^{cx_1}\big) + \\ + y_3 \cdot \big(e^{cx_1} - e^{cx_2}\big)\Big);$$

$$h = \frac{1}{D}\Big(y_1 \cdot \big(x_3 - x_2\big) \cdot e^{c(x_2+x_3)} + y_2 \cdot \big(x_1 - x_3\big) \cdot e^{c(x_3+x_1)} + \\ + y_3 \cdot \big(x_2 - x_1\big) \cdot e^{c(x_1+x_2)}\Big),$$

where

$$D = (x_3 - x_2) \cdot e^{c(x_2+x_3)} + (x_1 - x_3) \cdot e^{c(x_3+x_1)} + (x_2 - x_1) \cdot e^{c(x_1+x_2)}.$$

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \equiv \Delta x$, then the value $u \equiv \exp(c \cdot \Delta x)$ satisfies to the quadratic equation

$$(y_2 - y_1) \cdot u^2 + 2\,(y_2 - y_3) \cdot u + (y_4 - y_3) = 0.$$

Let us allow, that the ratios of differences of any given values of the argument are equal to some rational numbers. Then values of the argument can be represented by the way $x_k = X_c + r_k \cdot \Delta x$, where r_k $(k = 1, 2, 3)$ are some integers; X_c, $\Delta x = \text{const}$. In this case the value $u \equiv \exp(c \cdot \Delta x)$ is the radical of the algebraic equation

$$r_{43} \cdot (y_2 - y_1) \cdot u^{r_{21}+2r_{32}+r_{43}} + 2 \cdot (r_{32} + r_{43}) \cdot (y_1 - y_3) \cdot u^{r_{21}+r_{32}+r_{43}} + \\ + r_{32} \cdot (y_4 - y_1) \cdot u^{r_{21}+r_{32}} + (r_{21} + r_{32} + r_{43}) \cdot (y_3 - y_2) \cdot u^{r_{32}+r_{43}} + \\ + (r_{21} + r_{32}) \cdot (y_2 - y_4) \cdot u^{r_{32}} + r_{21} \cdot (y_4 - y_3) = 0,$$

where $r_{jk} \equiv r_j - r_k$; $j, k = 1, 2, 3$.

6. Function $a\,x^c \cdot (1 - b\,x)^d$

Let it be required to interpolate the function $a\,x^c \cdot (1 - b\,x)^d$ on four pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_4, y_4\}$, i.e., to solve a system of equations

$$a\,x_1^c \cdot (1 - b\,x_1)^d = y_1; \qquad a\,x_2^c \cdot (1 - b\,x_2)^d = y_2;$$
$$a\,x_3^c \cdot (1 - b\,x_3)^d = y_3; \qquad a\,x_4^c \cdot (1 - b\,x_4)^d = y_4$$

regarding parameters a, b, c, d. Let us denote

$$Z_k \equiv \ln x_k \qquad W_k \equiv \ln y_k, \quad k = 1, 2, 3, 4.$$

The parameter b is a nonzero solution of the equation $f(b) = 0$, where

$$f(b) = H_1 \cdot \ln(1 - b\,x_1) + H_2 \cdot \ln(1 - b\,x_2) + \\ + H_3 \cdot \ln(1 - b\,x_3) + H_4 \cdot \ln(1 - b\,x_4);$$

$$H_1 = W_2 \cdot (Z_4 - Z_3) + W_3 \cdot (Z_2 - Z_4) + W_4 \cdot (Z_3 - Z_2);$$
$$H_2 = W_1 \cdot (Z_3 - Z_4) + W_3 \cdot (Z_4 - Z_1) + W_4 \cdot (Z_1 - Z_3);$$
$$H_3 = W_1 \cdot (Z_4 - Z_2) + W_2 \cdot (Z_1 - Z_4) + W_4 \cdot (Z_2 - Z_1);$$
$$H_4 = W_1 \cdot (Z_2 - Z_3) + W_2 \cdot (Z_3 - Z_1) + W_3 \cdot (Z_1 - Z_2).$$

The number 0 always is zero point of the function $f(b)$.
The parameters a, c d are determined by relations

$$\ln a = T^{-1} \cdot (W_2 Z_3 - W_3 Z_2) \cdot \ln(1 - b\,x_1) + \\ + T^{-1} \cdot (W_3 Z_1 - W_1 Z_3) \cdot \ln(1 - b\,x_2) + T^{-1} \cdot (W_1 Z_2 - W_2 Z_1) \cdot \ln(1 - b\,x_3);$$
$$c = T^{-1} \cdot (W_3 - W_2) \cdot \ln(1 - b\,x_1) + \\ + T^{-1} \cdot (W_1 - W_3) \cdot \ln(1 - b\,x_2) + T^{-1} \cdot (W_2 - W_1) \cdot \ln(1 - b\,x_3);$$
$$d = T^{-1}\Big(W_1 \cdot (Z_3 - Z_2) + W_2 \cdot (Z_1 - Z_3) + W_3 \cdot (Z_2 - Z_1)\Big),$$

where

$$T = (Z_3 - Z_2) \cdot \ln(1 - b\,x_1) + (Z_1 - Z_3) \cdot \ln(1 - b\,x_2) + (Z_2 - Z_1) \cdot \ln(1 - b\,x_3).$$

7. Function $a\,e^{cx} + b\,e^{dx}$

Let it be required to interpolate the function $a\,e^{cx} + b\,e^{dx}$ on four pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_4, y_4\}$, i.e., to solve the system of equations

$$a\,e^{cx_1} + b\,e^{dx_1} = y_1; \qquad a\,e^{cx_2} + b\,e^{dx_2} = y_2;$$
$$a\,e^{cx_3} + b\,e^{dx_3} = y_3; \qquad a\,e^{cx_4} + b\,e^{dx_4} = y_4$$

regarding parameters a, b, c, d.

The pair of numbers $[c, d]$ can be determined as a solution of any pair from the following four equations

$$\begin{gathered}
y_1 \cdot \left(e^{cx_2+dx_3} - e^{cx_3+dx_2}\right) + y_2 \cdot \left(e^{cx_3+dx_1} - e^{cx_1+dx_3}\right) + \\
+y_3 \cdot \left(e^{cx_1+dx_2} - e^{cx_2+dx_1}\right) = 0; \\
y_2 \cdot \left(e^{cx_3+dx_4} - e^{cx_4+dx_3}\right) + y_3 \cdot \left(e^{cx_4+dx_2} - e^{cx_2+dx_4}\right) + \\
+y_4 \cdot \left(e^{cx_2+dx_3} - e^{cx_3+dx_2}\right) = 0; \\
y_1 \cdot \left(e^{cx_2+dx_4} - e^{cx_4+dx_2}\right) + y_2 \cdot \left(e^{cx_4+dx_1} - e^{cx_1+dx_4}\right) + \\
+y_4 \cdot \left(e^{cx_1+dx_2} - e^{cx_2+dx_1}\right) = 0; \\
y_1 \cdot \left(e^{cx_3+dx_4} - e^{cx_4+dx_3}\right) + y_3 \cdot \left(e^{cx_4+dx_1} - e^{cx_1+dx_4}\right) + \\
+y_4 \cdot \left(e^{cx_1+dx_3} - e^{cx_3+dx_1}\right) = 0
\end{gathered}$$

at additional condition $c \neq d$.

The parameters a and b are determined by relations

$$a = \frac{y_1 \cdot e^{dx_4} - y_4 \cdot e^{dx_1}}{e^{cx_1+dx_4} - e^{cx_4+dx_1}}; \quad b = \frac{-y_1 \cdot e^{cx_4} - y_4 \cdot e^{cx_1}}{e^{cx_1+dx_4} - e^{cx_4+dx_1}},$$

and in these formulas two pairs of variables $[x_1, y_1]$ and $[x_4, y_4]$ can be replaced by any other pairs $[x_j, y_j]$ and $[x_k, y_k]$ provided that $j \neq k$.

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \equiv \Delta x$, then the values $u \equiv \exp(c \cdot \Delta x)$ and $v \equiv \exp(d \cdot \Delta x)$ are radicals of the quadratic equation

$$(y_2^2 - y_1\, y_3) \cdot \xi^2 + (y_1\, y_4 - y_2\, y_3) \cdot \xi + (y_3^2 - y_2\, y_4) = 0.$$

The initial system of equations has the solution in that and only in that case, if this quadratic equation has two real, positive and not conterminous with each another radicals.

8. Function $h + a\, e^{cx} + b\, e^{dx}$

Let it be required to interpolate the function $h + a\, e^{cx} + b\, e^{dx}$ on five pairs of numbers $\{x_k, y_k\}$ $(k = 1, ..., 5)$, i.e. to solve the system of equations

$$h + a\, e^{cx_k} + b\, e^{dx_k} = y_k, \qquad k = 1, ..., 5$$

regarding the parameters h, a, b, c, d.

The pair of numbers $[c, d]$ can be determined as a radical of the system of equations at additional $c \neq d$, $cd \neq 0$. One of the equations of this system looks as

$$y_1 \cdot \left(e^{cx_3+dx_2} - e^{cx_2+dx_3} + e^{cx_2+dx_4} - e^{cx_4+dx_2} + e^{cx_4+dx_3} - e^{cx_3+dx_4}\right) +$$
$$+ y_2 \cdot \left(e^{cx_1+dx_3} - e^{cx_3+dx_1} + e^{cx_3+dx_4} - e^{cx_4+dx_3} + e^{cx_4+dx_1} - e^{cx_1+dx_4}\right) +$$
$$+ y_3 \cdot \left(e^{cx_2+dx_1} - e^{cx_1+dx_2} + e^{cx_1+dx_4} - e^{cx_4+dx_1} + e^{cx_4+dx_2} - e^{cx_2+dx_4}\right) +$$
$$+ y_4 \cdot \left(e^{cx_1+dx_2} - e^{cx_2+dx_1} + e^{cx_2+dx_3} - e^{cx_3+dx_2} + +e^{cx_3+dx_1} - e^{cx_1+dx_3}\right) = 0,$$

and the second equation can be obtained from the first by replacement the arbitrary pair of numbers $[x_j, y_j]$ $(j = 1, ..., 4)$ with $[x_5, y_5]$.

The parameters h, a, b are determined by the relation

$$a = \frac{1}{D}\left(y_1 \cdot \left(e^{dx_2} - e^{dx_3}\right) + y_2 \cdot \left(e^{dx_3} - e^{dx_1}\right) + y_3 \cdot \left(e^{dx_1} - e^{dx_2}\right)\right);$$
$$b = \frac{1}{D}\left(y_1 \cdot \left(e^{cx_3} - e^{cx_2}\right) + y_2 \cdot \left(e^{cx_1} - e^{cx_3}\right) + y_3 \cdot \left(e^{cx_2} - e^{cx_1}\right)\right);$$
$$h = \frac{1}{D}\Big(y_1 \cdot \left(e^{cx_2+dx_3} - e^{cx_3+dx_2}\right) + y_2 \cdot \left(e^{cx_3+dx_1} - e^{cx_1+dx_3}\right) +$$
$$+ y_3 \cdot \left(e^{cx_1+dx_2} - e^{cx_2+dx_1}\right)\Big);$$

where

$$D = e^{cx_1+dx_2} + e^{cx_2+dx_3} + e^{cx_3+dx_1} -$$
$$- e^{cx_2+dx_1} - e^{cx_3+dx_2} - e^{cx_1+dx_3},$$

and in these formulas three pairs of variables $[x_1, y_1], [x_2, y_2]$ and $[x_3, y_3]$ can be replaced by any other pairs $[x_j, y_j]$, $[x_k, y_k]$ and $[x_l, y_l]$ provided that $j \neq k \neq l$.

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = x_5 - x_4 \equiv \Delta x$, then the values $u \equiv \exp(c \cdot \Delta x)$ and $v \equiv \exp(d \cdot \Delta x)$ are the radicals of the quadratic equation

$$\xi^2 + p\,\xi + q = 0,$$

where

$$p = \frac{(y_3 - y_2)(y_3 - y_4) + (y_1 - y_2)(y_4 - y_5)}{(y_3 - y_2)^2 + (y_1 - y_2)(y_4 - y_3)};$$
$$q = \frac{(y_3 - y_4)^2 + (y_3 - y_2)(y_4 - y_5)}{(y_3 - y_2)^2 + (y_1 - y_2)(y_4 - y_3)}. \tag{8.4}$$

The initial system of equations has a solution in that and only in that case, when this quadratic equation has two real, positive and not conterminous with one another radicals.

9. Function $e^{sx} \cdot \big(A\,\cos(\omega x) + B\,\sin(\omega x)\big)$

Let it be required to interpolate the considered function on four pairs of numbers $\{x_1, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_4, y_4\}$, i.e. to solve the system of equations

$$e^{sx_k} \cdot \big(A\,\cos(\omega x_k) + B\,\sin(\omega x_k)\big) = y_k, \quad k = 1, ..., 4$$

regarding parameters A, B, s, ω.

This interpolation is equivalent to interpolation of the function

$$\alpha\, e^{\lambda x} + \beta\, e^{\mu x},$$

where

$$\alpha = \frac{1}{2}\,(A - iB); \quad \beta = \frac{1}{2}\,(A + iB); \qquad \lambda = s + i\omega; \quad \mu = s - i\omega; \quad (9.5)$$

Thus can be utilized the formulas given in the section 7..

The pair of numbers $[s, \omega]$ can be determined as a solution of any pair from the following four equations

$$\begin{aligned} y_1 \cdot e^{-sx_1}\,\sin(\omega x_2 - \omega x_3) + y_2 \cdot e^{-sx_2}\,\sin(\omega x_3 - \omega x_1) + \\ + y_3 \cdot e^{-sx_3}\,\sin(\omega x_1 - \omega x_2) = 0; \\ y_2 \cdot e^{-sx_2}\,\sin(\omega x_3 - \omega x_4) + y_3 \cdot e^{-sx_3}\,\sin(\omega x_4 - \omega x_2) + \\ + y_4 \cdot e^{-sx_4}\,\sin(\omega x_2 - \omega x_3) = 0; \\ y_1 \cdot e^{-sx_1}\,\sin(\omega x_2 - \omega x_4) + y_2 \cdot e^{-sx_2}\,\sin(\omega x_4 - \omega x_1) + \\ + y_4 \cdot e^{-sx_4}\,\sin(\omega x_1 - \omega x_2) = 0; \\ y_1 \cdot e^{-sx_1}\,\sin(\omega x_3 - \omega x_4) + y_3 \cdot e^{-sx_3}\,\sin(\omega x_4 - \omega x_1) + \\ + y_4 \cdot e^{-sx_4}\,\sin(\omega x_1 - \omega x_3) = 0 \end{aligned}$$

at additional $\omega \neq 0$.

The parameters A and B are determined by the relation

$$A = \frac{y_1 \cdot e^{-sx_1}\,\sin(\omega x_4) - y_4 \cdot e^{-sx_4}\,\sin(\omega x_1)}{\sin(\omega x_4 - \omega x_1)};$$
$$B = \frac{-y_1 \cdot e^{-sx_1}\,\cos(\omega x_4) + y_4 \cdot e^{-sx_4}\,\cos(\omega x_1)}{\sin(\omega x_4 - \omega x_1)},$$

and in these formulas two pairs of variables $[x_1, y_1]$ and $[x_4, y_4]$ can be replaced by any other pairs $[x_j, y_j]$ and $[x_k, y_k]$ provided that $j \neq k$.

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 \equiv \Delta x$, the values $\xi_1 \equiv \exp\big((s + i\omega) \cdot \Delta x\big)$ and $\xi_2 \equiv \exp\big((s - i\omega) \cdot \Delta x\big)$ are the radicals of the quadratic equation

$$\xi^2 + p\,\xi + q = 0, \quad (9.6)$$

where

$$p = \frac{y_1\, y_4 - y_2\, y_3}{y_2^2 - y_1\, y_3}; \qquad q = \frac{y_3^2 - y_2\, y_4}{y_2^2 - y_1\, y_3}.$$

The initial system of equations has a solution in that and only in that case, when the given quadratic equation has two complex conjugate radicals (with nonzero imaginary parts). Thus

$$s = \frac{1}{\Delta x} \cdot \ln|\xi_1| = \frac{1}{2\,\Delta x} \cdot \ln q, \tag{9.7}$$

and it is possible to represent the set of every possible values ω by the unions of members of two sequences $[\omega'_k]$ and $[\omega''_k]$, where

$$\omega'_k = \omega_0 + k \cdot \Delta\omega; \quad \omega''_k = -\omega_0 + (k+1) \cdot \Delta\omega; \quad \Delta\omega = 2\pi/|\Delta x|;$$

$$\omega_0 = \left|\frac{\arg \xi_1}{\Delta x}\right| = \frac{\Delta\omega}{4} + \frac{1}{|\Delta x|} \cdot \arctan\Big(p \cdot \big(4q - p^2\big)^{-1/2}\Big). \tag{9.8}$$

10. Function $h + e^{sx} \cdot \big(A\, \cos(\omega x) + B\, \sin(\omega x)\big)$

Let it be required to interpolate the considered function on five pairs of numbers $\{x_k, y_k\}$ $(k = 1, ..., 5)$, i.e. to solve the system of equations

$$h + e^{s x_k} \cdot \big(A\, \cos(\omega x_k) + B\, \sin(\omega x_k)\big) = y_k, \qquad k = 1, ..., 5$$

regarding parameters h, A, B, s, ω.

This interpolation is equivalent to interpolation of the function

$$h + \alpha\, e^{\lambda x} + \beta\, e^{\mu x},$$

where α and β are determined on relation (9.5), thus can be utilized the formulas, introduced in the section 8..

The pair of numbers $[s, \omega]$ can be determined as a radical of the system of equations at additional condition $\omega \neq 0$. One of the equations of this system looks like

$$\begin{aligned}
&y_1 \cdot \big(e^{s\,(x_3+x_2)} \cdot \sin(\omega x_3 - \omega x_2) + e^{s\,(x_2+x_4)} \cdot \sin(\omega x_2 - \omega x_4) + \\
&\qquad + e^{s\,(x_4+x_3)} \cdot \sin(\omega x_4 - \omega x_3)\big) + \\
&y_2 \cdot \big(e^{s\,(x_1+x_3)} \cdot \sin(\omega x_1 - \omega x_3) + e^{s\,(x_3+x_4)} \cdot \sin(\omega x_3 - \omega x_4) + \\
&\qquad + e^{s\,(x_4+x_1)} \cdot \sin(\omega x_4 - \omega x_1)\big) + \\
&y_3 \cdot \big(e^{s\,(x_2+x_1)} \cdot \sin(\omega x_2 - \omega x_1) + e^{s\,(x_1+x_4)} \cdot \sin(\omega x_1 - \omega x_4) + \\
&\qquad + e^{s\,(x_4+x_2)} \cdot \sin(\omega x_4 - \omega x_2)\big) + \\
&y_4 \cdot \big(e^{s\,(x_1+x_2)} \cdot \sin(\omega x_1 - \omega x_2) + e^{s\,(x_2+x_3)} \cdot \sin(\omega x_2 - \omega x_3) + \\
&\qquad + e^{s\,(x_3+x_1)} \cdot \sin(\omega x_3 - \omega x_1)\big) = 0,
\end{aligned}$$

and the second one can be obtained from the first by replacement of an arbitrary pair of numbers $[x_j, y_j]$ $(j = 1, ..., 4)$ with $[x_5, y_5]$.

The parameters h, A and B are determined by the relations

$$A = \frac{1}{D}\big((y_1 - y_2)\cdot e^{sx_3}\sin(\omega x_3) + (y_2 - y_3)\cdot e^{sx_1}\sin(\omega x_1) + \\ +(y_3 - y_1)\cdot e^{sx_2}\sin(\omega x_2)\big);$$

$$B = \frac{1}{D}\big((y_2 - y_1)\cdot e^{sx_3}\cos(\omega x_3) + (y_3 - y_2)\cdot e^{sx_1}\cos(\omega x_1) + \\ +(y_1 - y_3)\cdot e^{sx_2}\cos(\omega x_2)\big);$$

$$h = \frac{1}{D}\big(y_1\cdot e^{s\,(x_2+x_3)}\sin(\omega x_2 - \omega x_3) + y_2\cdot e^{s(x_3+x_1)}\sin(\omega x_3 - \omega x_1) + \\ +y_3\cdot e^{s\,(x_1+x_2)}\sin(\omega x_1 - \omega x_2)\big),$$

where

$$D = e^{s\,(x_1+x_2)}\sin(\omega x_1 - \omega x_2) + e^{s\,(x_2+x_3)}\sin(\omega x_2 - \omega x_3) + \\ +e^{s\,(x_3+x_1)}\sin(\omega x_3 - \omega x_1),$$

and in these formulas three pairs of variables $[x_1, y_1]$, $[x_2, y_2]$ and $[x_3, y_3]$ can be replaced by any other pairs $[x_j, y_j]$, $[x_k, y_k]$ and $[x_l, y_l]$, provided that $j \neq k \neq l$.

If values of the argument are equidistant from each other, i.e. $x_2 - x_1 = x_3 - x_2 = x_4 - x_3 = x_5 - x_4 \equiv \Delta x$, the values $\xi_1 \equiv \exp\big((s + i\omega)\cdot\Delta x\big)$ and $\xi_2 \equiv \exp\big((s - i\omega)\cdot\Delta x\big)$ are the radicals of the quadratic equation (9.6), where p and q are determined by the relation (8.4). The initial system of equations has a solution in that and only in that case, when the given quadratic equation has two complex conjugate radicals (with nonzero imaginary parts). Thus s is calculated according to (9.7), and the set of every possible values ω is possible to represent by unions of members of two sequences $[\omega'_k]$ and $[\omega''_k]$, where ω'_k, ω''_k, ω_0 and $\Delta\omega$ are determined on the relation (9.8).

The above described algorithms are realized by the way of programs on the programming language of Delphi for IBM-compatible personal computers and are used in the universal application package of processing of the experimental information in the section of restoration of functional dependencies at finding of optimum initial intervals at definition of unknowns parameters [6].

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