FOURTH ORDER OF ACCURACY SEQUENTIAL TYPE DECOMPOSITION SCHEME FOR EVOLUTION PROBLEM

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Abstract

In the present work sequential type decomposition scheme of the fourth order of accuracy for the solution of evolution problem is offered. For the considered scheme the explicit a priori estimations are obtained.

Key words and phrases: Decomposition method, Operator split, Semigroup, Trotter formula, Cauchy abstract problem.

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1. Introduction

One of the most effective methods to solve multi-dimensional evolution problems is a decomposition method. Decomposition schemes with first and second order accuracy were constructed in the sixties of the XX century (see [7], [11] and references therein). Q. Sheng has proved that in the real number field there do not exist automatically stable decomposition schemes with an accuracy order higher than two (see [12]). Decomposition schemes are called automatically stable if a sum of the absolute values of its split coefficients (coefficients of exponentials' products) equals to one, and the real parts of exponential powers are positive. In the work [1] there is constructed decomposition schemes with the higher order accuracy, but their corresponding decomposition formulas are not automatically stable. In the works [2]-[5] introducing the complex parameter, we have constructed automatically stable decomposition schemes with third order accuracy for two- and multi-dimensional evolution problems and with fourth order accuracy for two-dimensional evolution problem (evolution problem with the operator A is called m-dimensional, if it can be represented as a sum of m summands $A = A_1 + ... + A_m$). The new idea is an introduction of a complex parameter, which allows us to break the order 2 barrier.

Decomposition formulas constructed in the above mentioned works represent formulas of exponential splitting. Exponential splitting is called a splitting which approximates a semigroup by a combination of semigroups generated by the summands of the operator generating the given semigroup.

In the present work, we have constructed the fourth order precision exponential splitting for an evolution problem. For the For the considered scheme the explicit *a priori* estimate are obtained. In [12] we have constructed and ivestigated analogical type third order of accurate decomposition scheme.

2. Statement of the Problem

Let us consider the Cauchy abstract problem for an evolution equation in the Banach space X:

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = \varphi, \tag{2.1}$$

where A is a linear closed operator with a definition domain $D(A)$, which is everywhere dense in X, φ is a given element from $D\,(A)$.

Suppose that the operator $(-A)$ generates a strongly continuous semigroup $\{exp(-tA)\}_{t\geq 0}$. Then the solution of problem (2.1) is given by the following formula [8,9]:

$$u(t) = U(t, A)\varphi, \tag{2.2}$$

where $U(t, A) = exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where $A_i \quad (i = 1, 2)$ are closed operators, densely defined in X.

Let us introduce a grid set:

$$\overline{\omega}_\tau = \{t_k = k\tau, k = 1, 2, ..., \tau > 0\}.$$

Together with problem (2.1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned}
\frac{dv_k^{(1)}(t)}{dt} + \frac{\overline{\alpha}}{4}A_1 v_k^{(1)}(t) &= 0, \quad v_k^{(1)}(t_{k-1}) = u_{k-1}(t_{k-1}), \\
\frac{dv_k^{(2)}(t)}{dt} + \frac{\overline{\alpha}}{2}A_2 v_k^{(2)}(t) &= 0, \quad v_k^{(2)}(t_{k-1}) = v_k^{(1)}(t_k), \\
\frac{dv_k^{(3)}(t)}{dt} + \frac{1}{4}A_1 v_k^{(3)}(t) &= 0, \quad v_k^{(3)}(t_{k-1}) = v_k^{(2)}(t_k), \\
\frac{dv_k^{(4)}(t)}{dt} + \frac{\alpha}{2}A_2 v_k^{(4)}(t) &= 0, \quad v_k^{(4)}(t_{k-1}) = v_k^{(3)}(t_k),
\end{aligned}$$

$$\begin{aligned}
\frac{dv_k^{(5)}(t)}{dt}+\frac{\alpha}{2}A_1v_k^{(5)}(t) &= 0, \quad v_k^{(5)}(t_{k-1})=v_k^{(4)}(t_k),\\
\frac{dv_k^{(6)}(t)}{dt}+\frac{\alpha}{2}A_2v_k^{(6)}(t) &= 0, \quad v_k^{(6)}(t_{k-1})=v_k^{(5)}(t_k),\\
\frac{dv_k^{(7)}(t)}{dt}+\frac{1}{4}A_2v_k^{(7)}(t) &= 0, \quad v_k^{(7)}(t_{k-1})=v_k^{(6)}(t_k),\\
\frac{dv_k^{(8)}(t)}{dt}+\frac{\overline{\alpha}}{2}A_1v_k^{(8)}(t) &= 0, \quad v_k^{(8)}(t_{k-1})=v_k^{(7)}(t_k),\\
\frac{dv_k^{(9)}(t)}{dt}+\frac{\overline{\alpha}}{4}A_2v_k^{(9)}(t) &= 0, \quad v_k^{(9)}(t_{k-1})=v_k^{(8)}(t_k),
\end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha)>0$; $u_0(0)=\varphi$. Suppose that the operators $(-A_j),(-\alpha A_j),(-\overline{\alpha}A_j)$, $j=1,2$ generate strongly continuous semigroups.

$u_k(t)$, $k=1,2,..$, is defined on each interval $[t_{k-1},t_k]$ as follows:

$$u_k(t)=v_k^{(9)}(t). \tag{2.3}$$

We declare function $u_k(t)$ as an approximated solution of problem (2.1) on each interval $[t_{k-1},t_k]$.

3. Estimate of Error of the Approximated Solution

We need the natural powers (A^s, $s=2,3,4,5$) of the operator $A=A_1+A_2$. They are defined as usually. It is obvious that the definition domain $D(A^s)$ of the operator A^s represents an intersection of definition domains of its addends.

Let us introduce the following notations:

$$\begin{aligned}
\|\varphi\|_A &= \|A_1\varphi\|+\|A_2\varphi\|, \quad \varphi\in D(A);\\
\|\varphi\|_{A^2} &= \left\|A_1^2\varphi\right\|+\left\|A_2^2\varphi\right\|+\|A_1A_2\varphi\|\\
&+ \|A_2A_1\varphi\|, \quad \varphi\in D\left(A^2\right),
\end{aligned}$$

where $\|\cdot\|$ is a norm in X. $\|\varphi\|_{A^s}$, $(s=3,4,5)$ is defined analogously.

Theorem. *Let the following conditions be fulfilled:*

(a) $\alpha=\frac{1}{2}\pm i\frac{1}{2\sqrt{3}}$ $\left(i=\sqrt{-1}\right)$;

(b) Operators $(-\gamma A_j)$, $\gamma=1,\ \alpha,\ \overline{\alpha}$ $(j=1,2)$ *and* $(-A)$ *generate strongly continuous semigroups, for which the following estimates are true:*

$$\begin{aligned}
\|U(t,\gamma A_j)\| &\le e^{\omega t},\\
\|U(t,A)\| &\le Me^{\omega t}, \quad M=const>0;
\end{aligned}$$

(c) $U(s,A)\varphi \in D\left(A^5\right)$ *for each fixed* $s \geq 0$.
Then the following estimate holds:

$$\|u(t_k) - u_k(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[0,t_k]} \|U(s,A)\varphi\|_{A^5},$$

where c *and* ω_0 *are positive constants.*

Proof. According to the following formula (see [8], p. 603):

$$A\int_r^t U(s,A)\,ds = U(r,A) - U(t,A), \quad 0 \leq r \leq t,$$

we can obtain the expansion:

$$U(t,A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t,A), \tag{3.1}$$

where

$$R_k(t,A) = (-A)^k \int_0^t \int_0^{s_1} \ldots \int_0^{s_{k-1}} U(s,A) ds ds_{k-1} \ldots ds_1. \tag{3.2}$$

From formula (2.2) we obtain:

$$u_k(t_k) = V^k(\tau)\varphi, \tag{3.3}$$

where

$$\begin{aligned} V(\tau) &= U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right) \\ &\quad \times U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right) U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right). \end{aligned}$$

Remark 2.1. *Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the Theorem. In this case, for the solving operator, the following estimate holds:*

$$\left\|V^k(\tau)\right\| \leq e^{\omega_1 t_k}, \tag{3.4}$$

where ω_1 *is a positive constant.*

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ $(i = 1, 2)$. Let us decompose every semigroup included in operator $T(\tau)$ according to formula (3.1), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $\left(\tau^2 A_i A_j\right)$, $\left(-\tau^3 A_i A_j A_k\right)$ and $\left(-\tau^4 A_i A_j A_k A_l\right)$ $(i,j,k,l=1,2)$ respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (3.1) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^{2} [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^{2} [V(\tau)]_{i,j} A_i A_j \\ &\quad - \tau^3 \sum_{i,j,k=1}^{2} [V(\tau)]_{i,j,k} A_i A_j A_k \\ &\quad + \tau^4 \sum_{i,j,k,l=1}^{2} [V(\tau)]_{i,j,k,l} A_i A_j A_k A_l + R_5(\tau). \end{aligned} \tag{3.5}$$

According to the first inequality of the condition *(b)* of the Theorem, for $R_5(\tau)$, the following estimate holds:

$$\|R_5(\tau)\varphi\| \leq c e^{\omega_0 \tau} \tau^5 \|\varphi\|_{A^4}, \quad \varphi \in D\left(A^5\right), \tag{3.6}$$

where c and ω_0 are positive constants.

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (3.5). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i=1$ and $i=2$. Let us consider the case $i=1$. We obviously have:

$$[V(\tau)]_1 = [U(\tau, A_1)]_1 = 1. \tag{3.7}$$

Analogously for $i=2$ we have:

$$[V(\tau)]_2 = [U(\tau, A_2)]_2 = 1. \tag{3.8}$$

By combining formulas (3.7) and (3.8), we will obtain:

$$[V(\tau)]_i = 1, \quad i=1,2. \tag{3.9}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ $(i,j=1,2)$ corresponding to the second order members included in formula (3.5). On the whole we have two cases: $(i,j) = (1,1),\ (1,2),\ (2,1),\ (2,2)$. Let us consider the case $(i,j) = (1,1)$. We obviously have:

$$[V(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}. \tag{3.10}$$

Analogously for $(i,j) = (2,2)$ we have:

$$[V(\tau)]_{2,2} = [U(\tau, A_2)]_{2,2} = \frac{1}{2}. \tag{3.11}$$

Let us consider the case $(i,j) = (1,2)$, we obviously have:

$$\begin{aligned}
[V(\tau)]_{1,2} \;=\;& \left[U\left(\tau, \frac{\overline{\alpha}}{4}A_1\right)\right]_1 \left(2\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + 2\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
& + \left[U\left(\tau, \frac{1}{4}A_1\right)\right]_1 \left(\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + 2\left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
& + \left[U\left(\tau, \frac{\alpha}{2}A_1\right)\right]_1 \left(\left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 + \left[U\left(\tau, \frac{\alpha}{2}A_2\right)\right]_2\right) \\
& + \left[U\left(\tau, \frac{1}{4}A_1\right)\right]_1 \left[U\left(\tau, \frac{\overline{\alpha}}{2}A_2\right)\right]_2 \\
=\;& \frac{\overline{\alpha}\left(\overline{\alpha}+\alpha\right)}{4} + \frac{\overline{\alpha}+2\alpha}{8} + \frac{\alpha\left(\overline{\alpha}+\alpha\right)}{4} + \frac{\overline{\alpha}}{8} \\
=\;& \frac{2\overline{\alpha}+1+\alpha+2\alpha+\overline{\alpha}}{8} = \frac{1}{2}.
\end{aligned} \tag{3.12}$$

For $(i,j) = (2,1)$ we have:

$$[V(\tau)]_{2,1} = \frac{1}{2}. \tag{3.13}$$

Here we used the identity $\alpha + \overline{\alpha} = 1$.

By combining formulas (3.10) - (3.13), we will obtain:

$$[V(\tau)]_{i,j} = \frac{1}{2}, \quad i,j = 1,2. \tag{3.14}$$

Let us calculate the coefficients $[V(\tau)]_{i,,j,k}$ $(i,j,k = 1,2)$ corresponding to the third order members in formula (3.5). On the whole we have eight cases: $(i,j,k) = (1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (2,1,1),\ (2,1,2),\ (2,2,1),\ (2,2,2)$. Let us consider the case $(i,j,k) = (1,1,1)$. We obviously have:

$$[V(\tau)]_{1,1,1} = [U(\tau, A_1)]_{1,1,1} = \frac{1}{6}. \tag{3.15}$$

Analogously for $(i,j) = (2,2,2)$ we have:

$$[V(\tau)]_{2,2,2} = [U(\tau, A_2)]_{2,2,2} = \frac{1}{6}. \tag{3.16}$$

Thus Let us calculate the case $(i,j,k)=(1,1,2)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,1,2} &= \left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_{1,1}\left(2\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_{1,1}\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_{1,1}\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_{1,1}\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+2\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\\
&\times\left(\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2+\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2\right)\\
&+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\\
&= \frac{\overline{\alpha}^2\left(\overline{\alpha}+\alpha\right)}{32}+\frac{\overline{\alpha}+2\alpha}{64}+\frac{\alpha^2\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}}{64}+\frac{\overline{\alpha}\left(\overline{\alpha}+2\alpha\right)}{32}\\
&+\frac{\overline{\alpha}\alpha\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}^2}{32}+\frac{\alpha\left(\overline{\alpha}+\alpha\right)}{16}+\frac{\overline{\alpha}}{32}+\frac{\alpha\overline{\alpha}}{16}=\frac{1}{6}. \qquad (3.17)
\end{aligned}$$

For $(i,j,k)=(2,1,1)$ we have:

$$[V(\tau)]_{2,1,1}=\frac{1}{6} \qquad (3.18)$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,2)$. We have:

$$[V(\tau)]_{1,2,2}=\frac{1}{6}. \tag{3.19}$$

For $(i,j,k)=(2,1,1)$ we have:

$$[V(\tau)]_{2,1,1}=\frac{1}{6} \tag{3.20}$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

Thus Let us calculate the case $(i,j,k)=(1,2,1)$. We have:

$$\begin{aligned}
[V(\tau)]_{1,2,1} = {} & \left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\right. \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1+\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2 \\
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\alpha}{2}A_2\right)\right]_2
\end{aligned}$$

$$
\begin{aligned}
& \times\left(\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1+\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1\right) \\
& +\left[U\left(\tau,\frac{\alpha}{2}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
& +\left[U\left(\tau,\frac{1}{4}A_1\right)\right]_1\left[U\left(\tau,\frac{\overline{\alpha}}{2}A_2\right)\right]_2\left[U\left(\tau,\frac{\overline{\alpha}}{4}A_1\right)\right]_1 \\
= \; & \frac{\overline{\alpha}^2\left(2+2\alpha+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}\alpha\left(1+2\alpha+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}\alpha\left(1+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}^3}{32} \\
& +\frac{\alpha\left(1+2\alpha+\overline{\alpha}\right)}{32}+\frac{\alpha\left(1+\overline{\alpha}\right)}{32}+\frac{\overline{\alpha}^2}{32}+\frac{\alpha^2\left(1+\overline{\alpha}\right)}{16}+\frac{\alpha\overline{\alpha}^2}{16}+\frac{\overline{\alpha}^2}{32} \\
= \; & \frac{4+2\overline{\alpha}-\frac{2}{3}+2\alpha}{32}=\frac{6-\frac{2}{3}}{32}=\frac{1}{6}.
\end{aligned} \tag{3.21}
$$

For $(i,j,k)=(2,1,2)$ we have:

$$
\begin{aligned}
[V(\tau)]_{2,1,2} &= \left[U\left(\tau,\alpha A_2\right)\right]_2\left[U\left(\tau,\frac{1}{2}A_1\right)\right]_1\left[U\left(\tau,\overline{\alpha}A_2\right)\right]_2 \\
&= \alpha\frac{1}{2}\overline{\alpha}=\frac{1}{6}.
\end{aligned} \tag{3.22}
$$

Here we used the identities $\alpha+\overline{\alpha}=1,\ \alpha\overline{\alpha}=\frac{1}{3}$ and $\alpha^2+\overline{\alpha}^2=\frac{1}{3}$.

By combining formulas (3.15) - (3.22), we will obtain:

$$
[V(\tau)]_{i,j,k}=\frac{1}{6},\quad i,j,k=1,2. \tag{3.23}
$$

Analogouosly we can show that

$$
[V(\tau)]_{i,j,k,l}=\frac{1}{24},\quad i,j,k,l=1,2. \tag{3.24}
$$

From equality (3.5), taking into account formulas (3.9), (3.14), (3.23) and (3.24), we will obtain:

$$
\begin{aligned}
V\left(\tau\right) &= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\sum_{i,j=1}^{2}A_iA_j-\frac{1}{6}\tau^3\sum_{i,j,k=1}^{2}A_iA_jA_k \\
&\quad +\frac{1}{24}\tau^4\sum_{i,j,k,l=1}^{2}A_iA_jA_kA_l+R_5\left(\tau\right) \\
&= I-\tau\sum_{i=1}^{2}A_i+\frac{1}{2}\tau^2\left(\sum_{i=1}^{2}A_i\right)^2-\frac{1}{6}\tau^3\left(\sum_{i=1}^{2}A_i\right)^3 \\
&\quad +\frac{1}{24}\tau^4\left(\sum_{i=1}^{2}A_i\right)^4+R_5\left(\tau\right) \\
&= I-\tau A+\frac{1}{2}\tau^2A^2-\frac{1}{6}\tau^3A^3+\frac{1}{24}\tau^4A^4+R_5\left(\tau\right).
\end{aligned} \tag{3.25}
$$

According to formula (3.1) we have:

$$U(\tau, A) = I - \tau A + \frac{1}{2}\tau^2 A^2 - \frac{1}{6}\tau^3 A^3 + \frac{1}{24}\tau^4 A^4 + R_5(\tau, A). \qquad (3.26)$$

According to condition *(b)* of the second inequality of the Theorem, for $R_5(\tau, A)$, the following estimate holds:

$$\|R_5(\tau, A)\varphi\| \le c e^{\omega\tau}\tau^5 \left\|A^5\varphi\right\| \le c e^{\omega\tau}\tau^5 \|\varphi\|_{A^5}. \qquad (3.27)$$

According to equalities (3.25) and (3.26) we have:

$$U(\tau, A) - V(\tau) = R_5(\tau, A) - R_5(\tau).$$

From here, taking into account inequalities (3.6) and (3.27), we will obtain the following estimate:

$$\|[U(\tau, A) - V(\tau)]\varphi\| \le c e^{\omega_2\tau}\tau^5 \|\varphi\|_{A^5}. \qquad (3.28)$$

From equalities (2.2) and (3.3), taking into account inequalities (3.4) and (3.28), we will obtain:

$$\begin{aligned}
\|u(t_k) - u_k(t_k)\| &= \left\|\left[U(t_k, A) - V^k(\tau)\right]\varphi\right\| = \left\|\left[U^k(\tau, A) - V^k(\tau)\right]\varphi\right\| \\
&= \left\|\left[\sum_{i=1}^{k} V^{k-i}(\tau)\left[U(\tau, A) - V(\tau)\right]U((i-1)\tau, A)\right]\varphi\right\| \\
&\le \sum_{i=1}^{k} \|V(\tau)\|^{k-i} \left\|\left[U(\tau, A) - V(\tau)\right]U((i-1)\tau, A)\varphi\right\| \\
&\le \sum_{i=1}^{k} e^{\omega_1(k-i)\tau} c e^{\omega_2\tau}\tau^5 \|U((i-1)\tau, A)\varphi\|_{A^5} \\
&\le c e^{\omega_0 t_k}\tau^5 \sum_{i=1}^{k} \|U((i-1)\tau, A)\varphi\|_{A^5} \\
&\le k c e^{\omega_0 t_k}\tau^4 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^5} \\
&\le c e^{\omega_0 t_k} t_k \tau^4 \sup_{s\in[o,t_k]} \|U(s, A)\varphi\|_{A^5} \qquad \blacksquare
\end{aligned}$$

Remark 2.2. *The fourth order of accurate decomposition formula in case of Multidimensional problem has the following form:*

$$V^{(m)}(\tau) =, \quad m \ge 2.$$

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